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ON SOME PROPERTIES OF HOLOMORPHIC SPACES BASED ON BERGMAN METRIC BALL AND LUZIN AREA OPERATOR

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ABSTRACT. We provide new estimates and embedding theorems for holomorphic spaces in the unit ball defined with the help of Bergman metric ball and Luzin area operator. We also study the boundedness of integral operators similar to classical Bergman projections on spaces of mentioned type.

1. INTRODUCTION AND NOTATION

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B}^n = \{z \in \mathbb{C}^n | |z| < 1\}$. The boundary of $\mathbf{B}^n$ will be denoted by $\mathbf{S}^n$, $\mathbf{S}^n = \{z \in \mathbb{C}^n | |z| = 1\}$.

As usual, we denote by $H(\mathbf{B}^n)$ the class of all holomorphic functions on $\mathbf{B}^n$. For $0 < p < \infty$ we define the Hardy space $H^p(\mathbf{B}^n)$ consist of holomorphic functions f in $\mathbf{B}^n$ such that $\|f\|_p^p = \sup_{0<r<1} \int_{\mathbf{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty$. Here $d\sigma$ denotes the surface measure on $\mathbf{S}^n$ normalized so that $\sigma(\mathbf{S}^n) = 1$.

We denote by $H(\mathbf{D})$ the space of holomorphic functions in the unit disk $\mathbf{D}$.

For every function $f \in H(\mathbf{B}^n)$ having a series expansion $f(z) = \sum_{|k|\geq 0} a_k z^k$, we define the operator of fractional differentiation by

$$D^\alpha f(z) = \sum_{|k|\geq 0} (|k|+1)^\alpha a_k z^k,$$

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where α is any real number.

For a fixed $\alpha > 1$ let $\Gamma_\alpha(\xi) = \{z \in \mathbf{B}^n : |1 - \overline{\xi} z| < \alpha(1 - |z|)\}$ be the admissible approach region vertex is at $\xi \in \mathbf{S}^n$.

Let dv denote the volume measure on $\mathbf{B}^n$, normalized so that $v(\mathbf{B}^n) = 1$, and let $d\mu$ denote the positive Borel measure.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The well known Littlewood - Paley inequality in the unit ball of $\mathbb{C}^n$ for functions $f \in H^p(\mathbf{B}^n)$ is the following:

Theorem A. If $2 \le p < \infty$, $\beta > 0$, then for any $f \in H^p(\mathbf{B}^n)$

$$\int_{\mathbb{B}^n} |D^\beta f(z)|^p (1 - |z|)^{\beta p - 1} dv(z) \le C \|f\|^p_{H^p(\mathbf{B}^n)}. \tag{1.1}$$

Since

$$\|f\|^p_{H^p(\mathbf{B}^n)} \asymp \int_{\mathbf{S}^n} \left(\int_{\Gamma_\delta(\xi)} |D^k f(z)|^2 (1 - |z|)^{2k-2} dv(z) \right)^{\frac{p}{2}} d\sigma(\xi), \quad (\text{see } [8]), \tag{1.2}$$

and

$$\int_{\mathbf{B}^n} |f(z)|^p d\mu(z) \asymp \int_{\mathbf{S}^n} \left(\int_{\Gamma_\delta(\xi)} \frac{|f(z)|^p}{(1 - |z|)^n} d\mu(z) \right) d\sigma(\xi), \quad (\text{see } [4]), \tag{1.3}$$

for $0 < p < \infty$, then looking at estimate (1.1) it is natural to pose a general problem (see [2], [3], [4], [5], [12]).

Describe all positive Borel measures μ in the unit ball such that

$$\left(\int_{\mathbf{S}^n} \left(\int_{\Gamma_\delta(\xi)} \frac{|D^\beta f(z)|^p}{(1 - |z|)^n} d\mu(z) \right)^{\frac{q}{p}} d\sigma(\xi) \right)^{\frac{1}{q}} \le C \left(\int_X \left(\int_{\mathcal{J}} |f(z)|^{q_1} dv_\alpha(z) \right)^{\frac{s}{q_1}} dm \right)^{\frac{1}{s}}, \tag{1.4}$$

$\alpha > -1$, $\beta \ge 0$, $0 < p, q, q_1, s < \infty$ and $dv_\alpha = c_\alpha (1 - |z|^2)^\alpha dv(z)$ where c_α is a normalizing constant so that $v_\alpha(\mathbf{B}^n) = 1$.

$\mathcal{J}$ is a subset of $\mathbf{S}^n$ or $\mathbf{B}^n$, i.e. $\mathcal{J} = \mathcal{J}(\xi)$, $\xi \in \mathbf{S}^n$ or $\mathcal{J} = \mathcal{J}(z)$, $z \in \mathbf{B}^n$, $X = \mathbf{S}^n$ or $X = \mathbf{B}^n$ and dm is adequate measure.

For example $\mathcal{J} = Q_r(\xi) = \{z \in \mathbf{B}^n : d(z, \xi) < r\}$, where d is a non-isotropic metric on $\mathbf{S}^n$, $d(z, w) = |1 - \langle z, w \rangle|^{\frac{1}{2}}$, or $\mathcal{J} = \mathcal{D}(z, r) = \{w \in \mathbf{B}^n : \beta(z, w) < r\}$ where β is a Bergman metric on $\mathbf{B}^n$, $\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$.

Here the involution φ_z has the form

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle},$$

where $s_z = (1 - |z|^2)^{\frac{1}{2}}$, P_z is the ortogonal projection into the space spanned by $z \in \mathbf{B}^n$, i.e. $P_z w = \frac{\langle w, z \rangle z}{|z|^2}$, $P_0 w = 0$ and $Q_z = I - P_z$ (see, for example, [13]).

For $z \in \mathbf{B}^n$ and $r > 0$ set $\mathcal{D}(z, r)$ is called the Bergman metric ball at z, and for $\xi \in \mathbf{S}^n$ and $r > 0$ set $Q_r(\xi)$ is called the Carleson tube at ξ (see [13]).

For $\alpha > -1$ and $p > 0$ the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(\mathbf{B}^n, dv_\alpha)$, that is,

$$A^p_\alpha = L^p(\mathbf{B}^n, dv_\alpha) \cap H(\mathbf{B}^n).$$

See [7] and [13] for more details of weighted Bergman spaces. Various sharp embedding theorems in the unit ball and their numerous applications were given by many authors in recent years (see, for example, [2], [3], [4], [13]). The main purpose of this paper is to provide new estimates and sharp embedding theorems of mentioned type for the Luzin area operator and Bergman metric ball in the unit ball. Let us note that the study of similar to (1.4) embeddings in particular cases in the unit disk started recently in papers of W.Cohn [5] and Z.Wu [12]. We will also study general embeddings like (1.4) in the ball only in some concrete cases with some restrictions on parameters.

2. Preliminaries

This section is devoted mainly to proofs of auxiliary propositions. They will be used for proofs of main results of paper. We provide some new assertions for holomorphic spaces defined with the help of Bergman metric ball and Carleson tubes, they may have other applications and they are interesting from our point of view as separate statements. Also, we will introduce some further notations in order to formulate several auxiliary propositions.

Lemma 2.1. (a) *For every $\tau > 0$ there is a number δ, $\delta > 0$ such that the following inclusion $\mathcal{D}(z,\delta) \subset Q_{2\tau}(\xi)$ is continuous for all $z \in Q_\tau(\xi)$ and $\xi \in \mathbf{S}$.*
(b) *Let $\sigma > 1$, $t \geq 0$, $\xi \in \mathbf{S}^n$, $\widetilde{\Gamma}_\sigma(\xi) = \{z \| 1 - \overline{\xi} z| < \sigma(1-|z|)^{\frac{1}{n}}\}$. Then there exist $\widetilde{\sigma}(\sigma, t) > 1$ such that $\mathcal{D}(z,t) \subset \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$ for all $z \in \Gamma_\sigma(\xi)$.*

Proof. (a) Let $z \in Q_\tau(\xi)$ and $w \in \mathcal{D}(z,\delta)$, then

$$|1 - \langle w, \xi\rangle| \leq |1 - \langle z, \xi\rangle| + |\langle z - w, \xi\rangle| < \tau^2 + |z - w| \tag{2.1}$$

by definition of $Q_\tau(\xi)$ and triangle inequality.

Since $w \in \mathcal{D}(z,\delta)$ we have by definition $w = \varphi_z(u)$ with $|u| < \tanh(\delta)$. Now it is clear that if

$$|z - w| < 2\sqrt{|\overline{u}|} \tag{2.2}$$

then $|1 - \langle w, \xi\rangle| \leq 2\sqrt{\tanh(\delta)} + \tau^2$.

Hence choosing $2\sqrt{\tanh(\delta)} < 3\tau^2$ we get from (2.1) by definition of Carleson tube $|1 - \langle w, \xi\rangle|^{\frac{1}{2}} \leq 2\tau$ and hence $w \in Q_{2\tau}(\xi)$, this is what we need. Therefore it is enough to show (2.2). Since

$$|\varphi_a(w) - a|^2 = \frac{(1-|a|^2)(|w|^2 - |\langle w, a\rangle|^2)}{|1 - \langle a, w\rangle|^2} \tag{2.3}$$

and using Lemma 3.3 [13] we get

$$|z - w|^2 = |z - \varphi_z(u)|^2 \leq 4u, \quad \text{(see [13])}. \tag{2.4}$$

(*b*) Let $w \in \mathcal{D}(z,t)$, $z \in \Gamma_\sigma(\xi)$. We will show that $w \in \widetilde{\Gamma}_{\widetilde{\sigma}}(\xi)$, for some $\widetilde{\sigma} > 1$. Since $z \in \Gamma_\sigma(\xi)$, *then* $|1 - \overline{\xi} z| < \sigma(1-|z|)$ hence

$$\begin{aligned}|1 - \langle \xi, w\rangle| &\le |1 - \langle \xi, z\rangle| + |\langle \overline{\xi}, z\rangle - \langle \overline{\xi}, w\rangle| \\ &\le \sigma(1-|z|) + |z-w| \\ &\le \sigma(1-|w|) + (\sigma+1)|z-w|.\end{aligned}$$

We will show $|z-w| \le \sigma_1(1-|w|)^{\frac{1}{2}}$ for some $\sigma_1 > 1$. This is enough since $w \in \mathcal{D}(z,t)$ is the same to $z \in \mathcal{D}(w,t)$ we have by exercise 1.1 from [13]:

$$\frac{|P_w(z) - c|^2}{R^2\sigma_1^2} + \frac{|Q_z(w)|^2}{R^2\sigma_1} < 1, \tag{2.5}$$

where

$$R = tanh(t), \qquad c = \frac{(1-R^2)w}{1-R^2|w|^2}, \qquad \sigma_1 = \frac{1-|w|^2}{1-R^2|w|^2},$$

$$P_w(z) = \frac{\langle z, w\rangle w}{|w|^2}, \qquad Q_w(z) = z - \frac{\langle z, w\rangle w}{|w|^2}.$$

Hence

$$|z-w| \le C\left(|z-P_w| + \left|\frac{\langle z,w\rangle w}{|w|^2} - c\right| + |c-w|\right); \tag{2.6}$$

$$|c-w| \le |w|\left(1 - \frac{(1-R^2)}{1-R^2|w|^2}\right) \le \frac{C_2}{1-R^2}(1-|w|) = S_2. \tag{2.7}$$

It is enough to show

$$\left|z - \frac{\langle z,w\rangle w}{|w|^2}\right| + \left|\frac{\langle z,w\rangle w}{|w|^2} - c\right| \le R\left(\frac{1-|w|^2}{1-R^2|w|^2}\right)^{\frac{1}{2}} \widetilde{c}(R). \tag{2.8}$$

Note

$$|z-w|^2 \le C_3\left(|c-w|^2 + \left|z - \frac{\langle z,w\rangle w}{|w|^2}\right|^2 + \left|\frac{\langle z,w\rangle w}{|w|^2} - c\right|^2\right). \tag{2.9}$$

Hence from (2.5)

$$\begin{aligned}|z-w|^2 &\le C_4\left(|c-w|^2 + R^2\sigma_1\right) \\ &\le C_4\left(S_2(|w|,R)^2 + \frac{R^2(1-|w|^2)}{1-R^2|w|^2}\right) \\ &\le C_4\left(S_2 + \frac{R(1-|w|^2)^{\frac{1}{2}}}{1-R^2|w|^2}^{\frac{1}{2}}\right)^2.\end{aligned}$$

Hence $|z-w| \le C_5(1-|w|)^{\frac{1}{2}}$. Lemma is proved. □

Remark 2.2. For $n = 1$ the second part of lemma was proved in [12] and the first part can be found in [13], page 186. We gave the proof of the first part for completeness of our exposition. Both assertion in Lemma 2.1 obviously have similar structure.

Lets turn now to some observations concerning the action of integral operators similar to classical Bergman projections on spaces of mixed norm type discussed above. Below we present a useful argument that allows to get assertions of this type directly from one dimensional results under some natural conditions on function.

First we consider Toeplitz operators defined with the help of Bergman projection $T_\Phi(f) = P(\Phi f)$, $\Phi \in L^1(\mathbf{B})$, $f \in A^p(\mathbf{B})$. We consider the particular case of analytic symbol Φ. The general case can be considered similarly.

We have by formula from survey of A.B.Aleksandrov (see [1], [6]):

$$\begin{aligned}
&\int_{\mathbf{B}} |T_\Phi f(w)|^p (1-|w|)^\alpha dv(w) \\
&\quad = \frac{1}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |T_\Phi f(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le \frac{C}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |f(z\xi)|^p |\Phi(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le \frac{C}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |f(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le C \int_{\mathbf{B}} |f(w)|^p (1-|w|)^\alpha dv(w).
\end{aligned}$$

These estimates will hold if $\Phi_\xi(z) := \Phi(z\xi) \in X(\mathbf{D})$ for every fixed $\xi \in \mathbf{S}^n$.

We have proved that if $(f_\xi)(z) \in A^p_\alpha(\mathbf{D})$, $\xi \in \mathbf{S}$ and $(\Phi_\xi(z)) \in X(\mathbf{D}) \subset H(\mathbf{D})$ for all $\xi \in \mathbf{S}^n$ and $(\Phi_\xi)(z)(f_\xi)(z) = (T_\Phi f)(z)$ is acting and bounded from $A^p_\alpha(\mathbf{D})$ to $A^p_\alpha(\mathbf{D})$, $\alpha > -1$, $\xi \in \mathbf{S}$, $0 < p < \infty$, then $(T_\Phi f)(z) = \Phi(z) f(z)$, $z \in \mathbf{B}^n$ is also bounded operator which is acting from $A^p_\alpha(\mathbf{B})$ to $A^p_\alpha(\mathbf{B}^n)$, $0 < p < \infty$, $\alpha > -1$.

It is natural to question about operators invariant in

$$\int_{\mathbf{B}^n} \left(\int_{\mathcal{D}(z,r)} |f(w)|^q dv_\alpha(w) \right)^{\frac{p}{q}} dv_\beta(z) < \infty, \quad \alpha, \beta > -1, \quad p, q < \infty.$$

We now fix two real parameters $a > 0$ and $b > -1$ and define integral operator $S^{a,b}_r$ by

$$S^{a,b}_r f(z) = (1-|z|^2)^a \int_{\mathcal{D}(z,r)} \frac{(1-|w|^2)^b f(w)}{|1-\langle z, w\rangle|^{n+1+a+b}} dv(w), \ z \in \mathbf{B}^n, \ r > 0, \ f \in L^1(\mathbf{B}^n).$$

These Bergman projection type operators are invariant in spaces in which applies

$$\int_{\mathbf{B}^n} \left(\int_{\mathcal{D}(z,r)} |f(w)|^q dv_\alpha(w) \right)^{\frac{p}{q}} dv_\beta(z) < \infty.$$

Proposition 2.3. *Let* $1 \le p < \infty$, $s > 0$ *and* $t > -1$. *Then*

$$\begin{aligned}
&\int_{\mathbf{B}^n} \left(\int_{\mathcal{D}(z,r)} |S^{a,b}_r f(\widetilde{z})|^p dv_t(\widetilde{z}) \right)^{\frac{s}{p}} dv_\alpha(z) \\
&\qquad \le C \int_{\mathbf{B}^n} \left(\int_{\mathcal{D}(z,r)} |f(\widetilde{z})|^p dv_t(\widetilde{z}) \right)^{\frac{s}{p}} dv_\alpha(z).
\end{aligned}$$

Proof. We use properties of $\mathcal{D}(z,r)$.

Let $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and fix $\widetilde{s}>0$. Let $\widetilde{z}\in\mathcal{D}(z,r)$. Then

$$\int_{\mathcal{D}(z,r)}\frac{(1-|w|^2)^{\widetilde{s}q}(1-|\widetilde{z}|)^a(1-|w|^2)^b}{|1-\langle w,\widetilde{z}\rangle|^{n+1+a+b}}dv(w)\le C_1\left(1-|\widetilde{z}|^2\right)^{\widetilde{s}q},\quad \text{see [13].}$$

Let $w\in\mathcal{D}(z,r)$. Then

$$\int_{\mathcal{D}(z,r)}\frac{(1-|\widetilde{z}|^2)^{\widetilde{s}p}(1-|\widetilde{z}|)^a(1-|w|^2)^b}{|1-\langle \widetilde{z},w\rangle|^{n+1+a+b}}dv(\widetilde{z})\le C_2\left(1-|w|^2\right)^{\widetilde{s}p},\quad \text{see [13].}$$

Hence using Theorem 2.9 of [13] (Schur's test) we get

$$\int_{\mathcal{D}(z,r)}|S_r^{a,b}f(z)|^p dv_t(z))\le C\int_{\mathcal{D}(z,r)}|f(z)|^p dv_t(z). \tag{2.10}$$

The rest is clear. For $p=1$ the result following from Fubinis theorem. The proposition is proved. □

It is easy to see that we can replace in the statement of proposition the ball $\mathcal{D}(z,r)$ by r-lattice balls $\mathcal{D}(a_k,r)$, see [13], Chapter 2, replacing integration by ball by $\sum$.

3. New estimates and embeddings for holomorphic spaces in the unit ball defined with the help of Luzin area operator and Bergman metric ball

One if the main intention of this section is to find necessary and sufficient conditions on positive Borel measure μ in the unit ball such that

$$\left(\int_{\mathbf{B}^n}|f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\le C\left(\int_{\mathbf{B}^n}\left(\int_{\mathcal{D}(z,r)}|f(z)|^s dv_\alpha(z)\right)^{\frac{t}{s}}dv_\beta(z)\right)^{\frac{1}{t}},\ f\in H(\mathbf{B}^n), \tag{3.1}$$

with some restriction on $q,\ s,\ t,\ \alpha,\ \beta$ and sharp conditions on measure μ such that

$$\int_{\mathbf{S}^n}\left(\int_{\Gamma_\delta(\xi)}|f(z)|^s\frac{d\mu}{(1-|z|)^n}\right)^{\frac{t}{s}}d\sigma(\xi)\le C\left(\int_{\mathbf{B}^n}|f(z)|^p dv(z)\right)^{\frac{1}{p}},\ f\in H(\mathbf{B}^n), \tag{3.2}$$

with some restriction on parameters.

The classical Hardy's maximal theorem in the unit disk states that (see [8], [9], [10], [13])

$$\int_T \sup_{z\in\Gamma_t(\xi)}|f(z)|^p dm(\xi)\le C\|f\|^p_{H^p},\quad 0<p<\infty, \tag{3.3}$$

where T is a unit circle in the complex plane $\mathbb{C}$.

This estimate has many applications in complex analysis. Various generalizations of this estimate are also well known (see [8], [9], [10], [13])

The following general maximal theorem is based on Lemma 2.1 and gives estimates like (3.2) in the limit case $s=\infty$.

Theorem 3.1. (1) *Let* $\alpha > -(n+1)$, $\beta \in \mathbb{R}$, $0 < p < \infty$, $\alpha > \beta$, $f \in H(\mathbf{B})$. *Then*

$$\left\| \left(\sup_{w \in \Gamma_\sigma(\xi)} |D^\beta f(w)|^p (1-|w|)^{\alpha+n+1} \right)^{\frac{1}{p}} \right\|_{X(\mathbf{S})} \leq C \left\| \left(\int_{\widetilde{\Gamma}_{\sigma_t}(\xi)} |f(z)|^p (1-|z|)^{\alpha-\beta} dv(z) \right)^{\frac{1}{p}} \right\|_{X(\mathbf{S})}$$

where $X(\mathbf{S})$ *is such a functional space on* $\mathbf{S}$ *such that* $|g(\xi)| \leq C|g_1(\xi)|$, $\xi \in \mathbf{S}$ *gives* $\|g\|_{X(\mathbf{S})} \leq \|g_1\|_{X(\mathbf{S})}$, *for any two functions* g *and* g_1 *in* $X(\mathbf{S})$.

(2) *Let* μ *be positive Borel measure on* $\mathbf{B}$ *and let* $\mu = g dv(z)$ *and*

$$f \in \widetilde{A}^p_{\widetilde{\alpha}}(\mathbf{B}) = \{f \in H(\mathbf{B}) : \int_{\mathbf{S}} \left(\int_{\widetilde{\Gamma}_{\sigma_t}(\xi)} |f(z)|^p (1-|z|)^{\widetilde{\alpha}} dv(z) \right) d\xi < \infty\},$$

$\widetilde{\alpha} > -(n+1)$, $0 < p < \infty$. *Then for every* $\beta > 0$, $\beta = n+1+\widetilde{\alpha}$,

$$\int_{\mathbf{B}} |f(z)|^p (1-|z|)^\beta d\mu(z) \leq C \|\mu\|_{\mathcal{C}} \|f\|^p_{\widetilde{A}^p_{\widetilde{\alpha}}(\mathbf{B})}, \tag{3.4}$$

where $\|\mu\|_{\mathcal{C}}$ *is a known Carleson norm of* μ.

Proof. Proof of Theorem 3.1 based on Lemma 2.1.
(1) From Lemma 2.1 and [13], Chapter 2, we have the following estimates

$$|D^\beta f(w)|^p (1-|w|)^{\alpha+n+1} \leq C \int_{\mathcal{D}(w,r)} |f(\widetilde{w})|^p (1-|\widetilde{w}|)^{\alpha-\beta} dv(\widetilde{w}),$$

$0 < p < \infty$, $w \in \Gamma_\sigma(\xi)$, $\alpha > \max\{-(n+1), \beta-(n+1)\}$. It remains to use the fact that $\mathcal{D}(w,r) \subset \widetilde{\Gamma}_{\sigma_t}(\xi)$ by Lemma 2.1 and property of $X(\mathbf{S})$ class.
(2) Since (see [8])

$$G^\beta_p(f,\mu) = \int_{\mathbf{B}} |\widetilde{f}(z)|^p d\mu(z) \leq C \|\mu\|_{\mathcal{C}} \int_{\mathbf{S}} \sup_{z \in \Gamma_\sigma(\xi)} |\widetilde{f}(z)|^p d\sigma(\xi),$$

$\widetilde{f} = f(1-|z|)^\beta$, $f \in H(\mathbf{B})$, $0 < p < \infty$, we have by first part of the theorem

$$G^\beta_p(f,\mu) \leq C \|\mu\|_{\mathcal{C}} \|f\|^p_{\widetilde{A}^p_{\widetilde{\alpha}}(\mathbf{B})}.$$

□

Remark 3.2. Similar arguments with Carleson tubes based on first part of Lemma 2.1 can be applied to get assertions like in Theorem 3.1 but for Carlesom tubes.

Remark 3.3. Note when formally $n = 1$, $\beta \to 0$ and $\widetilde{\alpha} \to -2$ estimate (3.4) coincide with classical Carleson embedding theorem for $\mu = g dv(z)$.

Remark 3.4. For $X = L^p(\mathbf{S})$, $n = 1$, $\beta = 0$ and $\alpha \to -2$ the first estimate is a classical maximal theorem for Hardy classes in the unit disk (it obviously coincide with (3.3)).

Theorem 3.5. *Assume $1 \le p < \infty$ and μ is a nonnegative Borel measure on* **B**. *Then*

$$\int_{\mathbf{S}} \left(\int_{\Gamma_\gamma(\xi)} \frac{|f| d\mu(z)}{(1-|z|)^n} \right)^p d\sigma(\xi) \le C\|f\|^p_{H^p} \tag{3.5}$$

if and only if μ is a Carleson measure.

Proof. For $p = 1$ our theorem is a classical Carleson embedding theorem for the unit ball **B** (see [13]).

Let now $p > 1$. Suppose that the estimate (3.5) is true.
Let m is a big enough natural number. For $\xi \in \mathbf{S}$ and $0 < r < 1$, let

$$f(z) = |1 - \langle z, w \rangle|^{\frac{-mn}{p}}, \quad w = (1-r)\xi.$$

Then using this test function by standard arguments we get (see [13])

$$\|f\|^p_{H^p} \asymp r^{-(m-1)n}.$$

By Jensen's inequality and known estimates for the Poisson kernel on Carleson tubes (see [13], page 163)

$$\int_{\mathbf{S}} \left(\int_{Q_r(\xi) \cap \Gamma_\gamma(\widetilde{\xi})} \frac{d\mu(z) d\sigma}{r^n (1-|z|)^n} \right) \le C.$$

Hence by Fubinis theorem, and known property of admissible approach region and obvious multiplicative property of products characteristic functions we have

$$\frac{1}{r^n} \int_{\mathbf{B}} \chi_{Q_r(\xi)}(z) \frac{d\mu(z)}{(1-|z|)^n} \int_{\mathbf{S}} \chi_{\Gamma_\gamma(\widetilde{\xi})}(z) d\sigma \le C,$$

$$\frac{\mu(Q_r(\xi))}{r^n} \le C.$$

Now we repeat these arguments with $w = (1-r^2)(\xi)$ and hence we get μ is a Carleson measure in the unit ball.

We now prove the reverse. By duality it is enough to show

$$\begin{aligned} &\int_{\mathbf{S}} g(z) \int_{\Gamma_\sigma(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} d\sigma(\xi) = \\ &\int_{\mathbf{B}} |f(z)| \left(\frac{1}{(1-|z|)^n} \int_{\mathbf{S}} \chi_{\Gamma_\sigma(\xi)}(z) g(z) d\sigma(\xi) \right) d\mu(z) = \\ &\int_{\mathbf{B}} |f(z)||Kg(z)| d\mu(z) \le C\|f\|_{H^p}, \end{aligned}$$

where $g \in L^q(\mathbf{S})$, $\frac{1}{p} + \frac{1}{q} = 1, \quad q > 1$.

Since $|Kg(z)| \le CP[g](z)$ where P is Poisson integral of g (see [13]) we have by using standard estimates of Poisson integral and Hölder's inequality

$$M \le C\|f\|_{H^p}\|g\|_{L^q}.$$

The theorem is proved. □

The proof of Theorem 3.5 shows that Theorem 3.5 is actually true for any region $X(\xi)$, $\xi \in \mathbf{S}$, $X(\xi)$ in $\mathbf{B}$, instead of $\Gamma_\gamma(\xi)$ in (3.5) with two conditions on the growth of characteristic function of $X(\xi)$.

Remark 3.6. For $n = 1$ this theorem was obtained by W. Cohn in [5].

Remark 3.7. Theorem 3.5 can be expanded to mixed norm spaces

$$F_s^{pq} = \{f \in H(\mathbf{B}) : \int_{\mathbf{S}} \left(\int_0^1 |D^k f(z)|^q (1-|z|)^{(k-s)q-1} d|z| \right)^{\frac{p}{q}} d\sigma(\xi) < \infty,$$
$$0 < p, q < \infty,\ s \in \mathbb{R}\}.$$

$$B_s^{pq} = \{f \in H(\mathbf{B}) : \int_0^1 \left(\int_{\mathbf{S}} |D^k f(z)|^q (1-|z|)^{(k-s)q-1} d\sigma(\xi) \right)^{\frac{p}{q}} d|z| < \infty,$$
$$0 < p, q < \infty,\ s \in \mathbb{R}\}.$$

These classes were studied in [8] and [9]. Note $F_0^{p,2} = H^p$, $0 < p < \infty$ (see [8]).

We give only the expansion of Theorem 3.5 for F_s^{pq} classes.

Theorem 3.8. *Let* $1 \le p < \infty$ *and* $0 < q \le 2$. *Then*

$$\int_{\mathbf{S}} \left(\int_{\Gamma_\sigma(\xi)} \frac{|f(z)| d\mu(z)}{(1-|z|)^n} \right)^p d\sigma(\xi) \le C \|f\|_{F_0^{pq}}^p$$

if and only if μ *is a Carleson measure.*

The proof needs small modification of the above arguments and is based on some embeddings obtained in [8].

Let as now look again at $\int_X |f(w)|^p dv_\alpha(w)$, $\alpha > -1$, $0 < p < \infty$. X is a subset of $\mathbf{B}^n$, $f \in H(\mathbf{B}^n)$, and $\{a_k\}$ is an r-lattice in the Bergman metric (or sampling sequence) with the following known properties.

Lemma A. [13] There exists a positive integer N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}^n$ with the following properties:
(1) $\mathbf{B}^n = \bigcup_k \mathcal{D}(a_k, r)$;
(2) The sets $\mathcal{D}(a_k, \frac{r}{4})$ are mutually disjoint;
(3) Each point $z \in \mathbf{B}^n$ belongs to at most N of the sets $\mathcal{D}(a_k, 4r)$

Lemma B. [13] For each $r > 0$ there exists a positive constant C_r such that

$$C_r^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_r, \qquad C_r^{-1} \le \frac{1-|a|^2}{|1-\langle z, a\rangle|} \le C_r,$$

for all a and z such that $\beta(a, z) < r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Using properties of sequence $\{a_k\}$ from Lemma A, we have

$$\begin{aligned}\|f\|_X &= \int_X |f(w)|^q dv_\alpha(w)\\ &= \int_{\mathbf{B}^n} |f(w)|^q \chi_X(w) dv_\alpha(w)\\ &= \sum_k \int_{\mathcal{D}(a_k,r)} |f(w)|^q \chi_X(w) dv_\alpha(w).\end{aligned}$$

So, if $X = \mathcal{D}(z,r);\ Q_r(\xi);\ \Gamma_\delta(\xi)$ and $z \in \mathbf{B},\ \xi \in \mathbf{S},\ \mathbf{r} > \mathbf{0}$ we can generalize $\|f\|_X$:

$$\|f\|_X = \sum_k \left(\int_{\mathcal{D}(a_k,r)} |f(w)|^q \chi_X(w) dv_\alpha(w)\right)^{\frac{p}{q}},\ 0 < p,q < \infty,\ \alpha > -1.$$

Various embedding theorems for spaces with quasinorm of the type

$$N_q^p(f, \mathbf{S}^n, \Gamma_t(\xi)) = \int_{\mathbf{S}^n} \left(\sum_{k\geq 0} \left(\int_{\Gamma_t(\xi)\cap\mathcal{D}(a_k,r)} \frac{|f(z)|}{(1-|z|)^n} d\mu(z)\right)^p\right)^{\frac{q}{p}} d\sigma(z)$$

or $N_q^p(f, \mathbf{B}^n, \mathcal{D}(z,r))$ or $N_q^p(f, \mathbf{B}^n, Q_r(\xi))$, where $0 < p,q < \infty$, can be obtained immediately from known embedding theorems by „cutting measure“ argument. First we consider already known embeddings. For example

$$\sum_{k\geq 0} \left(\int_{\mathcal{D}(a_k,r)} |f(z)| d\mu_1(z)\right)^p \leq C\|f\|_Y^p, \tag{3.6}$$

$$\int_{\mathbf{S}} \left(\int_{\Gamma_t(\xi)} \frac{|f(z)|}{(1-|z|)^n} d\mu_2(z)\right)^q d\sigma(z) \leq C\|f\|_Y^q, \tag{3.7}$$

where Y is a subspace of $H(\mathbf{B}^n)$, or $Y = H^p$ or A_α^p and then cut the measure putting

$$\mu_2(z) = \sum_{k\in M} \chi_{\mathcal{D}(a_k,r)}\mu(z),$$

where $M = \{k|\Gamma_t(\xi)\cap\mathcal{D}(a_k,r) \neq 0\}$, μ is an positive Borel measure on $\mathbf{B}^n$, or $\mu_1(z) = \chi_{\Gamma_t(\xi)}(z)\mu(z)$, for some positive Borel measure μ.

Finally we give sharp embedding theorem for classes based on Bergman metric ball.

Theorem 3.9. *Let* $q, s, r, q_1 \in (0,\infty)$, $q_1 \leq q$, $\alpha > -1$. *Let* $\{a_k\}$ *be a sampling sequence defined above.Let* μ *be positive Borel measure on* $\mathbf{B}$. *Then*

$$\left(\int_{\mathbf{B}} |f(z)|^q d\mu(z)\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{B}} \left(\int_{\mathcal{D}(w,r)} |f(z)|^s dv_\alpha(z)\right)^{\frac{q_1}{s}} dv(w)\right)^{\frac{1}{q_1}}$$

if and only if

$$\mu\left(\mathcal{D}(a_k,r)\right) \leq C(1-|a_k|)^{q(\frac{n+1+\alpha}{s}+\frac{n+1}{q_1})}$$

for some constant $C > 0$.

Proof. Sufficiency. We use systematically Lemma A and B. Obviously, we have the following estimates

$$\left(\int_{\mathbf{B}} |f(z)|^q d\mu(z)\right)^{\frac{q_1}{q}} \leq C \sum_{k=1}^{\infty} \left(\max_{\mathcal{D}_{a_k,r}} |f(z)|^{q_1}\right) (1-|a_k|)^{q_1(\frac{n+1+\alpha}{s}+\frac{n+1}{q_1})}.$$

Since

$$|f(z)|^{\widetilde{s}} \leq C \left(\int_{\mathcal{D}(z,r)} \frac{|f(\widetilde{w})|^s dv_\alpha(\widetilde{w})}{(1-|z|)^{n+1+\alpha}}\right)^{\frac{\widetilde{s}}{s}}, \ z \in \mathbf{B}.$$

(see [13], Chapter 2) we get from above and Lemma A and B

$$\left(\int_{\mathbf{B}} |f(z)|^q d\mu(z)\right)^{\frac{q_1}{q}} \leq C \sum_{k=1}^{\infty} \int_{\mathcal{D}(a_k,r)} \left(\int_{\mathcal{D}(z,r)} |f(\widetilde{w})|^s dv_\alpha(\widetilde{w})\right)^{\frac{q_1}{s}} dv(z),$$

so we get what we need.

To get the converse we use standard test function (see [13])

$$f_k(z) = \frac{1}{(1-\langle z, a_k\rangle)^\beta}, \quad a_k \in \mathbf{B}, \quad z \in \mathbf{B},$$

where a_k is a r-lattice, β is large enough positive number, and Lemma A and B and standard arguments for estimation of test functions. □

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