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# A SHORT PROOF OF A FIBRE CRITERION FOR POLYNOMIALS TO BELONG TO AN IDEAL

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*Dedicated to Professor Tadeusz Winiarski on the occasion of his 60th birthday*

**Abstract.** The purpose of this paper is to give a short, purely algebraic proof of a fibre criterion for polynomials to belong to an ideal.

W. Jarnicki–L. O'Carrol–T. Winiarski [**2**] present a method of expressing a given ideal $I$ in the polynomial ring $k[X_1, \dots, X_n]$ as an intersection of zero-dimensional ideals, which is based on the theory of comprehensive Gröbner bases. A generalization of this result (with no conditions on fibres) is the following criterion for polynomials to belong to an ideal:

FIBRE CRITERION. *Let $k$ be an algebraically closed field. For a polynomial $f = f(X,Y) \in k[X,Y]$ in two sets of variables $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$, an ideal $I \subset k[X,Y]$ and any point $a = (a_1, \dots, a_m) \in k^m$, set*

$$f_a = f_a(X) := f(X,a) \in k[X] \quad \text{and} \quad I_a := \{f_a : f \in I\} \subset k[X].$$

*Suppose $I \subset k[X,Y]$ is a primary ideal such that the ideal $I \cap k[Y]$ is radical (whence prime). Then in order for a polynomial $f = f(X,Y)$ to belong to $I$, it is necessary and sufficient that $f_a \in I_a$ for every $a \in k^m$.*

The above criterion is presented by M. Dumnicki [**1**], whose proof also uses Gröbner bases, namely reduced ones. In this paper we provide a purely algebraic proof of the fibre criterion, which does not make use of Gröbner bases and may be applied to carrying over the criterion to analytic geometry, for holomorphic functions and primary coherent ideal sheaves in an arbitrary domain in $\mathbb{C}^{n+m}$.

We now proceed with proving the fibre criterion. Observe that the proof comes down to showing that

$$\bigcap_{a \in k^m} (I + \mathfrak{m}_a \cdot k[X,Y]) \subset I,$$

where $\mathfrak{m}_a := (Y_1 - a_1, \ldots, Y_m - a_m) \subset k[Y]$ is the maximal ideal corresponding to a point $a = (a_1, \ldots, a_m) \in k^m$. We are therefore to prove that

$$\bigcap \{\mathfrak{m} \cdot B : \mathfrak{m} \in \operatorname{Max}(A)\} = (0),$$

where

$$A := k[Y]/(I \cap k[Y]) \quad \text{and} \quad B := k[X,Y]/I.$$

The rings $A \subset B$ are affine $k$-algebras, $B$ is an $A$-algebra of finite type, and — under the assumptions of the fibre criterion — $A$ is a domain.

In the circumstances a well known lemma (see e.g. [**3**], Chapt. VIII, Sect. 22, Lemma 1) asserts that there exists an element $s \in A$, $s \neq 0$ such that $B_s := B \otimes_A A_s$ is a free $A_s$-module; here $A_s = A[1/s]$ denotes the localization of $A$ with respect to the multiplicative set $\{1, s, s^2, \ldots\}$. Hence

$$\bigcap \{\mathfrak{m} \cdot B_s : \mathfrak{m} \in \operatorname{Max}(A_s)\} = (\bigcap \{\mathfrak{m} : \mathfrak{m} \in \operatorname{Max}(A_s)\}) \cdot B_s = (0) \cdot B_s = (0);$$

the equality

$$\bigcap \{\mathfrak{m} : \mathfrak{m} \in \operatorname{Max}(A_s)\} = (0)$$

holds as $A_s$ is a Jacobson ring, and thus its Jacobson radical and nilradical coincide. Observe that the zero divisors in $B$ form the ideal $\sqrt{I} \pmod{I}$. Since the ideal $I \cap k[Y]$ is radical (whence prime) by assumption, we get

$$\sqrt{I} \cap k[Y] = I \cap k[Y].$$

Therefore, every element $a \in A$, $a \neq 0$ is a non-zero divisor in $B$. Consequently, the canonical homomorphism $B \longrightarrow B_s$ is injective, and we may regard $B$ as a subring of $B_s$. Hence

$$\bigcap \{\mathfrak{m} \cdot B : \mathfrak{m} \in \operatorname{Max}(A)\} \subset \bigcap \{\mathfrak{m} \cdot B_s : \mathfrak{m} \in \operatorname{Max}(A_s)\} = (0),$$

which completes the proof.

Remarks.

1) Clearly, the necessary and sufficient condition of the fibre criterion can be weakened as follows:

*$f_a \in I_a$ for every $a \in U \subset k^m$, where $U$ is any polynomial identity subset of the irreducible affine variety $V := V(I \cap k[Y]) \subset k^m$; in particular, $U$ can be chosen as an arbitrary non-empty open subset of $V$.*

2) Let $I = \cap Q_i$ be such a primary decomposition of an ideal $I$ in the polynomial ring $k[X, Y]$ that every primary component $Q_i$ fulfils the assumption of the fibre criterion. Then obviously a polynomial $f = f(X, Y)$ belongs to $I$ iff $f_a \in I_a$ for every $a \in k^m$.

## References

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