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AN EXAMPLE CONCERNING THE EMBEDDED PRIMES ASSOCIATED WITH AN IDEAL IN THE RING OF POLYNOMIALS

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Abstract. This paper presents an ideal I, generated by three elements in the ring $\mathbb{C}[x_1, x_2, x_4, x_5]$ of polynomials in four variables, which has the maximal ideal $\mathfrak{m} := (x_1, x_2, x_4, x_5)$ of height 4 among its associated (necessarily embedded) primes.

It is well-known that the height of the minimal primes of an ideal, generated in a noetherian ring by n elements, cannot be greater than n (see e.g. [**2**], Chapt. V). No such bound exists for the embedded primes, irrespective of any assumptions imposed on the ambient ring (whether Cohen-Macaulay or regular). Below, we wish to demonstrate this by giving an example concerning the ring of polynomials.

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We first consider the commutative diagram of polynomial mappings

$$\begin{tikzcd} & \mathbb{C}^5 \arrow[dd, "\pi"] \\ \mathbb{C}^2 \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & \\ & \mathbb{C}^4 \end{tikzcd}$$

where

$$\phi(u,v) = (u^4, u^3v, u^2v^2, uv^3, v^4), \quad \psi(u,v) = (u^4, u^3v, uv^3, v^4)$$

and $\pi(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_4, x_5)$ is the canonical projection. Observe that ϕ is the homogeneous polynomial mapping which determines the 4-fold Veronese embedding (see e.g. [**1**], Chap. I, § 2):

$$\mathbb{P}_1(\mathbb{C}) \longrightarrow \mathbb{P}_4(\mathbb{C}).$$

The above mappings induce the ring homomorphisms ϕ^*, ψ^* and π^*:

$$\begin{tikzcd} & \mathbb{C}[x_1, x_2, x_3, x_4, x_5] \arrow[dl, "\phi^*"'] \\ \mathbb{C}[u,v] & \\ & \mathbb{C}[x_1, x_2, x_4, x_5] \arrow[ul, "\psi^*"] \arrow[uu, "\pi^*"'] \end{tikzcd}$$

Clearly, the kernel of ϕ^* is the ideal

$$\ker \phi^* = (x_1x_5 - x_2x_4,\ x_1x_5 - x_3^2,\ x_1x_4 - x_2x_3,\ x_1x_3 - x_2^2,\ x_2x_5 - x_3x_4,\ x_3x_5 - x_4^2)$$

in the ring of polynomials $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$, and

$$\ker \psi^* = (\pi^*)^{-1}(\ker \phi^*) = \ker \phi^* \cap \mathbb{C}[x_1, x_2, x_4, x_5].$$

It follows, by a standard elimination of the variable x_3, that the prime ideal $\mathfrak{p} := \ker \psi^*$ is of the form

$$\mathfrak{p} = (x_1x_5 - x_2x_4,\ x_1^2x_4 - x_2^3,\ x_2x_5^2 - x_4^3,\ x_1x_4^2 - x_2^2x_5).$$

Now, let $I \subset \mathfrak{p} \subset \mathbb{C}[x_1, x_2, x_4, x_5]$ be the ideal generated by the first three generators of the ideal $\mathfrak{p}$:

$$I := (x_1x_5 - x_2x_4,\ x_1^2x_4 - x_2^3,\ x_2x_5^2 - x_4^3).$$

The homogeneous polynomial

$$s = s(x_1, x_2, x_4, x_5) := x_1x_4^2 - x_2^2x_5 \in \mathfrak{p} \setminus I$$

does not belong to the ideal I. Otherwise, taking $x_1 = x_4 = 0$, one would obtain $x_2^2x_5 \in (x_2^3, x_2x_5^2)$, which is impossible.

Further, an easy computation shows that

$$s \cdot x_1 = s \cdot x_2 = s \cdot x_4 = s \cdot x_5 \equiv 0 \pmod{I}.$$

Hence

$$I : s = (x_1, x_2, x_4, x_5),$$

and thus the maximal ideal

$$\mathfrak{m} := (x_1, x_2, x_4, x_5) \subset \mathbb{C}[x_1, x_2, x_4, x_5],$$

being obviously of height 4, is an associated prime of I (see e.g. [**2**], Chap. III). Moreover, since $s^2 \in I$ and

$$\mathfrak{p} = I + s \cdot \mathbb{C}[x_1, x_2, x_4, x_5],$$

the prime ideal $\mathfrak{p}$ coincides with the radical of I: $\sqrt{I} = \mathfrak{p}$, and consequently $\mathfrak{p}$ is the unique minimal prime of I.

Notice that the ideal I has the two associated primes $\mathfrak{p}$ and $\mathfrak{m}$ only. Indeed, the associated primes of I are exactly the prime ideals of the form $I : a$, where $a = a(x_1, x_2, x_4, x_5)$ is a polynomial. If $a \notin \mathfrak{p}$, then

$$I : a \subset \mathfrak{p} : a = \mathfrak{p},$$

and therefore the prime ideal $I : a = \mathfrak{p}$. If $a \in \mathfrak{p}$, then $a = b + c \cdot s$ with $b \in I$, whence

$$I : a = I : cs \supset I : s = \mathfrak{m},$$

and thus the prime ideal $I : a = \mathfrak{m}$.

Remark. All the above reasoning will remain valid when we consider polynomials over an arbitrary ground field k.

References

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