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COMMON FIXED POINT THEOREM OF ALTMAN INTEGRAL TYPE MAPPINGS

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ABSTRACT. In this paper, by virtue of some analysis techniques, we prove a new common fixed point theorem of Altman type for four mappings satisfying a contractive condition of integral type in complete metric spaces, which improves and extends several previous results obtained by others.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we assume that (X, d) is a complete metric space.

In 1975, Altman [2] considered a fixed point theorem for one mapping f satisfying the following contractive condition:

$$d(fx, fy) \leq Q(d(x, y)), \ \forall x, y \in X,$$

where $Q : [0, +\infty) \to [0, +\infty)$ is an increasing function that satisfies the following conditions:

(i) $0 < Q(t) < t, \ t \in (0, +\infty)$;

(ii) $p(t) = t/(t - Q(t))$ is a decreasing function;

(iii) $\int_0^{t_1} p(t)dt < +\infty$ for some positive number t_1.

The common fixed point of Altman type mappings mentioned above has been studied by many authors (see, [2, 3, 4, 5, 7], etc.) and has been extended to some extent respectively. In

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[1], Aliouche prove common fixed point theorems for two pairs of hybrid mappings satisfying generalized contractive conditions. In this paper, we further discuss the common fixed point problem for two pairs of Altman type mappings satisfying contractive conditions of integral type. The results presented in this paper generalize and extend results of Altman [2], Chen and Gu [3], Garbone and Singh [4], Gu and Deng [5], Liu [7], etc.

We begin with some known definitions.

Definition 1.1. ([9]) Two self-maps S and T of a metric space (X, d) are weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx),\ \forall x \in X$$

Definition 1.2. ([8]) Two self-maps S and T of a metric space (X, d) are R weakly commuting if there exists some positive real number R such that

$$d(STx, TSx) \leq Rd(Sx, Tx),\ \forall x \in X.$$

Definition 1.3. ([6]) Two self-maps S and T of a metric space (X, d) are compatible, if for any $\{x_n\} \subset X$, when $Sx_n \to x,\ Tx_n \to x, x \in X$. we have

$$d(STx_n, TSx_n) \to 0\ (n \to \infty).$$

Remark 1.4. By condition (i) and that Q is increasing implie $Q(0) = 0$ and $Q(t) = t \Leftrightarrow t = 0$.

Remark 1.5. Obviously, from the definitions above, we can see that if two self-maps S and T of a metric space (X, d) are weakly commuting, then S and T must be R weakly commuting; R weakly commuting implies compatible. But the converse of these statements are not true in general.

In order to prove the main results of this paper, we need the following lemma:

Lemma 1.6. *Let $\delta(t)$ be Lebesgue integrable function, and $\delta(t) > 0,\ \forall t > 0$. Let $F(x) = \int_0^x \delta(t)dt$, then $F(x)$ is an increasing function in $[0, +\infty)$.*

In this paper, we let $\phi : [0, \infty) \to [0, \infty)$ denote the function satisfying$0 < \phi(t) < t, \forall t > 0$.

2. Main Results

In this section, we shall prove our main theorems.

Theorem 2.1. *Suppose (S, A) and (T, B) be two pairs of self-maps of complete metric spaces (X, d) which are continuous and compatible. If there exists an increasing function $Q : [0, +\infty) \to [0, +\infty)$ satisfying the condition (i)-(iii) and the following conditions:*

(iv) There exists two sequences $\{x_n\}, \{y_n\} \subset X$ such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n},\ y_{2n} = Sx_{2n} = Bx_{2n+1},\ y_n \neq y_{n+1},\ \forall n \in Z^+$$

(v) For all $x, y \in X$, have

$$\int_0^{d(Sx,Ty)} \delta(t)dt \leq \phi\left(\int_0^{Q(d(Ax,By))} \delta(t)dt\right),$$

where $\delta(t)$ is a Lebesgue integrable function which is summable nonnegative and such that

$$\int_0^{\varepsilon} \delta(t) > 0,\ \forall \varepsilon > 0.$$

Then the sequence $\{Sy_n\}$ converges to the unique common fixed point of S, T, A and B in X.

Proof. Let $t_n = d(y_n, y_{n+1})$, from condition (iv), (v) and the property of ϕ, we have

$$\int_0^{t_{2n}} \delta(t)dt = \int_0^{d(y_{2n},y_{2n+1})} \delta(t)dt = \int_0^{d(Sx_{2n},Tx_{2n+1})} \delta(t)dt$$
$$\leq \phi\left(\int_0^{Q(d(Ax_{2n},Bx_{2n+1})} \delta(t)dt\right) = \phi\left(\int_0^{Q(d(y_{2n-1},y_{2n})} \delta(t)dt\right)$$
$$= \phi\left(\int_0^{Q(t_{2n-1})} \delta(t)dt\right) \leq \int_0^{Q(t_{2n-1})} \delta(t)dt.$$

Hence from condition (i), Remark 1.2 and Lemma 1.1 we get $t_{2n} \leq Q(t_{2n-1}) < t_{2n-1}$. The fact $t_{2n+1} \leq Q(t_{2n}) < t_{2n}$ can be proved similarly as well. So $\{t_n\}$ is a nonnegative sequence that is strictly decreasing, hence $\{t_n\}$ is convergent and $t_{n+1} \leq Q(t_n) < t_n, n \in Z^+$.

Next we will prove $\{y_n\}$ is a Cauchy sequence in X. In fact, by condition (i), (ii) and triangular inequality we get

$$d(y_n, y_m) \leq \sum_{i=n}^{m-1} d(y_i, y_{i+1}) = \sum_{i=n}^{m-1} t_i = \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - t_{i+1}} \leq \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - Q(t_i)}$$
$$\leq \sum_{i=n}^{m-1} \int_{t_{i+1}}^{t_i} \frac{t}{t - Q(t)} dt = \int_{t_m}^{t_n} \frac{t}{t - Q(t)} dt = \int_{t_m}^{t_n} p(t)dt,$$

where $m, n \in Z^+,\ n \leq m$. It follows from the convergence of the sequence $\{t_n\}$ and condition (iii) that

$$\lim_{n\to\infty} \int_{t_m}^{t_n} p(t)dt = 0.$$

Thus, $\{y_n\}$ is a cauchy sequence in X. Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$, hence $\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} Ax_{2n} = u$, $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n} = u$. Since (S, A) be compatible mappings, thus, $\lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = 0$. Notice that (S, A) be continuous mappings, we have $d(Su, Au) = 0$, this is $Su = Au$. The fact that $Tu = Bu$ can be proved similarly as well

By the condition (v) and the property of function ϕ, we get

$$\int_0^{d(Su,Tu)} \delta(t)dt \leq \phi\left(\int_0^{Q(d(Au,Bu))} \delta(t)dt\right) \leq \int_0^{Q(d(Au,Bu))} \delta(t)dt.$$

It follows from $Su = Au$ and $Tu = Bu$ that

$$\begin{aligned} d(Su,Tu) &\leq Q(d(Au,Bu)) \\ &\leq d(Au,Bu) \\ &\leq d(Au,Su) + d(Su,Tu) + d(Tu,Bu) \\ &= d(Su,Tu), \end{aligned}$$

thus

$$Q(d(Au,Bu) = d(Su,Tu) = d(Au,Bu).$$

From the condition (i) we know that $d(Au,Bu) = 0$, and so $Au = Bu$. Setting

$$z = Su = Au = Tu = Bu,$$

since (S,A) are compatible mappings, hence

$$d(Sz,Az) = d(SAu,ASu) = 0,$$

so that $Sz = Az$. Similarly we get $Tz = Bz$, thus $SAu = ASu$, $TBu = BTu$. By the condition (v) we have

$$\int_0^{d(Sz,z)} \delta(t)dt = \int_0^{d(S^2u,Tu)} \delta(t)dt \leq \phi\left(\int_0^{Q(d(ASu,Bu))} \delta(t)dt\right) \leq \int_0^{d(ASu,Bu)} \delta(t)dt.$$

Hence we have

$$d(Sz,z) \leq Q(d(ASu,Bu)) \leq d(ASu,Bu) = d(SAu,Bu) = d(Sz,z),$$

this implies

$$Q(d(ASu,Bu)) = d(Sz,z) = d(ASu,Bu),$$

hence $ASu = Bu$, and so $Az = z$. Similarly, we can prove $Bz = z$. Thus we come to the conclusion that

$$z = Sz = Az = Tz = Bz,$$

hence z is the common fixed point of S, T, A and B.

In the following part, we will show the common fixed point of S, T, A and B is unique. In fact, assume w is another common fixed point of S, T, A, B and $z \neq w$. So $z = Sz = Az$, $w = Tw = Bw$. From the condition (v) we have

$$\int_0^{d(z,w)} \delta(t)dt = \int_0^{d(Sz,Tw)} \delta(t)dt \leq \phi\left(\int_0^{Q(d(Az,Bw))} \delta(t)dt\right)$$

$$= \phi\left(\int_0^{Q(d(z,w))} \delta(t)dt\right) \leq \int_0^{Q(d(z,w))} \delta(t)dt.$$

This leads to $d(z,w) \leq Q(d(z,w))$. By (i) we have $Q(d(z,w)) < d(z,w)$, a contradiction unless $z = w$, that is S, T, A and B have a unique common fixed point. Meanwhile we have $\lim_{n\to\infty} Sy_n = Su = z$.

This completes the proof of Theorem 1.1.

Remark 2.1. Taking $\delta(t) = 1$ in condition (v), we have

$$d(Sx,Ty) \leq \phi(Q(d(Ax,By)) \leq Q(d(Ax,By)),$$

thus Theorem 1.1 improves and extends related results in papers [2, 3, 4, 5, 7].

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References

[1] A. Aliouche, Common fixed point theorems for hybrid mappings satisfying generalized contractive conditions, J. Nonlinear Sci. Appl., 2 (2009), 136–145.
[2] M. Altman, A fixed point theorem in compact metric spaces, Amer. Math. Monthly, 82 (1975), 827–829.
[3] J. Chen and F. Gu, The common fixed point theorems satisfying a contractive condition of integral type,Natural Science Edition, Journal of Hangzhou Normal University, 7 (2008), 338–344.
[4] A. Garbone and S. P. Singh, Common fixed point theorem for Altman type mappings, Indian J. Pure Appl. Math., 18 (1987), 1082–1087.
[5] F. Gu and B. Deng, Common fixed point for Altman type mappings ,Natural Science Edition, Journal of Harbin Normal University, 17 (2001), 44–46.
[6] G. Jungck, Compatible mappings and common fixed points, Internat J. Math. Math. Sci., 9 (1986), 771–779.
[7] Z. Liu, On common fixed points of Altman type mappings ,Natural Science Edition, Journal of Liaoning University, 16 (1993), 1–4.
[8] R. P. Pant, Common fixed points of noncommuting mapping, J. Math. Anal. Appl., 188 (1994), 436–440.
[9] S. Sessa, On a weak commutitivity condition of mappings in fixed point considerations, Publ. Inst. Math., 32 (1982), 149–153.

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