UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XL
2002

FINITE DIFFERENCE APPROXIMATIONS FOR NONLINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Classical solutions of nonlinear partial differential equations are approximated in the paper by solutions of quasilinear systems of difference equations. Sufficient conditions for the convergence of the method are given. The proof of the stability of the difference problem is based on a comparison method.

This new approach to the numerical solving of nonlinear equations is generated by a linearization method for initial problems. Numerical examples are given.

1. Difference systems corresponding to nonlinear equations. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let E be the Haar pyramid

$$E = \{ (t,x) = (t,x_1,\dots,x_n) \in R^{1+n} : t \in [0,a],\ -b + Mt \le x \le b - Mt \}$$

where $a > 0$, $M = (M_1,\dots,M_n) \in R^n_+$, $R_+ = [0,+\infty)$, $b = (b_1,\dots,b_n) \in R^n$ and $b \ge Ma$. Write $\Omega = E \times R \times R^n$ and suppose that $f\colon \Omega \to R$ is a given function of the variables (t,x,p,q), $q = (q_1,\dots,q_n)$. We consider the nonlinear first order partial differential equation

$$\partial_t z(t,x) = f(\, t, x, z(t,x), \partial_x z(t,x)\,) \tag{1}$$

with the initial condition

$$z(0,x) = \varphi(x), \quad x \in [-b,b], \tag{2}$$

where $\varphi\colon [-b,b] \to R$ is a given function and $\partial_x z = (\partial_{x_1} z,\dots,\partial_{x_n} z)$. We are interested in the construction of a method for the approximation of solutions to problem (1), (2) with solutions of associated difference equations and in the estimation of the difference between these solutions. The classical difference methods for nonlinear partial differential equations consist in replacing partial derivatives with difference expressions. Then, under suitable assumptions on given functions and on the mesh, solutions of difference equations approximate solutions of the original problem.

Let **N** and **Z** be the sets of natural numbers and integers, respectively. For $x, y \in R^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we write

$$x \diamond y = (x_1 y_1, \ldots, x_n y_n) \in R^n, \;\; \|x\| = \sum_{i=1}^{n} |x_i|.$$

We define a mesh on the set E in the following way. Suppose that (h_0, h') where $h' = (h_1, \ldots, h_n)$ stand for steps of the mesh. For $h = (h_0, h')$ and $(i, m) \in \mathbf{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(i)} = ih_0, \;\; x^{(m)} = m \diamond h', \;\; x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}).$$

Denote by Δ the set of all $h = (h_0, h')$ such that there is $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ with the property $N \diamond h' = b$. We assume that $\Delta \neq \emptyset$ and that there is a sequence $\{ h^{(j)} \}$, $h^{(j)} \in \Delta$, such that $\lim_{j\to\infty} h^{(j)} = 0$. There is $N_0 \in \mathbf{N}$ such that $N_0 h_0 \leq a < (N_0 + 1)h_0$. Let

$$R_h^{1+n} = \{ (t^{(i)}, x^{(m)}) : \;\; (i, m) \in \mathbf{Z}^{1+n} \}$$

and

$$E_h = E \cap R_h^{1+n}, \;\; E'_h = \{ (t^{(i)}, x^{(m)}) \in E_h : \;\; (t^{(i)} + h_0, x^{(m)}) \in E_h \},$$

$$E_{0.h} = \{ x^{(m)} : \;\; -N \leq m \leq N \}, \;\; I_h = \{ t^{(i)} : \;\; 0 \leq i \leq N_0 \}.$$

For a function $z \colon E_h \to R$ and for a point $(t^{(i)}, x^{(m)}) \in E_h$ we write $z^{(i,m)} = z(t^{(i)}, x^{(m)})$. For $1 \leq j \leq n$ we put $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, 1 standing on the j-th place. We define difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ in the following way. For $z \colon E_h \to R$ we put

$$\delta_0 z^{(i,m)} = \frac{1}{h_0} \left(z^{(i+1,m)} - z^{(i,m)} \right), \tag{3}$$

$$\delta_j z^{(i,m)} = \frac{1}{h_j} \left(z^{(i,m+e_j)} - z^{(i,m)} \right), \;\; 1 \leq j \leq \kappa, \tag{4}$$

$$\delta_j z^{(i,m)} = \frac{1}{h_j} \left(z^{(i,m)} - z^{(i,m-e_j)} \right), \;\; \kappa + 1 \leq j \leq n, \tag{5}$$

where $0 \leq \kappa \leq n$ is fixed. If $\kappa = 0$ then δ is given by (5), for $\kappa = n$, δ is defined by (4). Write

$$\delta z^{(i,m)} = (\delta_1 z^{(i,m)}, \ldots, \delta_n z^{(i,m)}).$$

Suppose that problem (1), (2) is solved numerically by the difference method

$$\delta_0 z^{(i,m)} = f(\,t^{(i)}, x^{(m)}, z^{(i,m)}, \delta z^{(i,m)}\,), \tag{6}$$

$$z^{(0,m)} = \varphi_h^{(m)}, \;\; x^{(m)} \in E_{0.h}, \tag{7}$$

where $\varphi_h \colon E_{0.h} \to R$ is a given function. If we assume that $h' \le h_0 M$ then the set E_h has the following property: if $(t^{(i)}, x^{(m)}) \in E'_h$ then $(t^{(i)}, x^{(m+e_j)})$, $(t^{(i)}, x^{(m-e_j)}) \in E_h$ for $1 \le j \le n$ and consequently, there exists exactly one solution $z_h \colon E_h \to R$ of problem (6), (7). Sufficient conditions for the convergence of the method (6), (7) to a solution of (1), (2) are given in the following theorem.

THEOREM 1.1. *Suppose that*

1) $f \in C(\Omega, R)$ and the derivatives $(\,\partial_{q_1} f, \dots, \partial_{q_n} f\,) = \partial_q f$ exist on Ω and $\partial_q f \in C(\Omega, R^n)$,

2) there is $A \in R_+$ such that

$$|\, f(t,x,p,q) - f(t,x,\bar p,q)\,| \le A|p - \bar p| \;\; on \;\; \Omega, \tag{8}$$

3) $h \in \Delta$, $h' \le h_0 M$ and

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j}\,|\,\partial_{q_j} f(\,P\,)\,| \ge 0$$

where $P = (t,x,p,q)$ and

$$\partial_{q_j} f(\,P\,) \ge 0 \; for \; 1 \le j \le \kappa, \;\; \partial_{q_j} f(\,P\,) \le 0 \; for \; \kappa + 1 \le j \le n, \tag{9}$$

4) $v \colon E \to R$ is a solution of (1), (2), v is of class C^1 and there is a function $\alpha_0 \colon \Delta \to R_+$ such that

$$|\,\varphi^{(m)} - \varphi_h^{(m)}\,| \le \alpha_0(h) \; on \; E_{0.h} \;\; and \;\; \lim_{h\to 0} \alpha_0(h) = 0,$$

5) $z_h \colon E_h \to R$ is a solution of (6), (7).

Under these assumptions there is a function $\alpha \colon \Delta \to R_+$ such that

$$|\,v^{(i,m)} - z_h^{(i,m)}\,| \le \alpha(h) \; on \; E_h \;\; and \;\; \lim_{h\to 0} \alpha(h) = 0. \tag{10}$$

The above theorem is a consequence of results presented in [**1**]–[**3**], see also [**4**]. Note that the Lipschitz condition (8) may be replaced in the theorem by a nonlinear estimate of the Perron type.

The following condition is important in these considerations. Write

$$\operatorname{sign} \partial_q f = (\,\operatorname{sign} \partial_{q_1} f, \dots, \operatorname{sign} \partial_{q_n} f\,)\,. \tag{11}$$

We have assumed in Theorem 1.1 that function (11) is constant on Ω.

Remark 1.2. *Suppose that all the assumptions of Theorem 1.1 are satisfied and*

1) *the solution* $v\colon E \to R$ *of (1), (2) is of class* C^2 *and* $\tilde{c} \in R_+$ *is such a constant that*

$$|\partial_{x_j} v(t,x)| \le \tilde{c},\ |\partial_{tt} v(t,x)| \le \tilde{c},\ |\partial_{x_j x_j} v(t,x)| \le \tilde{c}, 1 \le j \le n,$$

where $(t,x) \in E$,

2) *there is* $A_0 \in R_+$ *such that* $\|\partial_q f(t,x,p,q)\| \le A_0$ *on* Ω.

Then we have the following error estimate for the method (6), (7):

$$|v^{(i,m)} - z_h^{(i,m)}| \le \bar{\alpha}(h) \ \text{ on } \ E_h \tag{12}$$

where

$$\bar{\alpha}(h) = \alpha_0(h) e^{Aa} + h_0 \frac{\tilde{c}}{2} (1 + A_0\, M_\star)\, \theta(A), \quad M_\star = \max\{M_i : \ \ 1 \le i \le n\},$$

and

$$\theta(A) = \frac{e^{Aa} - 1}{A} \ \text{ if } A > 0, \quad \theta(A) = a \ \text{ if } \ A = 0.$$

The above result can be proved by the methods used in [**1**]–[**2**].

Consider now another difference method for problem (1), (2). Let the operators δ_0, $\delta = (\delta_1, \dots, \delta_n)$ be defined by

$$\begin{aligned} \delta_0 z^{(i,m)} &= \frac{1}{h_0} \left(z^{(i+1,m)} - D z^{(i,m)} \right), \\ D z^{(i,m)} &= \frac{1}{2n} \sum_{j=1}^{n} \left(z^{(i,m+e_j)} + z^{(i,m-e_j)} \right), \end{aligned} \tag{13}$$

$$\delta_j z^{(i,m)} = \frac{1}{2h_j} \left(z^{(i,m+e_j)} - z^{(i,m-e_j)} \right), \quad 1 \le j \le n, \tag{14}$$

where $(t^{(i)}, x^{(m)}) \in E'_h$ and $z\colon E_h \to R$. Consider difference problem (6), (7) with δ_0 and δ given by (13), (14).

Theorem 1.3. *Suppose that conditions 1), 2) of Theorem 1.1 are satisfied and*

1) $h \in \Delta$, $h' \le h_0 M$ *and for* $P = (t,x,p,q) \in \Omega$ *we have*

$$\frac{1}{n} - \frac{h_0}{h_j} \left| \partial_{q_j} f(P) \right| \ge 0, \ \ 1 \le j \le n,$$

2) $v\colon E \to R$ *is a solution of (1), (2),* v *is of class* C^1 *and there is a function* $\alpha_0\colon \Delta \to R_+$ *such that* $|\varphi^{(m)} - \varphi_h^{(m)}| \le \alpha_0(h)$ *on* $E_{0.h}$ *and* $\lim_{h\to 0} \alpha_0(h) = 0$,

3) $z_h\colon E_h \to R$ *is a solution of (6), (7) with* δ_0, δ *given by (13), (14).*

Then there is a function $\alpha\colon \Delta \to R_+$ *such that condition (10) is satisfied.*

This theorem can be proved with use of the methods presented in [**1**], [**4**].

Remark 1.4. *Suppose that all the assumtions of Theoren 1.3 are satisfied and the solution* $v\colon E \to$ *of (1), (2) is of class* C^2*. Then we have the following error estimate for the method: there are* c_0, $c_1 \in R_+$ *such that*

$$|v^{(i,m)} - z_h^{(i,m)}| \le c_0\alpha_0(h) + c_1\, h_0 \;\; on \;\; E_h.$$

Now we formulate a new class of difference problems corresponding to (1), (2). We transform the nonlinear differential equation into a quasilinear system of difference equations. We will use a linearization method for equation (1) with respect to the last variable. We omit the condition that function (11) is constant on Ω and we consider difference operators of the form (3)–(5). We will need the following assumption.

Assumption $\mathbf{H}_0[f]$**.** Suppose that $f \in C(\Omega, R)$ and the derivatives

$$\partial_x f = (\partial_{x_1} f, \ldots, \partial_{x_n} f),\; \partial_p f,\; \partial_q f = (\partial_{q_1} f, \ldots, \partial_{q_n} f)$$

exist on Ω and $\partial_x f$, $\partial_q f \in C(\Omega, R^n)$, $\partial_p f \in C(\Omega, R)$.

Denote by (z, u), $u = (u_1, \ldots, u_n)$ the unknown functions of the variables $(t^{(i)}, x^{(m)})$. Write $u^{(i,m)} = (u_1^{(i,m)}, \ldots, u_n^{(i,m)})$ and

$$P^{(i,m)}[z, u] = \Big(t^{(i)}, x^{(m)}, z^{(i,m)}, u^{(i,m)} \Big).$$

We consider the system of difference equations

$$\delta_0 z^{(i,m)} = f(\, P^{(i,m)}[z,u]\,) + \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[z,u]\,) \Big(\delta_j z^{(i,m)} - u_j^{(i,m)} \Big), \tag{15}$$

$$\begin{aligned} \delta_0 u_r^{(i,m)} &= \partial_{x_r} f(\, P^{(i,m)}[z,u]\,) + \partial_p f(\, P^{(i,m)}[z,u]\,)\, u_r^{(i,m)} \\ &\quad + \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[z,u]\,)\, \delta_j u_r^{(i,m)}, \quad r = 1, \ldots, n, \end{aligned} \tag{16}$$

with the initial condition

$$z^{(0,m)} = \varphi_h^{(m)}, \quad u^{(0,m)} = \psi_h^{(m)}, \;\; -N \le m \le N, \tag{17}$$

where $\varphi_h\colon E_{0.h} \to R$ and $\psi_h = (\psi_{h.1}, \ldots, \psi_{h.n})\colon E_{0.h} \to R^n$ are given functions. The operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ are defined now in the following way. If the functions z and $u = (u_1, \ldots, u_n)$ are calculated on the set $E_h \cap (\, [0, t^{(i)}] \times R^n\,)$ then we put

$$\delta_0 z^{(i,m)} = \frac{1}{h_0} \Big(z^{(i+1,m)} - z^{(i,m)} \Big). \tag{18}$$

The difference operators with respect to the spatial variables are given in the following way:

$$\delta_j z^{(i,m)} = \frac{1}{h_j}\left(z^{(i,m+e_j)} - z^{(i,m)}\right) \ \text{ if } \ \partial_{q_j} f(\,P^{(i,m)}[z,u]\,) \geq 0, \tag{19}$$

and

$$\delta_j z^{(i,m)} = \frac{1}{h_j}\left(z^{(i,m)} - z^{(i,m-e_j)}\right) \ \text{ if } \ \partial_{q_j} f(\,P^{(i,m)}[z,u]\,) < 0, \tag{20}$$

where $j = 1, \ldots, n$. The difference expressions

$$\delta_0 u_r^{(i,m)}, \quad (\,\delta_1 u_r^{(i,m)}, \ldots, \delta_n u_r^{(i,m)}\,), \ \ 1 \leq r \leq n,$$

are defined in the same way.

Note that if $h' \leq h_0 M$ then there exists exactly one solution (z_h, u_h), $z_h\colon E_h \to R$, $u_h = (u_{h.1}, \ldots, u_{h.n}) : E_h \to R^n$, of problem (15)–(20).

It is essential in our considerations that we approximate solutions of nonlinear problem (1), (2) with solutions of the quasilinear difference system. More precisely: we will use (15)–(17) for approximation of the solution $v\colon E \to R$ of problem (1), (2) and the derivative $\partial_x v\colon E \to R^n$.

System (15), (16) is obtained in the following way. We first introduce an additional unknown function $u = \partial_x z$, $u = (u_1, \ldots, u_n)$ in (1). Then we consider the following linearization of (1) with respect to u:

$$\partial_t z(t,x) = f(\,U\,) + \sum_{j=1}^{n} \partial_{q_j} f(\,U\,)\ \left(\,\partial_{x_j} z(t,x) - u_j(t,x)\,\right), \tag{21}$$

where $U = (t, x, z(t,x), u(t,x))$. By differentiating equation (1) with respect to x_r, $1 \leq r \leq n$, we get the differential system in the unknown function u :

$$\partial_t u_r(t,x) = \partial_{x_r} f(\,U\,) + \partial_p f(\,U\,)\, u_r(t,x) + \sum_{j=1}^{n} \partial_{q_j} f(\,U\,)\, \partial_{x_j} u_r(t,x), \qquad 1 \leq r \leq n. \tag{22}$$

Assume that $\partial_x \varphi = (\,\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi\,)$ exists on $[-b, b]$. It is natural to consider the following initial condition for system (21), (22):

$$z(0,x) = \varphi(x), \quad u(0,x) = \partial_x \varphi(x), \ \ x \in [-b,b]. \tag{23}$$

Difference problem (15)–(17) is a discretization of system (21), (22) with initial condition (23). In our approach, the discretization method for system (21), (22) depends on the point of the mesh and on the previous values of z and u.

2. Convergence of difference methods. We will denote by $\mathbf{F}(X,Y)$ the class of all functions defined on X and taking values in Y, X and Y being arbitrary sets. We will need the following assumption throughout the paper.

Assumption H$[f]$**.** Suppose that Assumption $H_0[f]$ is satisfied and

1) there is $A \in R_+$ such that

$$\| \partial_x f(P) \|, \ \ |\partial_p f(P)|, \ \ \| \partial_q f(P) \| \leq A \ \text{ on } \ \Omega$$

where $P = (t,x,p,q)$,

2) there is $B \in R_+$ such that the terms

$$\| \partial_x f(t,x,p,q) - \partial_x f(t,x,\bar{p},\bar{q}) \|, \ \ | \partial_p f(t,x,p,q) - \partial_p f(t,x,\bar{p},\bar{q}) |,$$
$$\| \partial_q f(t,x,p,q) - \partial_q f(t,x,\bar{p},\bar{q}) \|$$

are bounded from above by $B \, [\, |p-\bar{p}| + \|q-\bar{q}\| \,]$.

Theorem 2.1. *Suppose that Assumption H[f] is satisfied and*

1) $h \in \Delta$, $h' \leq h_0 M$ and for $P=(t,x,p,q) \in \Omega$ we have

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \, \big| \partial_{q_j} f(P) \big| \geq 0, \tag{24}$$

2) the function $\varphi\colon [-b,b] \to R$ is of class C^2 and $v\colon E \to R$ is the solution of (1), (2) and v is of class C^2 on E,

3) $(z_h,u_h) = (z_h, u_{h.1}, \ldots, u_{h.n})\colon E_h \to R^{1+n}$ is the solution of problem (15)–(20) and there is $\alpha_0\colon \Delta \to R$ such that

$$|\varphi^{(m)} - \varphi_h^{(m)}| + \|\partial_x \varphi^{(m)} - \psi_h^{(m)} \| \leq \alpha_0(h), \ \ -N \leq m \leq N,$$

and $\lim_{h\to 0} \alpha_0(h) = 0$.

Then there is a function $\alpha\colon \Delta \to R_+$ such that

$$|v^{(i,m)} - z_h^{(i,m)}| + \| \partial_x v^{(i,m)} - u_h^{(i,m)} \| \leq \alpha(h) \ \text{ on } \ E_h$$

and $\lim_{h\to 0} \alpha(h) = 0$.

Proof. Write $w = \partial_x v$ and $w = (w_1, \ldots, w_n)$. Then the functions $(v,w)\colon E \to R^{1+n}$ are the solution of problem (21)–(23). Let the functions

$$\Gamma_{h.0}\colon E'_h \to R, \ \ \Gamma_h\colon E'_h \to R^n, \ \ \Gamma_h = (\Gamma_{h.1}, \ldots, \Gamma_{h.n})$$
$$\Lambda_{h.0}\colon E'_h \to R, \ \ \Lambda_h\colon E'_h \to R^n, \ \ \Lambda_h = (\Lambda_{h.1}, \ldots, \Lambda_{h.n})$$

be defined by

$$\begin{aligned} \Gamma_{h.0}^{(i,m)} &= \delta_0 v^{(i,m)} - \partial_t v^{(i,m)} \\ &+ \sum_{j=1}^{n} \partial_{q_j} f(P^{(i,m)}[v,w]) \left[\partial_{x_j} v^{(i,m)} - \delta_j v^{(i,m)} \right], \end{aligned} \tag{25}$$

$$\Gamma^{(i,m)}_{h.r} = \delta_0 w^{(i,m)}_r - \partial_t w^{(i,m)}_r + \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[v,w]\,)\, \Big[\, \partial_{x_j} w^{(i,m)}_r - \delta_j w^{(i,m)}_r \Big], \quad r = 1,\ldots,n, \tag{26}$$

and

$$\begin{aligned}\Lambda^{(i,m)}_{h.0} &= f(\, P^{(i,m)}[v,w]\,) - f(\, P^{(i,m)}[z_h,u_h]\,) \\ &- \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[v,w]\,)\, w^{(i,m)}_j + \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[z_h,u_h]\,)\, u^{(i,m)}_{h.j} \\ &+ \sum_{j=1}^{n} \Big[\, \partial_{q_j} f(\, P^{(i,m)}[v,w]\,) - \partial_{q_j} f(\, P^{(i,m)}[z_h,u_h]\,)\Big]\, \delta_j v^{(i,m)},\end{aligned} \tag{27}$$

$$\begin{aligned}\Lambda^{(i,m)}_{h.r} &= \partial_{x_r} f(\, P^{(i,m)}[v,w]\,) + \partial_p f(\, P^{(i,m)}[v,w]\,)\, w^{(i,m)}_r \\ &- \partial_{x_r} f(\, P^{(i,m)}[z_h,u_h]\,) - \partial_p f(\, P^{(i,m)}[z_h,u_h]\,) u^{(i,m)}_{h.r} \\ &+ \sum_{j=1}^{n} \Big[\, \partial_{q_j} f(\, P^{(i,m)}[v,w]\,) - \partial_{q_j} f(\, P^{(i,m)}[z_h,u_h]\,)\Big]\, \delta_j w^{(i,m)}_r, \\ & \qquad r = 1,\ldots,n.\end{aligned} \tag{28}$$

Write

$$\xi^{(i,m)}_h = v^{(i,m)} - z^{(i,m)}_h,$$

$$\lambda^{(i,m)}_h = w^{(i,m)} - u^{(i,m)}_h, \;\; \lambda^{(i,m)}_h = \Big(\, \lambda^{(i,m)}_{h.1}, \ldots, \lambda^{(i,m)}_{h.n} \Big).$$

It follows from (15), (16) and from (21), (22) that ξ_h and λ_h satisfy the difference equations

$$\begin{aligned}\xi^{(i+1,m)}_h &= \xi^{(i,m)}_h + h_0 \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[z_h,u_h]\,)\, \delta_j \xi^{(i,m)}_h \\ &+ h_0 \Big[\, \Gamma^{(i,m)}_{h.0} + \Lambda^{(i,m)}_{h.0} \Big],\end{aligned} \tag{29}$$

and

$$\begin{aligned}\lambda^{(i+1,m)}_{h.r} &= \lambda^{(i,m)}_{h.r} + h_0 \sum_{j=1}^{n} \partial_{q_j} f(\, P^{(i,m)}[z_h,u_h]\,)\, \delta_j \lambda^{(i,m)}_{h.r} \\ &+ h_0 \Big[\, \Gamma^{(i,m)}_{h.r} + \Lambda^{(i,m)}_{h.r} \Big], \quad r = 1,\ldots,n.\end{aligned} \tag{30}$$

Let $\omega_{h.0},\ \omega_h\colon I_h \to R$ be the functions defined by

$$\omega_{h.0}^{(i)} = \max\{\, |\xi_h^{(i,m)}| : \ (t^{(i)}, x^{(m)}) \in E_h \,\}, \tag{31}$$

$$\omega_h^{(i)} = \max\{\, \|\lambda_h^{(i,m)}\| : \ (t^{(i)}, x^{(m)}) \in E_h \,\}, \tag{32}$$

where $0 \le i \le N_0$. We will write a difference inequality for the function $\omega_{h.0} + \omega_h$. Put

$$J_+[i,m] = \{\, j \in \{1,\dots,n\} : \ \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,) \ge 0 \,\},$$
$$J_-[i,m] = \{\, 1,\dots,n \,\} \setminus J_+[i,m].$$

Consider the operator $W_h\colon \mathbf{F}(E_h, R) \to \mathbf{F}(E'_h, R)$ defined by

$$\begin{aligned} W_h[\xi]^{(i,m)} = \xi^{(i,m)} & \left[1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left| \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,) \right| \right] \\ & + h_0 \sum_{j \in J_+[i,m]} \frac{1}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,)\, \xi^{(i,m+e_j)} \\ & - h_0 \sum_{j \in J_-[i,m]} \frac{1}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,)\, \xi^{(i,m-e_j)} \end{aligned}$$

where $\xi \in \mathbf{F}(E_h, R)$ and $(t^{(i)}, x^{(m)}) \in E'_h$. It follows from (19), (20) and (29), (30) that

$$\xi_h^{(i+1,m)} = W_h[\xi_h]^{(i,m)} + h_0 \left[\Lambda_{h.0}^{(i,m)} + \Gamma_{h.0}^{(i,m)} \right], \ \ (t^{(i)}, x^{(m)}) \in E'_h. \tag{33}$$

For the function $\lambda_h = (\lambda_{h.1}, \dots, \lambda_{h.n})$ we write

$$W_h[\lambda_h]^{(i,m)} = \Big(W_h[\lambda_{h.1}]^{(i,m)}, \dots, W_h[\lambda_{h.n}]^{(i,m)} \Big).$$

According to (30) and the definition of the difference operators $(\delta_1, \dots, \delta_n)$ we have

$$\lambda_h^{(i+1,m)} = W_h[\lambda_h]^{(i,m)} + h_0 \left[\Lambda_h^{(i,m)} + \Gamma_h^{(i,m)} \right], \ \ (t^{(i)}, x^{(m)}) \in E'_h. \tag{34}$$

We conclude from Assumption H[f] and from condition 2) of the theorem that there are functions $\gamma_0,\ \gamma\colon \Delta \to R_+$ and a constant $\tilde{c} \in R_+$ such that

$$|\Gamma_{h.0}^{(i,m)}| \le \gamma_0(h), \ \ \|\Gamma_h^{(i,m)}\| \le \gamma(h), \ \ \ (t^{(i)}, x^{(m)}) \in E'_h, \tag{35}$$

and

$$\|\, \partial_{x_j} v(t,x)\, \| \le \tilde{c}, \ \ |\partial_{x_j x_r} v(t,x)| \le \tilde{c}, \ \ (t,x) \in E, \ \ j,r = 1,\dots,n, \tag{36}$$

where

$$\lim_{h \to 0} \gamma_0(h) = 0, \ \ \lim_{h \to 0} \gamma(h) = 0.$$

According to Assumption H[f] and (31), (32) we have

$$\left| \Lambda_{h.0}^{(i,m)} \right| \le (A + 2\tilde{c}B) \left[\omega_{h.0}^{(i)} + \omega_h^{(i)}\right] + A\omega_h^{(i)}, \tag{37}$$

$$\| \Lambda_h^{(i,m)} \| \le B(1 + 2\tilde{c}) \left[\omega_{h.0}^{(i)} + \omega_h^{(i)}\right] + A\omega_h^{(i)}, \tag{38}$$

where $(t^{(i)}, x^{(m)}) \in E'_h$. We conclude from (24) and from (19), (20) that

$$|W_h[\xi_h]^{(i,m)}| \le \omega_{h.0}^{(i)}, \quad (t^{(i)}, x^{(m)}) \in E'_h, \tag{39}$$

and

$$\begin{aligned} \|W_h[\lambda_h]^{(i,m)}\| \le & \left[1 - h_0 \sum_{j=1}^n \frac{1}{h_j} \left| \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,) \right| \right] \|\lambda_h^{(i,m)}\| \\ & + h_0 \sum_{j \in J_+[i,m]} \frac{1}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,)\, \|\lambda_h^{(i,m+e_j)}\| \\ & - h_0 \sum_{j \in J_-[i,m]} \frac{1}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,)\, \|\lambda_h^{(i,m-e_j)}\| \le \omega_h^{(i)}, \end{aligned} \tag{40}$$

where $(t^{(i)}, x^{(m)}) \in E'_h$. It follows from (33) and from Assumption H[f] that

$$\omega_{h.0}^{(i+1)} \le \omega_{h.0}^{(i)} \left[1 + h_0(A + 2\tilde{c}B)\right] + 2h_0(A + \tilde{c}B)\, \omega_h^{(i)} + h_0\gamma_0(h), \tag{41}$$

where $0 \le i \le N_0 - 1$. In a similar way we obtain the difference inequality

$$\omega_h^{(i+1)} \le \omega_h^{(i)} \left[1 + h_0 B(1 + 2\tilde{c}) + h_0 A\right] + h_0 B(1 + 2\tilde{c})\omega_{h.0}^{(i)} + h_0\gamma(h), \tag{42}$$

where $0 \le i \le N_0 - 1$. Write $C = B + 3A + 4\tilde{c}B$. It follows from (41), (42) that the difference inequality

$$\begin{aligned} \omega_{h.0}^{(i+1)} + \omega_h^{(i+1)} \le \left(\omega_{h.0}^{(i)} + \omega_h^{(i)}\right) (1 + h_0 C) + h_0 \left[\gamma_0(h) + \gamma(h)\right], \\ i = 0, 1, \ldots, N_0 - 1, \end{aligned}$$

is satisfied. This gives

$$\omega_{h.0}^{(i)} + \omega_h^{(i)} \le \alpha(h), \quad i = 0, 1, \ldots, N_0, \tag{43}$$

with

$$\alpha(h) = \alpha_0(h) e^{Ca} + \left[\gamma_0(h) + \gamma(h)\right] \frac{e^{Ca} - 1}{C} \quad \text{if} \quad C > 0, \tag{44}$$

$$\alpha(h) = \alpha_0(h) + \left[\gamma_0(h) + \gamma(h)\right] a \quad \text{if} \quad C = 0. \tag{45}$$

This completes the proof of the theorem. □

Now we formulate a result on the error estimate for method (15)–(20).

LEMMA 2.2. *Suppose that all the assumptions of Theorem 2.1 are satisfied and*

1) *the solution* $v\colon E \to R$ *of (1), (2) is of class* C^3 *on* E,
2) *the constant* $\tilde{c} \in R_+$ *is such that*

$$|\partial_{x_j} v(t,x)|, |\partial_{x_i x_j} v(t,x)|, |\partial_{tt} v(t,x)|, |\partial_{tt x_j} v(t,x)|, |\partial_{x_i x_j x_r} v(t,x)| \le \tilde{c},$$

where $(t,x) \in E$ *and* $i,j,r = 1,\dots,n$.

Then

$$|v^{(i,m)} - z_h^{(i,m)}| + \|\partial_x v^{(i,m)} - u_h^{(i,m)}\| \le \tilde{\alpha}(h) \tag{46}$$

on E_h *where*

$$\tilde{\alpha}(h) = \alpha_0(h)e^{aC} + \tilde{\gamma}(h_0)\frac{e^{aC}-1}{C} \text{ if } C > 0,$$
$$\tilde{\alpha}(h) = \alpha_0(h) + a\tilde{\gamma}(h_0) \text{ if } C = 0,$$

and

$$C = B + 3A + 4B\tilde{c}, \quad \tilde{\gamma}(h_0) = h_0\tilde{c}\,[1 + A\, M_\star].$$

PROOF. It follows from assumption 2) that estimates (35) hold with

$$\gamma_0(h) = \gamma(h) = \frac{1}{2}\tilde{\gamma}(h_0).$$

Then we obtain the lemma from inequalities (43). □

REMARK 2.3. *If we apply method (6), (7) to solve problem (1), (2) numerically, then we approximate derivatives with respect to spatial variables with difference expressions which are calculated with use of the previous values of the approximate solution. If we use method (15)–(17) then we approximate the spatial derivatives of* z *with using adequate difference equations which are generated by the original problem. Therefore numerical results obtained by (15)–(17) are better than those obtained by method (6), (7).*

REMARK 2.4. *Results on the error estimates for methods (6), (7) and (15)–(17) can be characterized as follows. In (12) and (46) we have estimated the terms*

$$|v^{(i,m)} - z_h^{(i,m)}| \quad \text{and} \quad |v^{(i,m)} - z_h^{(i,m)}| + \|\partial_x v^{(i,m)} - u_h^{(i,m)}\|,$$

respectively. The functions $\bar{\alpha}$ *and* $\tilde{\alpha}$ *in (12) and (46) are similar. Therefore, numerical results obtained by (15)–(17) for initial problem (1), (2) are better that those obtained by (6), (7).*

We illustrate the above properties of difference methods by a numerical example.

Now we consider the system of difference equations (15), (16) with operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined by (13), (14) where $(t^{(i)}, x^{(m)}) \in E'_h$ and $z\colon E_h \to R$. The difference expressions

$$\delta_0 u_r^{(i,m)}, \quad (\, \delta_1 u_r^{(i,m)}, \ldots, \delta_n u_r^{(i,m)}\,), \;\; 1 \le r \le n,$$

are defined in the same way.

THEOREM 2.5. *Suppose that Assumption H[f] is satisfied and*

1) $h \in \Delta$, $h' \le h_0 M$ *and for* $P = (t, x, p, q) \in \Omega$ *we have*

$$\frac{1}{n} - \frac{h_0}{h_j} \,\big|\, \partial_{q_j} f(\,P\,) \,\big| \ge 0, \;\; 1 \le j \le n, \tag{47}$$

2) the function $\varphi\colon [-b, b] \to R$ *is of class* C^2 *and* $v\colon E \to R$ *is the solution of (1), (2) and* v *is of class* C^2 *on* E,

3) $(z_h, u_h) = (z_h, u_{h.1}, \ldots, u_{h.n})\colon E \to R^{1+n}$ *is the solution of problem (15)–(17) with* δ_0 *and* δ *given by (13), (14),*

4) there is $\alpha_0\colon \Delta \to R_+$ *such that*

$$|\varphi^{(m)} - \varphi_h^{(m)}| + \|\partial_x \varphi^{(m)} - \psi_h^{(m)}\| \le \alpha_0(h), \;\; -N \le m \le N,$$

and $\lim_{h\to 0} \alpha_0(h) = 0$.

Then there is $\alpha\colon \Delta \to R_+$ *such that*

$$|v^{(i,m)} - z_h^{(i,m)}| + \|\, \partial_x v^{(i,m)} - u_h^{(i,m)} \,\| \le \alpha(h) \;\; \text{on} \;\; E_h$$

and $\lim_{h\to 0} \alpha(h) = 0$.

PROOF. Write $w = \partial_x v$, $w = (w_1, \ldots, w_n)$. Then the functions $(v, w)\colon E \to R^n$ satisfy (21)–(23). Let the functions

$$\Gamma_{h.0}, \;\; \Gamma_h = (\, \Gamma_{h.1}, \ldots, \Gamma_{h.n}\,), \quad \Lambda_{h.0}, \;\; \Lambda_h = (\, \Lambda_{h.1}, \ldots, \Lambda_{h.n}\,),$$

be defined by (25)–(28) with δ_0 and $\delta = (\delta_1, \ldots, \delta_n$ given by (13), (14). Write

$$\xi_h^{(i,m)} = v^{(i,m)} - z_h^{(i,m)},$$

$$\lambda_h^{(i,m)} = w^{(i,m)} - u_h^{(i,m)}, \;\; \lambda_h^{(i,m)} = \left(\lambda_{h.1}^{(i,m)}, \ldots, \lambda_{h.n}^{(i,m)}\right).$$

Suppose that the functions $\omega_{h.0}, \omega_h\colon I_h \to R_+$ are defined by (31), (32) and the operator $W_h\colon \mathbf{F}(E_h, R) \to \mathbf{F}(E'_h, R)$ is given in the following way:

$$W_h[\xi]^{(i,m)} = \frac{1}{2} \sum_{j=1}^{n} \left[\frac{1}{n} + \frac{h_0}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,) \right] \xi^{(i,m+e_j)}$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \left[\frac{1}{n} - \frac{h_0}{h_j}\, \partial_{q_j} f(\, P^{(i,m)}[z_h, u_h]\,) \right] \xi^{(i,m-e_j)},$$

where $\xi \in \mathbf{F}(E_h, R)$ and $(t^{(i)}, x^{(m)}) \in E'_h$. It follows that relations (33), (34), (39), (40) are satisfied with the above given W_h and we get the difference inequality

$$\omega_{h.0}^{(i+1)} + \omega_h^{(i+1)} \leq (\omega_{h.0}^{(i)} + \omega_h^{(i)})(1 + h_0 C) + h_0[\gamma_0(h) + \gamma(h)],\ 0 \leq i \leq N_0 - 1,$$

with γ_0, γ, $\tilde{c}$ satisfying (35), (36) and $C = B + 3A + 4\tilde{c}B$. Then estimate (43) is satisfied with α defined by (44), (45). This completes the proof. □

It it easy to formulate a result on the error estimate for the method under the additional assumption that the solution of (1), (2) is of class C^3 on E.

In the results on error estimates we need estimates for the derivatives of the solution v of problem (1), (2). One may obtain them by the method of differential inequalities, see [**5**], Chapter VII.

3. Numerical examples. Let $n = 1$ and

$$E = \{\, (t, x) \in R^2 : \ \ t \in [0, 1], \ \ |x| \leq 2 - 2t \,\}.$$

Consider the differential equation

$$\partial_t z(t, x) = \frac{1}{2} \sin(\, 1 + \partial_x z(t, x)\,) + f(t, x) \tag{48}$$

with the initial condition

$$z(0, x) = 0, \ \ x \in [-2, 2], \tag{49}$$

where

$$f(t, x) = 1 + x^3 - \frac{1}{2} \sin(\, 1 + 3x^2 t\,).$$

The exact solution of this problem is $v(t, x) = t(1+x^3)$, $(t, x) \in E$. The classical difference method for (48), (49) has the form

$$\begin{aligned} z^{(i+1,m)} &= \frac{1}{2} \Big[\, z^{(i,m+1)} + z^{(i,m-1)} \,\Big] + h_0 f^{(i,m)} \\ &\quad + \frac{h_0}{2} \sin \Big[1 + (\, z^{(i,m+1)} - z^{(i,m-1)}\,)(\, 2h_1\,)^{-1} \,\Big], \end{aligned} \tag{50}$$

$$z^{(0,m)} = 0 \ \text{ for } \ x^{(m)} \in [-2, 2], \tag{51}$$

where $f^{(i,m)} = f(t^{(i)}, x^{(m)})$. Note that Theorem 1.1 does not apply to equation (48). The convergence of method (50), (51) follows from Theorem 1.3.

Now we consider method (15), (16) for problem (48), (49). Denote by (z, u) the unknown functions of the variables $(t^{(i)}, x^{(m)})$ and consider the system of difference equations

$$z^{(i+1,m)} = z^{(i,m)} + \frac{h_0}{2}\,\sin\left(1+u^{(i,m)}\right) + h_0\, f^{(i,m)} + \frac{h_0}{2}\,\cos\left(1+\left(u^{(i,m)}\right)\right)\left[\delta z^{(i,m)} - u^{(i,m)}\right], \tag{52}$$

$$u^{(i+1,m)} = u^{(i,m)} + h_0 F^{(i,m)} + \frac{h_0}{2}\cos\left(1+u^{(i,m)}\right)\delta u^{(i,m)} \tag{53}$$

with the initial condition

$$z^{(0,m)} = 0,\quad u^{(0,m)} = 0,\quad x^{(m)} \in [-2,2], \tag{54}$$

where

$$F^{(i,m)} = F(t^{(i)}, x^{(m)}),\quad F(t,x) = 3x^2 - 3xt\,\cos\left(1+3x^2t\right).$$

The difference expressions $\delta z^{(i,m)}$ and $\delta u^{(i,m)}$ are defined in the following way. If $\cos\left(1+u^{(i,m)}\right) \geq 0$ then

$$\delta z^{(i,m)} = \frac{z^{(i,m+1)} - z^{(i,m)}}{h_1} \quad\text{and}\quad \delta u^{(i,m)} = \frac{u^{(i,m+1)} - u^{(i,m)}}{h_1},$$

If $\cos\left(1+u^{(i,m)}\right) < 0$ then

$$\delta z^{(i,m)} = \frac{z^{(i,m)} - z^{(i,m-1)}}{h_1} \quad\text{and}\quad \delta u^{(i,m)} = \frac{u^{(i,m)} - u^{(i,m-1)}}{h_1}.$$

Denote by z_h and $(\tilde{z}_h, \tilde{u}_h)$ the solutions of problems (50), (51) and (52)–(54), respectively. Consider the errors

$$\varepsilon_h^{(i,m)} = v^{(i,m)} - z_h^{(i,m)},\quad \tilde{\varepsilon}_h^{(i,m)} = v^{(i,m)} - \tilde{z}_h^{(i,m)},\quad (t^{(i)}, x^{(m)}) \in E_h.$$

We put $h_0 = 0.001,\ h_1 = 0.002$ and we have the following experimental values for the errors ε and $\tilde{\varepsilon}$.

Table of errors, $\varepsilon = \mathbf{v} - \mathbf{z_h}$

	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.7$
$x = 0.5$	$-4.307\ 10^{-4}$	$-7.032\ 10^{-4}$	$-1.039\ 10^{-3}$	$-1.434\ 10^{-3}$
$x = 0$	$-3.284\ 10^{-5}$	$-6.422\ 10^{-5}$	$-1.109\ 10^{-4}$	$-1.760\ 10^{-4}$
$x = -0.5$	$5.095\ 10^{-4}$	$7.710\ 10^{-4}$	$1.089\ 10^{-3}$	$1.469\ 10^{-3}$

Table of errors, $\tilde{\varepsilon} = \mathbf{v} - \tilde{\mathbf{z}}_{\mathbf{h}}$

	$t = 0.4$	$t = 0.5$	$t = 0.6$	$t = 0.7$
$x = 0.5$	$2.500\ 10^{-5}$	$2.022\ 10^{-5}$	$1.477\ 10^{-5}$	$1.101\ 10^{-5}$
$x = 0$	$3.425\ 10^{-6}$	$-6.664\ 10^{-6}$	$-1.146\ 10^{-5}$	$-1.809\ 10^{-5}$
$x = -0.5$	$9.517\ 10^{-5}$	$1.174\ 10^{-5}$	$1.355\ 10^{-4}$	$1.469\ 10^{-4}$

Note that $|\tilde{\varepsilon}(t,x)| < |\varepsilon(t,x)|$ for all values of (t,x).
We also give the following information on the errors of methods (50), (51) and (52)–(54). Write

$$\eta^{(i)} = \max\{\, |\varepsilon^{(i,m)}| : \ (t^{(i)}, x^{(m)}) \in E_h \,\},$$
$$\tilde{\eta}^{(i)} = \max\{\, |\tilde{\varepsilon}^{(i,m)}| : \ (t^{(i)}, x^{(m)}) \in E_h \,\}, \ \ 0 \le i \le N_0.$$

In Table E, we give experimental values of the functions η and $\tilde{\eta}$ for $h_0 = 0.001,\ h_1 = 0.002$.

Table E

	$t = 0.40$	$t = 0.45$	$t = 0.50$	$t = 0.55$	$t = 0.60$	$t = 0.65$	$t = 0.70$
$\eta(t)$:	$1.28\ 10^{-3}$	$1.42\ 10^{-3}$	$1.52\ 10^{-3}$	$1.60\ 10^{-3}$	$1.76\ 10^{-3}$	$1.85\ 10^{-3}$	$1.83\ 10^{-3}$
$\tilde{\eta}(t)$:	$3.18\ 10^{-4}$	$2.85\ 10^{-4}$	$2.31\ 10^{-4}$	$1.65\ 10^{-4}$	$1.35\ 10^{-4}$	$1.42\ 10^{-4}$	$1.50\ 10^{-4}$

Note that $\tilde{\eta}(t) < \eta(t)$ for all t. Thus we see that the errors of method (50), (51) are larger than the errors of (52)–(54). This is due to the fact that the approximation of the spatial derivatives of z in (52)–(54) is better than the respective approximation of $\partial_x z$ in (50), (51). Methods described in Theorems 2.1 and 2.5 have the potential for applications in the numerical solving of first order nonlinear differential equations.

References

1. Kowalski Z., *A difference method for the nonlinear partial differential equations,* Ann. Polon. Math., **18** (1966), 235–242.
2. ______, *A difference method for certain hyperbolic systems of nonlinear partial differential equations of first order,* Ann Polon. Math., **19** (1967), 313–322.
3. ______, *On the difference method for certain hyperbolic systems of nonlinear partial differential equations of the first order,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **16** (1968), 297–302.
4. Pliś A., *On difference inequalities corresponding to partial differential inequalities of first order,* Ann. Polon. Math., **28** (1968), 179–181.
5. Szarski J., *Differential Inequalities,* Polish Sci. Publ., Warszawa, 1967.

Received November 23, 2001

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