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AN EXTENSION OF MARKOV PARTITIONS FOR A CERTAIN TORAL ENDOMORPHISM

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Abstract. We define and construct Markov partition for a certain toral endomorphism and then we use it to obtain a symbolic representation of the semidynamical system induced by the endomorphism.

1. Introduction. The increasing interest in the theory of symbolic dynamics is stimulated by the need of a simple tool for investigating complicated dynamical systems. Quite often it is much easier to prove that a given subshift of finite type has some dynamical property, than to show it directly for the considered dynamical system. By finding a semi-conjugacy between an appropriate subshift of finite type and a given dynamical system we often can draw conclusions about dynamical properties of the latter.

Markov partitions provide a way of constructing such semi-conjugaces. The method is based on a division of a space into a finite number of parts. As long as the partition is properly chosen it establishes a correspondence between an orbit and its itinerary, which records the parts visited by the successive points of the orbit. To construct an appropriate partition, stable and unstable manifolds are used to indicate the boundaries of these parts.

Markov partitions were first introduced by Adler and Weiss [**2**] for the hyperbolic automorphisms of the two-dimensional torus and the obtained symbolic model was used to investigate some measure theory problems.

The aim of this paper is to generalize the concept of Markov partitions for certain toral endomorphism. We search for the partition that gives a symbolic

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representation of the discrete semi-dynamical system induced by the endomorphism on a torus. It is natural to expect our representation to be a one-sided subshift of finite type, because in the case of the toral automorphism it is a subshift of finite type. Unfortunately the construction of Adler and Weiss does not apply in the non-invertible case. It is impossible to obtain in this way any symbolic representation, which will be explained in Chapter 8.

Our result is a construction of an extended Markov partition for the given toral endomorphism, which raises the expected symbolic representation of the semi-dynamical system of interest.

To reach our aim we generalize the concept of the good intersection property (Definition 3.6), which has so far been described (in a restricted way) geometrically only with pictures (compare Lind, Marcus [**14**]).

This provides a new method of obtaining the transition rule (Section 7), what is significant for the definition of the extended Markov matrix (Definition 3.9) and the extended Markov partition (Definition 3.10).

One of the most interesting problems which we approach is finding two suitable partitions such that there is a factor map between the one-sided subshifts of finite type associated with them.

Finally we would like to mention other generalizations of Markov partitions. Adler and Weiss's construction was first extended by Sinai [**21**], [**22**] for the Anosov diffeomorphisms. Then Bowen [**5**] constructed Markov partitions for Axiom A diffeomorphisms, using Smale's Spectral Decomposition Theorem. The detailed description of Bowen's construction is given by Shub [**19**] under slightly more general assumptions.

The recent generalizations of Markov condition is the weak local Markov condition (Blank [**4**], 1997).

Bowen's method, although applicable to a broaden class of maps, does not provide any useful information when an explicit dynamical system is considered.

Firstly, the rectangles that form the space division are too small and therefore the subshift of finite type which we obtain as a symbolic model for the considered dynamical system, acts on a very large number of symbols.

Secondly, at no boundary point of the rectangles is there a tangent space, so they have a fractal structure, [**7**] and it causes serious difficulties in constructing these sets in the higher-dimensional spaces (significant problems appear already in dimension three).

One can find more information on the topic in [**1**].

In spite of the fact that Markov partitions were created for the hyperbolic systems, similar methods are used for the systems that lack this property, [**13**]. It is worth mentioning that there are also some attempts to construct Markov partitions with the use of computer methods [**9**].

2. Acknowledgements. I would like to thank Professor Jerzy Ombach for introducing me to the theory of dynamical systems. Numerous conversations, participation in the seminar and also the pointed literature encouraged me to write this paper, which forms a part of my Master Thesis.

I wish to express my gratitude to Professor Anthony Manning of the University of Warwick, who raised the problem of defining Markov partitions for noninvertible maps and led the inspiring discussions that contributed significantly to my thesis.

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3. Definition of the extended Markov partition. Let us consider the toral endomorphism given by the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

All the definitions and results are stated for this matrix A.

Matrix A induces a map on $\mathbf{T}^2$, because it has integer entries, but it is not invertible because $\det A = 2$. We will denote a map induced on the torus with A, too.

Matrix A has the following eigenvalues and corresponding eigenvectors

$$\lambda_u = 2 + \sqrt{2}, \quad \lambda_s = 2 - \sqrt{2},$$
$$x_u = [1 + \sqrt{2}, 1],$$
$$x_s = [1 - \sqrt{2}, 1].$$

It is a hyperbolic matrix, because its eigenvalues are not on the unit circle. Let E^u be a linear subspace of $\mathbb{R}^2$ spanned by the vector x_u and E^s respectively – by the vector x_s.

Let us denote by W^s and W^u projections of E^s and E^u respectively, onto the torus. By $W^s(x)$ and $W^u(x)$ we mean projections of the subspaces $x + E^s$ and $x + E^u$ on the torus.

Let us introduce some preliminary definitions.

Definition 3.1. By the interval in $\mathbf{T}^2$ we mean the projection of an interval in $\mathbb{R}^2$.

Definition 3.2. By the rectangle in $\mathbf{T}^2$ we mean the projection of a rectangle in $\mathbb{R}^2$.

We consider only such rectangles whose sides are parallel to the stable and unstable direction.

Observation 3.3. Let R be a rectangle on $\mathbf{T}^2$. Then $A(R)$ is also a rectangle and $A^{-1}(R)$ is a union of two congruent rectangles P and Q, such that $P = Q + [\frac{1}{2}, \frac{1}{2}]$.

Definition 3.4. An interval J^u is called an unstable section of the rectangle P, if there exists an $x \in J^u$, such that J^u is the maximal interval included in $W^u(x) \cap P$.

By the *partition of the torus* we mean a finite set of open disjoint rectangles (with the sides parallel to the stable and unstable directions), such that the closure of their union is the whole torus.

Let us now define the notion of good intersections for a toral automorphism M, that captures the idea illustrated in the pictures in [**14**].

Definition 3.5. Let P and R be two open rectangles in T^2 with the sides parallel to the stable and unstable directions (these sides are denoted respectively by $\partial^s(R)$, $\partial^s(P)$ and $\partial^u(R)$, $\partial^u(P)$). We say that P and $M(R)$ satisfy the good intersection property if $M(R) \cap P = \emptyset$ or $M(R) \cap P$ is a rectangle and $M(R) \cap \partial^u(P) = \emptyset$ and $\partial^s(M(R)) \cap P = \emptyset$.

Unfortunately, following Adler and Weiss's method in the case of the endomorphism we obtain a partition which lacks the good intersection property (Section 8).

As we said in the introduction, we want to generalize the concept of the good intersection property in such a way that it would lead us to a partition which satisfies transition rule (Definition 3.8), because only such partitions can give a symbolic representation.

Let us define a new notion of a good intersection property, which uses preimages instead of images of rectangles and therefore it is suitable for our purpose.

Definition 3.6. Let $\mathcal{R}$ be a partition of the torus. We say that rectangles $P \in \mathcal{R}$ and $A^{-1}(R)$, where $R \in \mathcal{R}$, satisfy the good intersection property if for any unstable section J^u of the rectangle P the set $J^u \cap A^{-1}(R)$ is an interval or for any unstable section J^u of the rectangle P we have $J^u \cap A^{-1}(R) = \emptyset$.

Definition 3.6 is correct by virtue of Observation 3.3.

Definition 3.7. Let $\mathcal{R}$ be a partition of the torus. We say that $\mathcal{R}$ satisfies a good intersection property for the preimages if for any two rectangles $P, R \in \mathcal{R}$, the sets P and $A^{-1}(R)$ satisfy the good intersection property in the meaning of Definition 3.6.

One can easily check that the partition $\mathcal{P}$ constructed in Section 5 satisfies Definition 3.7, but does not satisfy Definition 3.5, which proves that the generalization is substantial and therefore gives a new method of obtaining the transition rule (the appropriate theorem is stated in Section 7).

Let us generalize the concept of the transition rule, [**1**].

DEFINITION 3.8. Let $\mathcal{R} = \{R_i : 1 \leq i \leq n\}$ be a partition of the torus. We say that $\mathcal{R}$ satisfies the transition rule for A if for any $m \in \mathbb{N}$, $m \geq 3$, the following condition is satisfied:

$R_{a_k} \cap A^{-1}(R_{a_{k+1}}) \neq \emptyset,\ 0 \leq k \leq m-1 \ \Rightarrow \bigcap_{k=0}^{m} A^{-k}(R_{a_k}) \neq \emptyset$

for any $R_{a_k} \in \mathcal{R}$, $0 \leq k \leq m$.

Let us generalize the concept of a Markov matrix, corresponding to the new notion of a good intersection property. By S^T we denote the transpose of a matrix S.

DEFINITION 3.9. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a partition of the torus, satisfying a good intersection property for the preimages.

Let us define a matrix $T = S^T$, where $S = [s_{ij}]_{i,j=1}^{n}$, $s_{ij} = 1$ if $A^{-1}(P_i) \cap J^u \neq \emptyset$ for all J^u unstable sections of the rectangle P_j, $1 \leq i, j \leq n$ and $s_{ij} = 0$ otherwise. The matrix T is called the extended Markov matrix.

Let us remind that the topological Cantor set is a topological space homeomorphic to the classic Cantor set.

DEFINITION 3.10. An extended Markov partition for the endomorphism A is a pair of the partitions of the torus $(\mathcal{P}, \mathcal{R})$ satisfying the following conditions

1. Partition $\mathcal{P}$ satisfies the transition rule.
2. Partition $\mathcal{R}$ satisfies the following condition

$$(*) \quad \forall (a_n)_{n\geq 0} \in \Sigma_{T'}^{+} : \bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) = I_{(a_n)} \times \mathcal{C},$$

where $I_{(a_n)} \subset W^s(x) \cap R_{a_0}$, for some $x \in \mathbf{T}^2$, $\mathcal{C}$ is a topological Cantor set, and T' is the extended Markov matrix associated with the partition $\mathcal{R}$.
3. There exists a factor map $F : \Sigma_{T'}^{+} \to \Sigma_{T}^{+}$, where T' and T are extended Markov matrices associated with the partitions $\mathcal{R}$ and $\mathcal{P}$ respectively.

4. Main results. Let $A : \mathbf{T}^2 \to \mathbf{T}^2$ be an endomorphism of the torus induced by the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

THEOREM 4.1. *There exists an extended Markov partition for the endomorphism A.*

THEOREM 4.2. *There exists a one-sided subshift of finite type* Σ_T^+ *and a factor map* $\Phi : \mathbf{T}^2 \setminus B \to \Sigma_T^+$ *such that the following diagram commutes*

$$\begin{array}{ccc} \Sigma_T^+ & \xrightarrow{\sigma_T} & \Sigma_T^+ \\ \Phi\big\uparrow & & \big\uparrow\Phi \\ \mathbf{T}^2 \setminus B & \xrightarrow[A]{} & \mathbf{T}^2 \setminus B \end{array}$$

$B = \{x \in \mathbf{T}^2 : A^n x \in \partial\mathcal{R} \; for \; some \; n \geq 0\}.$

5. Proof of the existence of the extended Markov partition.

CONSTRUCTION:

STEP 1.

We start by repeating the first step of Adler and Weiss's construction (compare [**2**]), which gives us two rectangles. Let us denote the larger one by R_1^* and the smaller one by R_2.

STEP 2. (Partition A)

Let us divide rectangle R_1^* into two pieces, named R_1 and R_3, by cutting the original rectangle along the unstable manifold of 0.

STEP 2a. (Partition A')

Let us construct the partition consisting of the preimages of the rectangles from the previous step. By Corollary 3.3 each of the preimages consists of two rectangles. Let us call them $R_{1'}$ and $R_{1"}$ respectively for the preimage of R_1, $R_{2'}$ and $R_{2"}$ in a case of R_2 and $R_{3'}$, $R_{3"}$ when R_3 is considered.

They will be used to modify the partition obtained in Step 2.

STEP 3. (Partition B)

We want to rebuild Partition A, using Partition A' in order to get a partition satisfying the good intersection property for the preimages.

$R_d = R_{1"} \cap R_1$.

The rectangle R_2 remains unchanged.

$R_e = (R_{1'} \cup R_{2'} \cup R_{3'}) \cap R_3$

$R_f = R_{3"} \cap R_3$

$R_g = (R_{1'} \cup R_{2"} \cup R_{3'} \cup R_{3"}) \cap R_1$

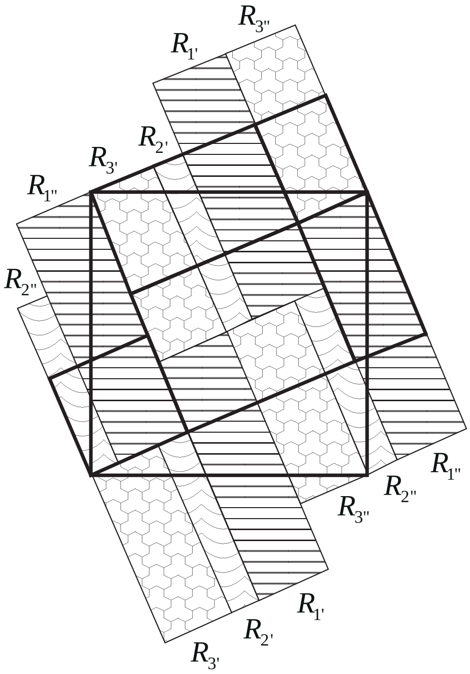
The partition which we obtained satisfies the good intersection property for the preimages and therefore satisfies also the transition rule, by Theorem 7.1.

This is the partition $\mathcal{P}$ required in the definition of the extended Markov partition.

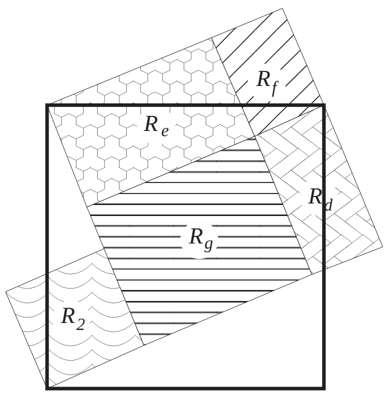
STEP 3a. (Partition C)

Let us build a new partition by taking the preimages of the rectangles R_d, R_2, R_e, R_f and R_g.

The apostrophe or double apostrophe are added to the symbol of the rectangle when the preimage is taken.

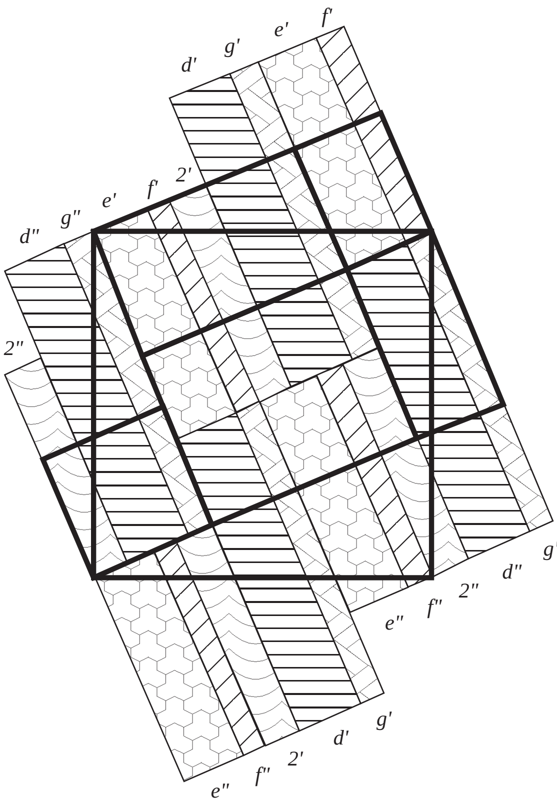


Picture 1. Partition A'



Picture 2. Partition B

This is the partition $\mathcal{R}$ required in the definition of the extended Markov partition.



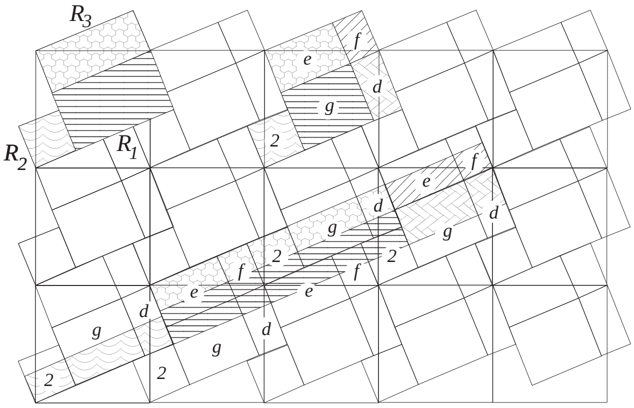
PICTURE 3. Partition C

This is the end of the construction. In the next step, we shell prove the required properties of the obtained partitions.

STEP 4.

The extended Markov matrix associated with the partition $\mathcal{P}$, referring to the following order of symbols $g, d, 2, e, f$ is as follows

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$



$$R_g = (R_{1'} \cup R_{2"} \cup R_{3'} \cup R_{3"}) \cap R_1$$
$$R_d = R_{1"} \cap R_1$$
$$R_e = (R_{1'} \cup R_{2'} \cup R_{3'}) \cap R_3$$
$$R_f = R_{3"} \cap R_3$$

Picture 4. Partitions A and B, and the image of Partition B

The extended Markov matrix for the partition $\mathcal{R}$, with the given order of symbols $g', g", d', d", 2', 2", e', e", f', f"$, is as follows

$$T' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Let us check if condition 3. of the definition of the extended Markov partition is satisfied. Let us define a map $F : \Sigma^+_{T'} \to \Sigma^+_T$ by the condition

$$(F(a_n)_{n=0}^{+\infty})_m = b_m :\Longleftrightarrow A(R_{a_m}) = R_{b_m},\ \text{for } m \geq 0.$$

Let us check if F commutes with the subshift of finite type that is if the following equality is valid

$$\sigma_T \circ F = F \circ \sigma_{T'}.$$

From the definition of F we obtain

$$\begin{aligned} &\sigma_T(F(a_n)_{n=0}^{+\infty}) = (c_n)_{n=0}^{+\infty} \\ \Longleftrightarrow\ &(F((a_n)_{n=0}^{+\infty}))_{m+1} = c_m,\ \text{for } m \geq 0 \\ \Longleftrightarrow\ &A(R_{a_{m+1}}) = R_{c_m},\ \text{for } m \geq 0 \end{aligned}$$

$$\begin{aligned} &F(\sigma_{T'}((a_n)_{n=0}^{+\infty})) = F((a_{n+1})_{n=0}^{+\infty}) = (d_n)_{n=0}^{+\infty} \\ \Longleftrightarrow\ &(F((a_{n+1})_{n=0}^{+\infty}))_m = d_m,\ \text{for } m \geq 0 \\ \Longleftrightarrow\ &A(R_{a_{m+1}}) = R_{d_m},\ \text{for } m \geq 0. \end{aligned}$$

From above it follows that $c_m = d_m,\ for\ m \geq 0$ and so F commutes with the subshift. Moreover by the definitions of the matrices T and T' the map F is surjective. This proves that condition 3. of the definition of the extended Markov partition is satisfied.

Remark 5.1. The map F is an infinite-to-one factor map.

Among the distinguished 2×2 submatrices of the matrix T' there is a matrix (corresponding to the symbols g', g" and $2', 2$") of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, in the sequences in which symbol g precedes 2, there are diamonds and that implies that T' induces a infinite-to-one factor map, because in the preimage under F of the sequence $(g, 2, g, 2, g, 2, ...)$ there are infinitely many sequences.

The definition of a diamond and an infinite-to-one factor map, as well as similar reasoning one can find in **[12]**, Chapter 'Embeddings and Factor Maps'.

STEP 5.

Now we will show that the condition $(*)$ is satisfied.

Theorem 5.2.

$$\forall (a_n)_{n\geq 0} \in \Sigma_{T'}^{+} : \bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) = I_{(a_n)} \times \mathcal{C},$$

where $I_{(a_n)} \subset W^s(x) \cap R_{a_0}$, *for some* $x \in \mathbf{T}^2$ *and* $\mathcal{C}$ *is a topological Cantor set.*

Let us switch to the coordinate system given by the vectors x_s, x_u. In this coordinate system the matrix A takes the form

$$J_A = \begin{bmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix}.$$

Let $V = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in $\mathbf{T}^2$. By $\mathcal{C}(V)$ we mean the set $\{v_{i_1} + v_{i_2} + \ldots + v_{i_s} : v_{i_1}, v_{i_2}, \ldots, v_{i_s} \in V, 1 \leq s \leq k, v_m \neq v_n \text{ for } m \neq n\}$.

Let V^k denote the set $\{0, [\frac{1}{4}, \frac{1}{4}], J_A^{-1}([\frac{1}{4}, \frac{1}{4}]), J_A^{-2}([\frac{1}{4}, \frac{1}{4}]), \ldots, J_A^{-(k-1)}([\frac{1}{4}, \frac{1}{4}])\}$. The following lemma describes the structure of the preimages of the given rectangle R_{a_k}.

Lemma 5.3. *$J_A^{-k} R_{a_k} = R_{a_k}^k + \mathcal{C}(V^k)$, where $R_{a_k}^k$ is a rectangle in the torus.*

Proof. Observation 3.3 holds in the coordinate system determined by vectors x_s and x_u, but rectangles in the preimage are displaced by the vector $[\frac{1}{4}, \frac{1}{4}]$ relative to each other. Then by induction the statement is true for any k. $\square$

Lemma 5.4. *$diam_{x_u} R_{a_k}^k \longrightarrow 0$, when $k \longrightarrow +\infty$.*

Proof. This is a straightforward consequence of the fact that the vector x_u indicates the stable direction for J_A^{-1}. $\square$

Now we can prove Theorem 5.2.

Proof of Theorem 5.2. We will use the coordinate system given by the vectors x_s and x_u. By Lemmas 5.3 and 5.4, for any $k \geq 0$ the set $J_A^{-k} R_{a_k}$ consists of a finite number of rectangles such that $diam_{x_u} R_{a_k}^k \longrightarrow 0$. Moreover, it is obvious that $\bigcap_{k \geq 0} J_A^{-k} R_{a_k} \subset R_{a_0}$. Therefore

$$\exists (x_n)_{n=0}^{+\infty},\ x_n \in \mathbf{T}^2 : \bigcap_{k \geq 0} J_A^{-k} R_{a_k} = \bigcup_{x_n} I_{x_n},$$

where $I_{x_n} \subset W^s(x_n) \cap R_{a_0}$.

From Lemma 5.3 and the definition of sets $\mathcal{C}(V^k)$, it follows that (x_n) is a topological Cantor set, which completes the proof. $\square$

6. Proof of the theorem on the correspondence to a one-sided subshift of finite type.

Let $(\mathcal{P}, \mathcal{R})$ be the extended Markov partition for A. Moreover, let T and T' be the extended Markov matrices associated with the partition $\mathcal{P}$ and $\mathcal{R}$ respectively. We denote by F the map appearing in the definition of the extended Markov partition.

We want to show that the following diagram commutes.

$$\begin{tikzcd}
\Sigma_T^+ \arrow[r, "\sigma_T"] & \Sigma_T^+ \\
\Sigma_{T'}^+ \arrow[u, "F"] & \Sigma_T^+ \arrow[u, "\sigma_T"'] \\
\Sigma_{T'}^+ \arrow[u, "\sigma_{T'}"] & \Sigma_{T'}^+ \arrow[u, "F"'] \\
\mathbf{T}^2 \setminus B \arrow[u, "\Pi"] \arrow[r, "A"'] & \mathbf{T}^2 \setminus B \arrow[u, "\Pi"']
\end{tikzcd}$$

Then we define the maps $\Phi_1 : \mathbf{T}^2 \setminus B \to \Sigma_T^+$ and $\Phi_2 : \mathbf{T}^2 \setminus B \to \Sigma_T^+$ by

$$\Phi_1 := F \circ \sigma_{T'} \circ \Pi$$

$$\Phi_2 := \sigma_T \circ F \circ \Pi.$$

Let us remind that the map $F : \Sigma_{T'}^+ \to \Sigma_T^+$ was defined in the previous chapter by

$$F((a_n)_{n\geq 0}) = (b_n)_{n\geq 0} \Longleftrightarrow A(R_{a_n}) = R_{b_n}, \text{ for } n \geq 0.$$

Let us define $\Pi : \mathbf{T}^2 \setminus B \to \Sigma_{T'}^+$ by

$$\Pi(x) = (a_n)_{n\geq 0} :\Longleftrightarrow A^n(x) \in R_{a_n}, \text{ for } n \geq 0.$$

Maps Φ_1 and Φ_2 are surjective because F is surjective and the set $\bigcap_{n=0}^{+\infty} A^{-n} R_{a_n}$ is nonempty, for $(a_n) \in \Sigma_{T'}^+$, which is a consequence of the condition $(*)$.

Maps Φ_1 and Φ_2 are equal (and this map is a map Φ in Theorem 4.2), because F is a shift commuting map.

To show that

$$\sigma_T \circ \Phi = \Phi \circ A,$$

it is enough to prove the equality

$$\sigma_T \circ F \circ \sigma_{T'} \circ \Pi(x) = \sigma_T \circ F \circ \Pi \circ A(x)$$

for any $x \in \mathbf{T}^2 \setminus B$.

Let us consider the left hand side of the equation, i.e. $\sigma_T \circ F \circ \sigma_{T'} \circ \Pi(x)$.

$$\Pi(x) = (a_n)_{n\geq 0} \Longleftrightarrow A^n(x) \in R_{a_n}, \text{ for } n \geq 0;$$

$$\sigma_{T'}((a_n)_{n\geq 0}) = (a_{n+1})_{n\geq 0};$$

$$F((a_{n+1})_{n\geq 0}) = (b_n)_{n\geq 0} \Longleftrightarrow A(R_{a_{n+1}}) = R_{b_n}, \text{ for } n \geq 0;$$

$$\sigma_T((b_n)_{n\geq 0}) = (b_{n+1})_{n\geq 0}.$$

Therefore the sequence $(b_n)_{n\geq 0}$ satisfies condition

$$A(R_{a_{n+1}}) = R_{b_n}, \text{ for } n \geq 0,$$
$$A^n(x) \in R_{a_n}, \text{ for } n \geq 0,$$

which is the same as

$$A^{n+1}(x) \in A(R_{a_n}) = R_{b_{n-1}}, \text{ for } n \geq 1,$$

hence

$$A^{n+2}(x) \in R_{b_n}, \text{ for } n \geq 0.$$

Let us in turn consider the right hand side of the equation : $\sigma_T \circ F \circ \Pi \circ A(x)$.

$$\Pi(A(x)) = (c_n)_{n\geq 0} \Longleftrightarrow A^{n+1}(x) \in R_{c_n}, \text{ for } n \geq 0;$$
$$F((c_n)_{n\geq 0}) = (d_n)_{n\geq 0} \Longleftrightarrow A(R_{c_n}) = R_{d_n}, \text{ for } n \geq 0;$$
$$\sigma_T((d_n)_{n\geq 0}) = (d_{n+1})_{n\geq 0}.$$

Therefore, the sequence $(d_n)_{n\geq 0}$ satisfies the condition

$$A(R_{c_n}) = R_{d_n}, \text{ for } n \geq 0,$$
$$A^{n+1}(x) \in R_{c_n}, \text{ for } n \geq 0.$$

Consequently

$$A^{n+2}(x) \in A(R_{c_n}) = R_{d_n}, \text{ for } n \geq 0,$$

which is equivalent to

$$A^{n+2}(x) \in R_{d_n}, \text{ for } n \geq 0.$$

To finish the proof it is enough to notice that $(b_n)_{n\geq 0} = (d_n)_{n\geq 0}$, because the rectangles in the partition are pairwise disjoint. □

7. A new method of obtaining the transition rule.

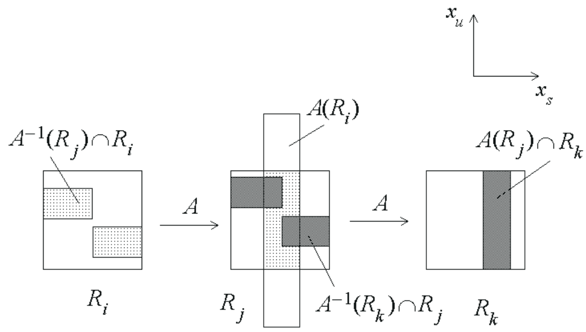
THEOREM 7.1. *Let $\mathcal{R}$ be a partition of the torus satisfying the good intersection property for the preimages. Then $\mathcal{R}$ satisfies the transition rule.*

PROOF. The transition rule is a consequence of the following two properties, which are satisfied for any three rectangles R_i, R_j, $R_k \in \mathcal{R}$.

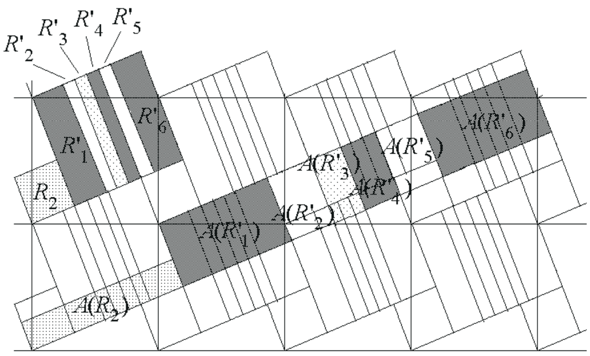
If $A(R_i) \cap R_j \neq \emptyset$ then the intersection is a rectangle that crosses R_j all the way along the unstable direction, because $A(\partial^s\mathcal{R}) \subset \partial^s\mathcal{R}$.

The set $A^{-1}(R_k) \cap R_j$, if non-empty, consists of two rectangles and $A^{-1}(R_k)$ crosses any unstable section of the rectangle R_j, due to the good intersection property.

Therefore if $A(R_i) \cap R_j \neq \emptyset$ and $A^{-1}(R_k) \cap R_j \neq \emptyset$ then $R_i \cap A^{-1}(R_j) \cap A^{-2}(R_k) \neq \emptyset$. The conclusion follows by induction. The idea of the proof is illustrated in Picture 5. □



PICTURE 5. The transition rule is a consequence of the good intersection property



PICTURE 6. Partition obtained by Adler and Weiss's method

8. Adler and Weiss's method fails in the case of an endomorphism. Adler and Weiss's method gives a partition that does not satisfy transition rule, which is necessary to obtain a symbolic representation of a dynamical system. More precisely it is described by

THEOREM 8.1. *Let $\mathcal{R} = \{R_i \subset \mathbf{T}^2 : 1 \leq i \leq n\}$ be a partition of the torus and let $T = [t_{ij}]_{i,j=1}^n$ be defined as follows*

$$t_{ij} = \begin{cases} 1 & \textit{if } A^{-1}(R_i) \cap R_j \neq \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover assume that there is at least one non-zero entry in each row of the matrix T and for any sequence $(a_k)_{k\geq 0}$ admissible by the matrix T the set $\bigcap_{k=0}^{+\infty} A^{-k}(R_{a_k})$ is non-empty.

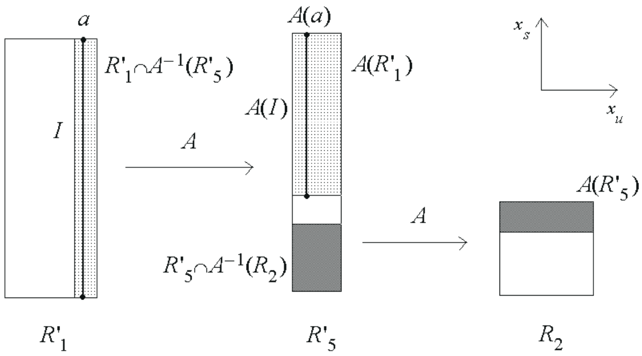
Then $\mathcal{R}$ satisfies the transition rule.

Proof. Let us assume that $R_{a_k} \cap A^{-1}(R_{a_{k+1}}) \neq \emptyset$, for any k such that $0 \leq k \leq m-1$ and for a given $m \geq 3$. Then the sequence $(a_0, a_1, \ldots, a_m)$ is admissible by the matrix T. It can be extended to a sequence $(a_k)_{k\geq 0}$ admissible by T, because in each row of the matrix T there is at lest one non-zero entry.

Finally the set $\bigcap_{k=0}^{m} A^{-k}(R_{a_k})$ is non-empty, because it includes the set $\bigcap_{k=0}^{+\infty} A^{-k}(R_{a_k})$ which is not empty by the assumption. □

Picture 6 illustrates partition obtained by Adler and Weiss's method and images of the rectangles from this partition.

Analyzing this picture one can notice that rectangles R'_1, R'_5, R_2 do not satisfy the transition rule, which is clearly visible in Picture 7. To see this one can compare Picture 7 with Picture 5 where the transition rule is satisfied. We omit technical details of the proof.



Picture 7. Transition rule not satisfied

9. Why do we choose such an extension of Markov partitions?

9.1. *The aim of the construction of the extended Markov partition.* Our aim is to find a partition of the torus, that could be used to obtain a symbolic model for the considered semi-dynamical system. Moreover we expect this model to be a one-sided subshift of finite type, because in the invertible case it was a subshift of finite type. Let us specify properties that will explain why we choose partitions constructed in Section 5 and why we propose such a definition of the extended Markov partition in Section 3.

9.2. *Properties that are significant for Markov partitions.* We already know from Section 8 that the transition rule is the necessary condition for the association of a one-sided subshift of finite type with a given toral endomorphism. A sequence $(a_n)_{n\geq 0}$ corresponds to all points in the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$ and this is why we want this set to be non-empty. Therefore, we search for a partition that satisfies

• PROPERTY A: Transition rule.

In the case of an automorphism, the transition rule is a consequence of the good intersection property, which follows from the Markov condition. In the non-invertible case, $A^{-1}(W^u(0,0)) \not\subset W^u(0,0)$ and Markov condition is not satisfied for any partition. That is why we must search for a new way to obtain the transition rule.

Moreover, we want to obtain the most accurate model, that is we want the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$ to be as small as possible for the sequences $(a_n)_{n\geq 0}$ admissible by the matrix T.

In the invertible case, we consider the set $\bigcap_{n=-\infty}^{+\infty} A^{-n}(R_{a_n})$, which is a single point and this is why we obtain a conjugacy. In the non-invertible case we consider the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n})$, which can at best be a union of intervals. Thus we need the following condition.

• PROPERTY B: For any sequence $(a_n)_{n=0}^{+\infty}$ admissible by the extended Markov matrix, the set $\bigcap_{n=0}^{+\infty} A^{-n}(R_{a_n}) \subset I \times \mathcal{C}$, where I is an interval parallel to the stable direction and $\mathcal{C}$ is a topological Cantor set.

Because we use two partitions, one satisfying property A and another that satisfies property B, to obtain Theorem 4.2 we need these partitions to be compatible in some way, which is

• PROPERTY C: There exists a shift commuting map from $\Sigma_{T'}^{+}$ to Σ_{T}^{+}, where T and T' are the extended Markov matrices associated with the partitions satisfying property A and B, respectively.

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