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IMPLICIT VARIATIONAL-LIKE INCLUSIONS INVOLVING GENERAL (H, η)-MONOTONE OPERATORS

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ABSTRACT. In this paper, using Lipschitz continuity of general (H, η)-monotone operators, a type of implicit variational-like inclusion problems in uniformly smooth Banach spaces are solved.

1. INTRODUCTION

A very powerful tool of mathematical technology is variational inequality theory. In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. Useful and important generalizations of variational inequalities are variational and quasi-variational inclusions. Variational inclusion theory is a branch of applicable mathematics with a wide range of applications in the fields of optimization, control, economics, transportation equilibrium, engineering science, industrial, physical, regional, social, pure and applied sciences. Recently, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, one can see [1–21] and the references therein.

This paper is organized as follows. In section 2 we study the required definitions about some generalized types of monotone operators and also some preliminary results which will be used in other sections are collecting. Section 3 is devoted to reviewing definition, examples and some results about the new class "general (H, η)-monotone operators", the proximal mapping associated with this type of monotone operators. Moreover, some fact about Lipschitz continuity of the proximal mapping associated with general (H, η)-monotone operators are proved.

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Finally, in section 4, using Lipschitz continuity of general proximal mappings associated with general (H, η)-monotone operators, a type of implicit variational-like inclusion problems in uniformly smooth Banach spaces are solved.

2. Preliminaries

Let X be a real Banach space with dual space X^* and let $\eta : X \times X \longrightarrow X$ is a single valued mapping. η is called *γ-Lipschitz continuous* mapping, if there exists some $\gamma > 0$ such that $\|\eta(x,y)\| \leq \gamma\|x-y\|$ for all $x, y \in X$. For a set-valued map $T : X \multimap Y$, the range of T is $Range(T) = \{y \in Y : \exists x \in X, (x,y) \in T\}$ and the inverse T^{-1} of T is $\{(y,x) : (x,y) \in T\}$. We say that T is *closed valued*, if $T(x)$ is closed for all $x \in X$. For a real number c, let $cT = \{(x, cy) : (x,y) \in T\}$. If S and T are any set-valued mappings, we define $S + T = \{(x, y+z) : (x,y) \in S, (x,z) \in T\}$.

Definition 2.1. A single valued map $H : X \longrightarrow X^*$ is said to be

(a) *monotone* if $\langle H(x) - H(y), x - y\rangle \geq 0$ for all $x, y \in X$.

(b) *η-monotone* if $\langle H(x) - H(y), \eta(x,y)\rangle \geq 0$ for all $x, y \in X$.

(c) *strictly monotone* if H is monotone and $\langle H(x) - H(y), x - y\rangle = 0$ if and only if $x = y$.

(d) *strictly η-monotone* if H is η-monotone and $\langle H(x) - H(y), \eta(x,y)\rangle = 0$ if and only if $x = y$.

(e) *r-strongly monotone* if there exists some constant $r > 0$ such that $\langle H(x) - H(y), x - y\rangle \geq r\|x-y\|^2$ for all $x, y \in X$.

(f) *r-strongly η-monotone* if there exists some constant $r > 0$ such that $\langle H(x) - H(y), \eta(x,y)\rangle \geq r\|x-y\|^2$ for all $x, y \in X$.

(g) *δ-Lipschitz* if $\|H(x) - H(y)\| \leq \delta\|x-y\|$ for all $x, y \in X$.

Definition 2.2. A set-valued map $T : X \multimap X$ is said to be

(a) *maximal monotone* if T is monotone and $(I + \lambda T)(X) = X$ holds for every $\lambda > 0$.

(b) *maximal η-monotone* if T is η-monotone and $(I + \lambda T)(X) = X$ holds for every $\lambda > 0$, if and only if T is η-monotone and there is no other η-monotone set-valued mapping whose graph strictly contains the graph of T [18].

Definition 2.3. [4, 5, 6, 15] A set-valued map $T : X \multimap X^*$ is said to be

(a) *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.

(b) *η-monotone* if $\langle x^* - y^*, \eta(x,y)\rangle \geq 0$ for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.

(c) *r-strongly monotone* if there exists some constant $r > 0$ such that $\langle x^* - y^*, x - y\rangle \geq r\|x-y\|^2$ for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.

(d) *r-strongly η-monotone* if there exists some constant $r > 0$ such that $\langle x^* - y^*, \eta(x,y)\rangle \geq r\|x-y\|^2$ for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.

The *duality mapping* $J : X \multimap X^*$ is defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in X$. The *module of smoothness* of X is the function $\rho_X : [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$\rho_X(t) = \sup\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called *uniformly smooth* if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$ if and only if there exists a constant $c > 0$ for which $\rho_X(t) \leq ct^2$.

Definition 2.4. A single valued map $H : X \longrightarrow X$ is said to be *k-strongly accretive* if for any $x, y \in X$ there exist $j \in J(x - y)$ for which $\langle j, H(x) - H(y)\rangle \geq k\|x - y\|^2$.

Xia and Huang, proving Theorem 3.4 in [18] showed the following useful inequality.

Proposition 2.5. *Suppose that X is a uniformly smooth Banach space with $\rho_X(t) \leq ct^2$ for some $c > 0$. If $\psi : X \longrightarrow X$ is k-strongly accretive and δ-Lipschitz continuous mapping, then* $\|x - y - \psi(x) - \psi(y)\| \leq \sqrt{1 - 2k + 64c\delta^2}\|x - y\|$.

Let $\mathcal{P}(X)$ be the power set of X. The function $D : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, +\infty]$ defined by

$$D(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

is called *Hausdorff pseudo-metric*. We note that, if D is restericted to closed bounded subsets of X, then it is Hausdorff metric.

Definition 2.6. A set valued mapping $M : X \multimap X$ is said to be *D-Lipschitz continuous* with constant ρ, if $D(M(x), M(y)) \leq \rho\|x - y\|$ for all $x, y \in X$.

3. General (H, η)-monotone operators

Definition 3.1. [1] The set-valued map $T : X \multimap X^*$ is said to be *general (H, η)-monotone operator* if T is η-monotone and $(H + \lambda T)(X) = X^*$ holds for every $\lambda > 0$.

Remark 3.2. The general (H, η)-monotone operator reduces to the

(a) general H-monotone operator which is introduced in [18], if $\eta(x, y) = x - y$.

(b) (H, η)-monotone operators [7], if X is a Hilbert space.

(c) g-η-monotone mapping which is considered in [20], when X is a Hilbert space and $H = g$.

(d) H-monotone operator [5], if X is a Hilbert space and $\eta(x, y) = x - y$.

(e) maximal η-monotone mapping considered in [2, 4], when X is a Hilbert space and $H = I$, the identity mapping.

Example 3.3. Let $X = \mathbb{R}$ and let $\eta : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\eta(x, y) = y^3 - x^3$. Consider the set-valued mapping $T : \mathbb{R} \multimap \mathbb{R}$ defined by

$$T(x) = \begin{cases} \{-x-1\} & x > 0 \\ \{-1, 1\} & x = 0 \\ \{-x+1\} & x < 0. \end{cases}$$

One can easily check that T is η-monotone and $(I + \lambda T)(\mathbb{R}) \neq \mathbb{R}$ for $\lambda \geq 1$. Therefore, T is not maximal η-monotone. Now, for single valued mapping $H : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$H(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0, \end{cases}$$

we have $(H + \lambda T)(\mathbb{R}) = \mathbb{R}$ for all $\lambda > 0$, which implies that T is general (H, η)-monotone operator.

Example 3.4. Let $X = \mathbb{R}$ and let $\eta : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\eta(x, y) = x^3 - y^3$. Consider the set-valued mapping $T : \mathbb{R} \multimap \mathbb{R}$ defined by $T(x) = \{x\}$. Then T is η-monotone and $(I + \lambda T)(x) = \{(1 + \lambda)x\}$ and hence $(I + \lambda T)(\mathbb{R}) = \mathbb{R}$ for all $\lambda > 0$. Therefore, T is maximal η-monotone. Now, for single valued mapping $H : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $H(x) = x^2$ we have $(H + \lambda T)(\mathbb{R}) = [\frac{-\lambda^2}{4}, +\infty) \neq \mathbb{R}$ for all $\lambda > 0$, which implies that T is not general (H, η)-monotone operator.

Theorem 3.5. [1] *Suppose that $H : X \longrightarrow X^*$ is a strictly η-monotone mapping and suppose that $T : X \multimap X^*$ is a general (H, η)-monotone operator. Then $(H + \lambda T)^{-1} : X^* \longrightarrow X$ is a single-valued operator for all $\lambda > 0$.*

Based on Theorem 3.5, we can define the *general proximal mapping* $R^{H,\eta}_{T,\lambda}$ as following.

Definition 3.6. [1] For a strictly η-monotone mapping $H : X \longrightarrow X^*$ and a general (H, η)-monotone operator $T : X \multimap X^*$, the *general proximal mapping* $R^{H,\eta}_{T,\lambda} : X^* \longrightarrow X$ is defined by $R^{H,\eta}_{T,\lambda}(x^*) = (H + \lambda T)^{-1}(x^*)$.

Remark 3.7. The general proximal mapping $R^{H,\eta}_{T,\lambda}$ reduces to the

(a) proximal mapping R^H_M which is introduced in [18], when $T = M$ and $\eta(x, y) = x - y$.

(b) resolvent operator $R^{H,\eta}_{M,\lambda}$ [7], when $T = M$ and X is a Hilbert space.

(c) resolvent operator $R^g_{M,\lambda}$ [20], if $T = M$, $H = g$ and X is a Hilbert space.

(d) resolvent operator $R^H_{M,\lambda}$ which is introduced in [5], if X is a Hilbert space, $T = M$ and $\eta(x, y) = x - y$.

(e) η-proximal mapping of φ [3], if X is a Hilbert space and $T = \partial\varphi$, where $\varphi : X \longrightarrow (-\infty, +\infty]$ is a lower semicontinuous subdifferentiable proper functional.

(f) resolvent of maximal monotone operator [19], when X is a Hilbert space, $H = I$ and $\eta(x, y) = x - y$.

Theorem 3.8. *Suppose that $\eta : X \times X \longrightarrow X$ is a γ-Lipschitz continuous mapping, $H : X \longrightarrow X^*$ is an r-strongly η-monotone operator and $T : X \multimap X^*$ is a general β-strongly (H, η)-monotone operator. Then the general proximal mapping $R^{H,\eta}_{T,\lambda} : X^* \longrightarrow X$ is a $\frac{\gamma}{r+\lambda\beta}$-Lipschitz continuous operator.*

Proof. For any two points $x^*, y^* \in X^*$ with $\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\| \neq 0$. Since

$$R^{H,\eta}_{T,\lambda}(x^*) = (H + \lambda T)^{-1}(x^*) \text{ and } R^{H,\eta}_{T,\lambda}(y^*) = (H + \lambda T)^{-1}(y^*),$$

so

$$\frac{x^* - H(R^{H,\eta}_{T,\lambda}(x^*))}{\lambda} \in T(R^{H,\eta}_{T,\lambda}(x^*)) \text{ and } \frac{y^* - H(R^{H,\eta}_{T,\lambda}(y^*))}{\lambda} \in T(R^{H,\eta}_{T,\lambda}(y^*)).$$

T is a general β-strongly (H, η)-monotone operator, so

$$\langle \frac{x^* - H(R^{H,\eta}_{T,\lambda}(x^*))}{\lambda} - \frac{y^* - H(R^{H,\eta}_{T,\lambda}(y^*))}{\lambda}, \eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\rangle \geq \beta\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\|^2.$$

Therefore,

$$\begin{aligned}\langle x^* - y^*, \eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\rangle \geq \; & \langle \; H(R^{H,\eta}_{T,\lambda}(x^*)) - H(R^{H,\eta}_{T,\lambda}(y^*)), \eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\rangle \\ & + \; \lambda\beta\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\|^2.\end{aligned}$$

Since $\eta : X \times X \longrightarrow X$ is γ-Lipschitz and H is r-strongly η-monotone,

$$\begin{aligned}\gamma\|x^* - y^*\|\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\| &\geq \|x^* - y^*\|\|\eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\| \\ &\geq \langle x^* - y^*, \eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\rangle \\ &\geq \langle H(R^{H,\eta}_{T,\lambda}(x^*)) - H(R^{H,\eta}_{T,\lambda}(y^*)), \eta(R^{H,\eta}_{T,\lambda}(x^*), R^{H,\eta}_{T,\lambda}(y^*))\rangle \\ &\quad +\lambda\beta\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\|^2 \\ &\geq (r + \lambda\beta)\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\|^2.\end{aligned}$$

That $R^{H,\eta}_{T,\lambda}$ is $\frac{\gamma}{r+\lambda\beta}$-Lipschitz continuous follows from the fact that $\|R^{H,\eta}_{T,\lambda}(x^*) - R^{H,\eta}_{T,\lambda}(y^*)\| \neq 0$.

Similar to the proof of Theorem 3.8, one can deduce the following results.

Theorem 3.9. [1] *Suppose $H : X \longrightarrow X^*$ is a strictly η-monotone operator, $\eta : X \times X \longrightarrow X$ is a γ-Lipschitz continuous mapping and also suppose that $T : X \multimap X^*$ is a general β-strongly (H, η)-monotone operator. Then the general proximal mapping $R^{H,\eta}_{T,\lambda} : X^* \longrightarrow X$ is a $\frac{\gamma}{\lambda\beta}$-Lipschitz continuous operator.*

Theorem 3.10. [1] *Suppose $H : X \longrightarrow X^*$ is an r-strongly η-monotone operator, $\eta : X \times X \longrightarrow X$ is a γ-Lipschitz continuous mapping and also suppose that $T : X \multimap X^*$ is a general (H, η)-monotone operator. Then the general proximal mapping $R^{H,\eta}_{T,\lambda} : X^* \longrightarrow X$ is a $\frac{\gamma}{r}$-Lipschitz continuous operator.*

Remark 3.11. Theorem 3.9 and Theorem 3.10 are extended versions of Theorem 2.2 in [5], Lemma 2.2 in [7], Theorem 2.2 in [20] and Theorem 3.2 in [18].

4. Variational Inclusion Problems

Problem 4.1. Consider three single valued mappings $\psi : X \longrightarrow X$, $H : X \longrightarrow X^*$ and $S : X \times X \longrightarrow X^*$. Also, consider two set-valued mappings $M : X \multimap X$ and $T : X \multimap X^*$. Suppose T is an (H, η)-monotone operator. Our problem is finding

$$x \in X \text{ and } y \in M(x) \text{ for which } 0 \in S(x, y) + T(\psi(x)).$$

Remark 4.2. For appropriate and suitable choices of X, ψ, H, S, M, η and T one can obtain many known and new classes of variational inequalities and variational inclusions as special cases of the Problem 4.1. For example see [5, 18].

Lemma 4.3. *Suppose $H : X \longrightarrow X^*$ is a strictly η-monotone operator and $T : X \multimap X^*$ is a general (H, η)-monotone operator. Then $(x, y) \in X \times M(x)$ is a solution of Problem* (4.1) *if and only if $\psi(x) = R^{H,\eta}_{T,\lambda}[H(\psi(x)) - \lambda S(x, y)]$.*

Proof. It is straightforward.

Algorithm 4.4. Consider the following four steps

Step 1: Choose $x_0 \in X$ and also choose $y_0 \in M(x_0)$.

Step 2: Let $x_n = x_{n-1} - \psi(x_{n-1}) + \mathcal{R}^H_{T,\lambda}[H(\psi(x_{n-1})) - \lambda S(x_{n-1}, y_{n-1})]$.

Step 3: Choose $y_n \in M(x_n)$ such that $\|y_n - y_{n-1}\| \leq \frac{n+1}{n} D(M(x_n), M(x_{n-1}))$.

Step 4: If (x_n, y_n) is a solution of Problem 4.1, stop, otherwise, set $n := n + 1$ and return to Step 2.

Definition 4.5. A single-valued map $S : X \times X \longrightarrow X^*$ is said to be

(a) *ρ-Lipschitz continuous in the first variable* if for all $y \in X$ the mapping $S(., y) : X \longrightarrow X$ is ρ-Lipschitz continuous.

(b) *ξ-Lipschitz continuous in the second variable* if for all $x \in X$ the mapping $S(x, .) : X \longrightarrow X$ is ξ-Lipschitz continuous.

Lemma 4.6. *Suppose X is a uniformly smooth Banach space with $\rho_X(t) \leq ct^2$ for some $c > 0$. Also, suppose that*

(a) *$\psi : X \longrightarrow X$ is k-strongly accretive and δ-Lipschitz continuous mapping.*

(b) *$H : X \longrightarrow X^*$ is an r-strongly η-monotone and s-Lipschitz continuous operator.*

(c) *$T : X \multimap X^*$ is a general β-strongly (H, η)-monotone operator.*

(d) *$S : X \times X \longrightarrow X$ is ρ-Lipschitz continuous in the first variable and ξ-Lipschitz continuous in the second variable.*

(e) *$\eta : X \times X \longrightarrow X$ is γ-Lipschitz continuous mapping.*

(f) *$M : X \multimap X$ is D-Lipschitz continuous with constant τ.*

If $\{x_n\}$ be as in Algorithm 4.4, *then $\|x_{n+1} - x_n\| \leq k_n \|x_n - x_{n-1}\|$, where $k_n = \sqrt{1 - 2k + 64c\delta^2} + \frac{\gamma}{r+\lambda\beta}(s\delta + \rho\lambda + \lambda\tau\xi\frac{n+1}{n})$.*

Proof. The hypothesis, Proposition 2.5, Theorem 3.8 and Step 3 of Algorithm 4.4 imply that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|x_n-\psi(x_n)+\mathcal{R}^{H,\eta}_{T,\lambda}[H(\psi(x_n))-\lambda S(x_n,y_n)]-(x_{n-1}-\psi(x_{n-1})\\
&\quad +\mathcal{R}^{H,\eta}_{T,\lambda}[H(\psi(x_{n-1}))-\lambda S(x_{n-1},y_{n-1})])\|\\
&\le \|x_n-x_{n-1}-\psi(x_n)+\psi(x_{n-1})\|\\
&\quad +\|\mathcal{R}^{H,\eta}_{T,\lambda}[H(\psi(x_n))-\lambda S(x_n,y_n)]-\mathcal{R}^{H,\eta}_{T,\lambda}[H(\psi(x_{n-1}))-\lambda S(x_{n-1},y_{n-1})]\|\\
&\le \sqrt{1-2k+64c\delta^2}\|x_n-x_{n-1}\|\\
&\quad +\frac{\gamma}{r+\lambda\beta}\|(H(\psi(x_n))-\lambda S(x_n,y_n))-(H(\psi(x_{n-1}))-\lambda S(x_{n-1},y_{n-1}))\|\\
&\le \sqrt{1-2k+64c\delta^2}\|x_n-x_{n-1}\|+\frac{\gamma}{r+\lambda\beta}\|H(\psi(x_n))-H(\psi(x_{n-1}))\|\\
&\quad +\frac{\gamma\lambda}{r+\lambda\beta}\|S(x_n,y_n)-S(x_{n-1},y_{n-1}))\|\\
&\le \sqrt{1-2k+64c\delta^2}\|x_n-x_{n-1}\|+\frac{s\delta\gamma}{r+\lambda\beta}\|x_n-x_{n-1}\|\\
&\quad +\frac{\gamma\lambda}{r+\lambda\beta}\|S(x_n,y_n)-S(x_{n-1},y_n)\|+\frac{\gamma\lambda}{r+\lambda\beta}\|S(x_{n-1},y_n)-S(x_{n-1},y_{n-1}))\|\\
&\le (\sqrt{1-2k+64c\delta^2}+\frac{s\delta\gamma}{r+\lambda\beta})\|x_n-x_{n-1}\|+\frac{\rho\lambda\gamma}{r+\lambda\beta}\|x_n-x_{n-1}\|\\
&\quad +\frac{\gamma\lambda\xi}{r+\lambda\beta}\|y_n-y_{n-1}\|\\
&\le (\sqrt{1-2k+64c\delta^2}+\frac{s\delta\gamma}{r+\lambda\beta}+\frac{\rho\lambda\gamma}{r+\lambda\beta})\|x_n-x_{n-1}\|+\frac{\gamma\lambda\xi}{r+\lambda\beta}\tau\frac{n+1}{n}\|x_n-x_{n-1}\|\\
&\le (\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{r+\lambda\beta}(s\delta+\rho\lambda+\lambda\tau\xi\frac{n+1}{n}))\|x_n-x_{n-1}\|
\end{aligned}$$

Then $\|x_{n+1}-x_n\|\le k_n\|x_n-x_{n-1}\|$.

Theorem 4.7. *Suppose X is a uniformly smooth Banach space with $\rho_X(t)\le ct^2$ for some $c>0$. Also, suppose that*

(a) *$\psi : X \longrightarrow X$ is k-strongly accretive and δ-Lipschitz continuous mapping.*

(b) *$H : X \longrightarrow X^*$ be an r-strongly η-monotone and s-Lipschitz continuous operator.*

(c) *$T : X \multimap X^*$ be a general β-strongly (H,η)-monotone operator.*

(d) *$S : X\times X \longrightarrow X$ is ρ-Lipschitz continuous in first variable and ξ-Lipschitz continuous in second variable.*

(e) *$\eta : X\times X \longrightarrow X$ is γ-Lipschitz continuous mapping.*

(f) *$M : X \multimap X$ is closed valued and also M is D-Lipschitz continuous with constant τ.*

If $\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{r+\lambda\beta}(s\delta+\rho\lambda+\lambda\tau\xi)<1$*, then the* Problem 4.1 *has a solution.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be as in Algorithm 4.4. According to Lemma 4.6, we have $\|x_{n+1}-x_n\|\leq k_n\|x_n-x_{n-1}\|$, where $k_n=\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{r+\lambda\beta}(s\delta+\rho\lambda+\lambda\tau\xi\frac{n+1}{n})$. Set $k=\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{r+\lambda\beta}(s\delta+\rho\lambda+\lambda\tau\xi)$. Obviously, $k_n\longrightarrow k$ as $n\longrightarrow+\infty$, which our assumption implies that $k<1$. Therefore, $\{x_n\}$ is a Cauchy sequence and hence there exists $x\in X$ for which $x_n\longrightarrow x$. On the other hand, by Step 3 of Algorithm 4.4, we have

$$\|y_n-y_{n-1}\|\leq\frac{n+1}{n}D(M(x_n),M(x_{n-1}))\leq\frac{n+1}{n}\tau\|x_n-x_{n-1}\|.$$

Consequently, $\{y_n\}$ is a Cauchy sequence and so there exists $y\in X$ for which $y_n\longrightarrow y$. Also

$$\begin{aligned}d(y,M(x)) &= \inf\{\|y-m\|:m\in M(x)\}\\ &\leq \|y-y_n\|+d(y_n,M(x))\\ &\leq \|y-y_n\|+D(M(x_n),M(x))\\ &\leq \|y-y_n\|+\tau\|x_n-x\|.\end{aligned}$$

$\|y-y_n\|+\tau\|x_n-x\|\longrightarrow 0$ as $n\longrightarrow+\infty$ and than $d(y,M(x))=0$. Now, because $M(x)$ is closed, we have $y\in M(x)$. Since $\mathcal{R}^{H,\eta}_{T,\lambda}$, H, ψ and S are Lipschitz continuous, so Step 3 of Algorithm 4.4 and since M is D-Lipschitz continuous with constant ρ, we have $\psi(x)=R^{H,\eta}_{T,\lambda}[H(\psi(x))-\lambda S(x,y)]$. That (x,y) is a solution of Problem 4.1 follows from Lemma 4.3.

Remark 4.8. According to Theorem 3.9 and Theorem 3.10, by an argument similar to the Theorem 4.7, if we consider the following condition instead of (b) in Theorem 4.7,

(b') $H:X\longrightarrow X^*$ is a strictly η-monotone operator. Then one can deduce that :

If $\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{\lambda\beta}(s\delta+\rho\lambda+\lambda\tau\xi)<1$, then the Problem 4.1 has a solution.

Similarly, if we consider the following condition instead of (c) in Theorem 4.7,

(c') $T:X\multimap X^*$ is a general (H,η)-monotone operator. Then it is easy to see that:

If $\sqrt{1-2k+64c\delta^2}+\frac{\gamma}{r}(s\delta+\rho\lambda+\lambda\tau\xi)<1$, then the Problem 4.1 has a solution.

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