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CERTAIN SUBCLASSES OF CONVEX FUNCTIONS WITH POSITIVE AND MISSING COEFFICIENTS BY USING A FIXED POINT

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ABSTRACT. By considering a fixed point in unit disk Δ, a new class of univalent convex functions is defined. Coefficient inequalities, integral operator and extreme points of this class are obtained.

1. INTRODUCTION

Let w be fixed point in $\Delta = \{z : |z| < 1\}$ and

$$A(w) = \{f \in H(\Delta) : f(w) = f'(w) - 1 = 0\}.$$

Let $M_w = \{f \in A(w)$: f is Univalent in $\Delta\}$, and M_w denoted the subclass of $A(w)$ consist of the functions of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=k}^{+\infty} a_n (z-w)^n, a_n \geq 0, z \neq w \tag{1.1}$$

The function $f(z)$ in M_w is said to be Convex of order $\eta(0 < \eta < 1)$ if and only if

$$Re\{1 + \frac{(z-w)f''(z)}{f'(z)}\} > \eta \quad , \quad (z \in \Delta).$$

The similar definitions for uniformly convex and starlike functions are introduced by Goodman in [4, 5]. For the function $f(z)$ in M_w, Ghanim and Darus have defined an operator as follows:

$$I^0 f(z) = f(z)$$

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$$I^1 f(z) = (z-w)f'(z) + \frac{2}{z-w}$$

$$I^2 f(z) = (z-w)(I^1 f(z))' + \frac{2}{z-w}$$

and for $m = 1, 2, ...$

$$I^m f(z) = (z-w)(I^{m-1} f(z))' + \frac{2}{z-w} \tag{1.2}$$

$$= \frac{1}{z-w} + \sum_{n=k}^{+\infty} n^m a_n (z-w)^n.$$

We note that various other subclasses have been studied rather extensively by many authors (see [1, 2, 3] and [6]).

Definition 1.1. A function $f(z)$ belonging to the class M_w is in the class $M_w(A, B, m)$ if it satisfies the condition

$$\left|\frac{\frac{1}{2}[(z-w)\frac{(I^m f(z))''}{(I^m f(z))'} + 1] + \frac{1}{2}}{\frac{B}{2}[(z-w)\frac{(I^m f(z))''}{(I^m f(z))'} + 1] + \frac{A+1}{2}}\right| \leq 1 \tag{1.3}$$

for some $-1 \leq B < A < 1$, $0 \leq A \leq 1$.

2. Main Results

First we obtain coefficient inequalities for functions in $M_w(A, B, m)$. Then we prove the linear combination property.

Theorem 2.1. *The function $f(z)$ of the form (1.1) belongs t $M_w(A, B, m)$ if and only if*

$$\sum_{n=k}^{+\infty} n^{m+1}[n(B+1) + A + 2]a_n \leq 1 + A - B. \tag{2.1}$$

Proof. Suppose (2.1) holds and

$$H = \left|\frac{1}{2}(z-w)(I^m f(z))'' + (I^m f(z))'\right| - \left|\frac{B}{2}(z-w)(I^m f(z))'' + \frac{B+A}{2}\right| \tag{2.2}$$

Replacing $(I^m f(z))'$ and $(I^m f(z))''$ by their series expansion for $0 < |z-w| = r < 1$, we have

$$H = \left|\sum_{n=k}^{+\infty} \frac{1}{2} n^{m+1}(n+1)a_n(z-w)^{n-1}\right| - \left|\frac{1}{2}(B-A-1)\frac{1}{(z-w)^2}\right.$$

$$\left. + \sum_{n=k}^{+\infty} n^{m+1}[\frac{1}{2}(Bn+A+1)]a_n(z-w)^{n-1}\right|$$

$$\leq \left|\sum_{n=k}^{+\infty} \frac{1}{2} n^{m+1}(n+1)a_n r^{n-1} - \frac{1}{2}(1+A-B)\frac{1}{r^2} + \sum_{n=k}^{+\infty} n^{m+1}[\frac{1}{2}(Bn+A+1)]a_n r^{n-1}\right|$$

Since this inequality holds for all r $(0<r<1)$, making $r\to 1$, we have

$$H \le \sum_{n=k}^{+\infty} \frac{1}{2} n^{m+1}[n(B+1)+A+2]a_n - \frac{1}{2}(1+A-B)$$

by (2.1)

$$H \le 0.$$

So we have the required result.
Conversely, let
$f(z)=\frac{1}{z-w}+\sum_{n=k}^{+\infty} a_n(z-w)^n$ and (1.3) holds, then we have

$$\left|\frac{\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(n+1)a_n(z-w)^{n-1}}{\frac{1}{2}(B-A-1)\frac{1}{(z-w)^2}+\sum_{n=k}^{+\infty} n^{m+1}[\frac{1}{2}(Bn+A+1)]a_n(z-w)^{n-1}}\right| \le 1,$$

hence,

$$\left|\frac{\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(n+1)a_n(z-w)^{n+1}}{\frac{1}{2}(1+A+B)-\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(Bn+A+1)a_n(z-w)^{n+1}}\right| \le 1.$$

Since $Re(z)\le |z|$ for all z, then it follows from above inequality that

$$Re\left\{\frac{\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(n+1)a_n(z-w)^{n+1}}{\frac{1}{2}(1+A+B)-\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(Bn+A+1)a_n(z-w)^{n+1}}\right\} \le 1.$$

By putting $z-w=r$ with $0<r<1$ in above inequality, we obtain

$$\frac{\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(n+1)a_n r^{n+1}}{\frac{1}{2}(1+A-B)-\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(Bn+A+1)a_n r^{n+1}} \le 1. \tag{2.3}$$

Upon clearing the denominator in (2.3) and letting $r\to 1$, we get

$$\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(n+1)a_n \le \frac{1}{2}(1+A-B)-\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}(Bn+A+1)a_n,$$

so

$$\sum_{n=k}^{+\infty}\frac{1}{2}n^{m+1}[n(B+1)+A+2]a_n \le \frac{1}{2}(1+A-B).$$

This completes the proof. □

Theorem 2.2. *Let $f_j(z)$ defined by*

$$f_j(z)=\frac{1}{z-w}+\sum_{n=k}^{+\infty} a_{n,j}(z-w)^n, j=1,2,... \tag{2.4}$$

be in the class $M_w(A,B,m)$, then the function

$$F(z)=\sum_{j=1}^{t} d_j f_j(z) \quad , \quad d_j \ge 0$$

is also in $M_w(A,B,m)$, where $\sum_{j=1}^{t} d_j = 1$.

Proof. Since $f_j(z) \in M_w(A,B,m)$, by (2.1) we have,

$$\sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2]a_{n,j} \leq 1+A-B \quad ,j=1,2,... \tag{2.5}$$

also

$$F(z) = \sum_{j=1}^{t} d_j(\frac{1}{z-w} + \sum_{n=k}^{+\infty} a_{n,j}(z-w)^n)$$

$$= \frac{1}{z-w}\sum_{j=1}^{t} d_j + \sum_{n=k}^{+\infty}(\sum_{j=1}^{t} d_j a_{n,j})(z-w)^n$$

$$= \frac{1}{z-w} + \sum_{n=k}^{+\infty} s_n(z-w)^n \quad where \quad s_n = \sum_{j=1}^{t} d_j a_{n,j}$$

But

$$\sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2]s_n = \sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2][\sum_{j=1}^{t} d_j a_{n,j}]$$

$$= \sum_{j=1}^{t} d_j\{\sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2]a_{n,j}\}$$

by (2.5)

$$\leq \sum_{j=1}^{t} d_j(1+A-B) = 1+A-B.$$

Now the proof is complete. □

3. Extreme points and Integral operators

In the last section we investigate about extreme points of $M_w(A,B,m)$ and verify the effect of two operators on functions in the class $M_w(A,B,m)$.

Theorem 3.1. *Let*

$$f_0(z) = \frac{1}{z-w}$$

and

$$f_n(z) = \frac{1}{z-w} + \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]}(z-w)^n \ , \quad n \geq k \tag{3.1}$$

then $f(z) \in M_w(A,B,m)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} C_n f_n(z)$ where $C_n \geq 0$, $C_i = 0$ $(i=1,2,...,k-1)$, $\sum_0^{+\infty} C_n = 1$

Proof. Let $f(z) = \sum_0^{+\infty} C_n f_n(z)$, so,

$$f(z) = \frac{C_0}{z-w} + \sum_{n=1}^{k-1} C_n[\frac{1}{z-w} + \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]}(z-w)^n] +$$

$$\sum_{n=k}^{+\infty} C_n[\frac{1}{z-w} + \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]}(z-w)^n]$$

$$= \frac{1}{z-w} + \sum_{n=k}^{+\infty} \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]} C_n (z-w)^n.$$

Since

$$\sum_{n=k}^{+\infty} \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]} \frac{n^{m+1}[n(B+1)+A+2]}{1+A-B} C_n$$

$$= \sum_{n=k}^{+\infty} C_n = \sum_{n=1}^{+\infty} C_n = 1 - C_0 \le 1,$$

so $f(z) \in M_w(A,B,m)$.

Conversely , suppose that $f(z) \in M_w(A,B,m)$. Then by (2.1) we have

$$0 \le a_n \le \frac{1+A-B}{n^{m+1}[n(B+1)+A+2]}.$$

By setting

$$C_n = \frac{n^{m+1}[n(B+1)+A+2]}{1+A-B} a_n \quad , \ n \ge 1$$

$C_i = 0 \ \ (i = 1, 2, ..., k-1)$, $C_0 = 1 - \sum_{n=1}^{+\infty} C_n$, we obtain the required result. □

Theorem 3.2. *Let γ be a real number such that $\gamma > 1$. If $f(z) \in M_w(A,B,m)$, then the functions*

$$H_1(z) = \frac{\gamma - 1}{(z-w)^\gamma} \int_w^z (t-w)^{\gamma-1} f(t) dt \quad \textit{and}$$

$$H_2(z) = C \int_0^1 \nu^C f(\nu(z-w)+w) d\nu \ , \ C \ge 1$$

are also in the same class.

Proof. Let $f(z) \in M_w(A,B,m)$, then a simple calculation shows that,

$$H_1(z) = \frac{1}{z-w} + \sum_{n=k}^{+\infty} \frac{1}{\gamma+n} a_n (z-w)^n$$

$$H_2(z) = \frac{1}{z-w} + \sum_{n=k}^{+\infty} \frac{C}{C+n+1} a_n (z-w)^n$$

Since $\frac{C}{C+n+1} < 1$ and $\frac{1}{\gamma+n} < 1$, by using Theorem 2.1, we get,

$$\sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2] a_n \frac{1}{\gamma+n} < \sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2] a_n < 1+A-B$$

and

$$\sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2] a_n \frac{C}{C+n+1} < \sum_{n=k}^{+\infty} n^{m+1}[n(B+1)+A+2] a_n < 1+A-B$$

Hence by Theorem (2.1) we conclude that $H_1(z)$ and $H_2(z)$ are in the class $M_w(A,B,m)$. So the proof is complete. □

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