J. Nonlinear Sci. Appl. 2 (2009), no. 3, 174–179

The Journal of Nonlinear Sciences and Applications
http://www.tjnsa.com

SOME PROPERTIES OF $L_{p,w}(0 < p \leq 1)$

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Communicated by R. Shamoyan

Abstract. In this article we explain some properties of $L_{p,w}$ when $0 < p \leq 1$ and w is weight. These properties are general and we derive them from L_p spaces.

1. Introduction and preliminaries

The concept of coorbit spaces theory was originally developed by Feichtinger and Gröchenig [5,6,7] in the late 1980's with the aim to provide a unified and group theoretical approach to function spaces and their atomic decompositions. After that S. Dahlke, G. Steidl and G. Teschke have studied coorbit spaces in [2,3,4]. We should know about $L_{p,w}$ for concept of coorbit spaces. Then if we introduce some properties of $L_{p,w}$ so it will be useful for coorbit spaces. Really the idea of this article has made when we was researching about coorbit spaces.

Definition 1.1. Let G be a separable, locally compact, topological Hausdorff group, then $X = \frac{G}{P}$ is a homogeneous space, where P is a closed subgroup of G.

Definition 1.2. Let G be a separable, locally compact, topological Hausdorff group with right Haar measure v. A unitary representation of G in a Hilbert space H is defined as a mapping U of G into the space of unitary operators on H such that $\mathrm{U}(gog') = \mathrm{U}(g)\mathrm{U}(g')$ for all $g, g\prime \in G$ and U(e) = Id which e is identity element in G.

Definition 1.3. If the right and the left Haar measure coincide, Simply it is called Haar measure, and G is said to be unimodular.

Date: Received: 10 March 2009, Revised: 20 May 2009.

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2000 *Mathematics Subject Classification.* 57S25, 42C15, 46E15, 28E99.

Key words and phrases. Coorbit space, homogeneous, invariant, representation theory, square integrable, unimodular, unitary.

Definition 1.4. A unitary representation U is called irreducible, if the only closed subspaces of H which are invariant under all operators $U(g)$ $(g \in G\ )$ are $\{0\}$ and H.

Definition 1.5. The representation U is said to be square integrable, if there exists a nonzero vector $\psi \in H$ which fulfills the admissibility condition

$$\int_G |<\psi, U(g)\psi>_H|^2 dv(g) < \infty$$

Let X is a homogeneous space. Because U is not directly defined on X, it is necessary to embed X in G. This can be realized by using the canonical fiber bundle structure of G with projection $\Pi : G \to X$. Let $\sigma : X \to G$ be a Borel section of this fiber bundle[3], i.e., $\Pi o\sigma(h) = h$ for all $h \in H$. In this article, we always assume that X is homogeneous space, and carries a $G-invariant$ measure μ, i.e., a measure invariant under the action $h \to hg (h \in X, g \in G)$ and $<.,.>$ always denotes the L_2-inner product

$$<F, K> = \int_X F(x)\overline{K(x)} d\mu(x)$$

whenever integral is defined.

2. Main results

Proposition 2.1. *Let G be a locally compact group with left Haar measure μ, and assume that Π is a square integrable representation of G on H, then there exists a unique positive self-adjoint operator U with domain $D(U)$, such that*

1. $V_g(g) \in L^2(G) \Leftrightarrow g \in D(U)$
2. *For all $g_1, g_2 \in D(U)$ and $f_1, f_2 \in H$*

$$\int_G \overline{<f_1, \Pi(x)g_1>} <f_2, \Pi(x)g_2> d\mu(x) = <Ug_1, Ug_2><f_2, f_1>$$

$D(U)$ is dense in H. If G is unimodular, then $D(U) = H$ and U is a multiple of the identity on H.

Proof. We refer the readers to[1, Theorem 17.1.4]. □

Definition 2.2. An irreducible, unitary representation U of G on H is called square integrable mod (P, σ), if there exists $\psi \in H$ such that the integral

$$\int_X <f, U(\sigma(h)^{-1}\psi)>_H U(\sigma(h)^{-1}\psi) d\mu(h)$$

converges weakly to a positive, bounded operator A_σ (dependent on σ and ψ) which has a bounded inverse A_σ^{-1}, in the sense that

$$<A_\sigma f, g>_H = \int_X <f, U(\sigma(h)^{-1}\psi)>_H \overline{<g, U(\sigma(h)^{-1}\psi)>_H} d\mu(h).$$

Definition 2.3. If $A_\sigma = \lambda Id$ for some $\lambda > 0$, then we call U strictly square integrable mod (P, σ) and (ψ, σ) is a strictly admissible pair.

Theorem 2.4. *Let $X = \frac{G}{P}$ be a homogeneous space and unimodular. If Π is square integrable mod (P, σ) which σ is a section from X to G, then Π is strictly square integrable.*

Proof. By use of proposition 2.1, there exists positive , self-adjoint operator U which $D(U) = H$. Suppose that $f_2 = f$, $f_1 = g$, and, $g_1 = g_2 = \psi \neq 0$. We have

$$\int_X \overline{< g, \Pi(h)\psi) >} < f, \Pi(h)\psi) > d\mu(h) =< U\psi, U\psi) >< f, g >$$

Since Π is square integrable mod (P, σ) then there exists A_σ that is invertible, bounded with bounded inverse A_σ^{-1}. Hence, for all $g \in H$ we have

$$< A_\sigma f, g >= \int_X < f, \Pi(h)\psi) > \overline{< g, \Pi(h)\psi) >} d\mu(h) =< U\psi, U\psi) >< f, g >$$

$$\Rightarrow < A_\sigma f, g >= \|U\psi\|^2 < f, g >$$
$$\Rightarrow < A_\sigma f, g >=< \|U\psi\|^2 f, g >$$

Because g is an arbitrary element in H so we have $A_\sigma = \|U\psi\|^2 Id$. If $\lambda = \|U\psi\|^2$, then Π is strictly square integrable. □

Definition 2.5. If w be positive, continuous function on G and for all $g \in G$, $0 < w(g) \leq 1$ so we say that w is weight function on G.

Definition 2.6. Let X be homogeneous space, Similar to [3] we introduce weighted $L_p - spaces$ on X for $0 < p \leq 1$ by

$$L_{p,w} = \{f \ measurable \ onX : \|f\|_{L_{p,w}} = (\int_X |f(h)|^p w^p(\sigma(h)) d\mu(h))^{\frac{1}{p}} < \infty\}.$$

Theorem 2.7. *Suppose that X is homogeneous and topological vector space, then $L_{p,w}(X)$ for $0 < p < 1$ is a locally bounded F-space.*

Proof. In the beginning we define

$$\Delta(f) = \int_X |f(h)|^p w^p(\sigma(h)) d\mu(h)$$

For more details we refer the readers to [8]. □

Corollary 2.8. *Let X be measurable, homogeneous and topological vector space, $r > 0$ and $0 < p < 1$. We know that there exists n belong to natural numbers such that $n^{p-1}\Delta(f) < r$. If $X = \bigcup_{i=1}^n A_i$ such that A_i for $1 \leq i \leq n$ are mesurable sets and for $i \neq j$, $A_i \cap A_j = \emptyset$ and $\int_{A_i} |f(h)|^p w^p(\sigma(h)) d\mu(h) = \frac{\Delta f}{n}$, then $L_{p,w}(X)$ contains no convex open sets, other $\emptyset$ and $L_{p,w}(X)$.*

Proof. Suppose $V \neq \emptyset$ is open and convex in $L_{p,w}$. Assume $0 \in V$, without loss of generality. Then $B_r \subset V$, for some $r > 0$. Define $g_i(h) = nf(h)$ if $h \in A_i$, $g_i(h) = 0$ otherwise. Then by use of hypothesis $\Delta(g_i) = n^{p-1}\Delta(f) < r$ for $1 \leq i \leq n$ we have $g_i \in V$. Since V is convex and $f = \frac{1}{n}(g_1 + ... + g_n)$ is follows that $f \in V$. Hence $V = L_{p,w}$. □

Corollary 2.9. *Suppose that hypothesis in previous corollary are true, then* $(L_{p,w}(X))^* = 0$.

Proof. Suppose that $\Lambda : L_{p,w} \to Y$ is a continuous linear mapping of $L_{p,w}$ into some locally convex space Y. Let β be a convex local base for Y and $V \in \beta$, then $\Lambda^{-1}(V)$ is convex, open and not empty. Hence by use of previous corollary $\Lambda^{-1}(V) = L_{p,w}$. Consequently $\Lambda(L_{p,w}) \subset V$ for every $V \in \beta$ we conclude that $\Lambda f = 0$ for every $f \in L_{p,w}$. Thus 0 is the only continuous linear mapping of $L_{p,w}$ into any locally convex space Y. If Y be complex scalers then $(L_{p,w}(X))^* = 0$. □

Definition 2.10. Let U is strictly square integrable mod (P, σ) , then for $\psi \in H$ $V_\psi(f)$ and $H_{1,w}$ are defined in the following

$$V_\psi : H \to L_2(X)$$
$$V_\psi f(h) :=< f, U(\sigma(h)^{-1})\psi) >_H$$

$$H_{1,w} := \{f \in H : V_\psi f \in L_{1,w}(X)\}.$$

Corollary 2.11. *$H_{1,w}$ is dense in H.*

Proof. We refer readers to [3, Lemma 3.1]. □

Theorem 2.12. *Let $\Lambda : H_{1,w} \to L_{p,w}$ is continuous(relative to the topology that $H_{1,w}$ inherits from H)and linear, then Λ has a continuous linear extension $\widetilde{\Lambda}$ so that $\widetilde{\Lambda} : H \to L_{p,w}$.*

Proof. H is a Hilbert space, hence, it is topological vector space. Now suppose that V_n be balanced neighborhoods of 0 in H such that $V_n + V_n \subset V_{n-1}$. Now $\Lambda : H_{1,w} \to L_{p,w}$ is continuous, then by the use of continuity definition we know that Λ is continuous in 0. Hence for $\varepsilon = 2^{-n}$ there exists $V_n \cap H_{1,w}$ so that

$$\forall x \in V_n \cap H_{1,w} : d(\Lambda 0, \Lambda x) < \varepsilon = 2^{-n}$$

It is remarkable that $V_n \cap H_{1,w}$ is neighborhoods of 0 relative to the topology that $H_{1,w}$ inherits from H. By corollary 2.11, $H_{1,w}$ is dense in H. Then we have two choices for all $x \in H$

(1) $x \in H_{1,w}$: In this form for all n we define $x_n = x$ and

$$\widetilde{\Lambda}x = \lim_{n\to\infty} \Lambda x_n = \Lambda x$$

(2) x is a limit point for $H_{1,w}$. Then

$$(x + V_n) \cap H_{1,w} \neq \emptyset \Longrightarrow x_n \in (x + V_n) \cap H_{1,w}.$$

We intend to show $\{\Lambda x_n\}$ is Cauchy sequence in $L_{p,w}$. d is invariant so

$$d(0, \Lambda(x_n - x)) < 2^{-n} \Longrightarrow d(\Lambda x, \Lambda x_n) < 2^{-n}$$

Hence

$$d(\Lambda x_n, \Lambda x_m) \leq d(\Lambda x_n, \Lambda x) + d(\Lambda x, \Lambda x_m) \leq 2^{-n} + 2^{-m}$$

For $\varepsilon > 0$, there exists $k \in N$ such that $\frac{1}{2^k} < \varepsilon$. If $n, m \geq k$, then we have

$$2^n, 2^m \geq 2^k \Longrightarrow d(\Lambda x_n, \Lambda x_m) \leq \frac{1}{2^n} + \frac{1}{2^m} \leq 2\varepsilon.$$

We know that $L_{p,w}$ is F-space and d is invariant and complete, then limit for $\{\Lambda x_n\}$ exists. If we show this limit with $\widetilde{\Lambda} x$, it will be welldefined, linear and continuous. □

Theorem 2.13. *Let μ is positive, unimodular and σ-finite measure with σ-algebra Σ . If w be weight function such that $\int_X w(\sigma(h))d\mu(h) \leq 1$, then there will exist $S(x) \in L_{1,w}$ so that $0 < S(x) < 1$.*

Proof. μ is σ-finite, then there exists $\{E_i\}_{i \in I} \subseteq \Sigma$ such that $X = \cup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty$.
Now, we define $S_i(x) = \frac{2^{-i}}{(1+\mu(E_i))(1+w(\sigma(x)))}$ for $x \in E_i$ and $S_i(x) = 0$ otherwise. We define $S(x) = \sum_{n=1}^{\infty} S_n(x)$. Then

$$0 < S(x) = \sum_{n=1}^{\infty} S_n(x) \leq \sum_{n=1}^{\infty} \frac{2^{-n}}{(1+\mu(E_n))(1+w(\sigma(x)))} \leq 1$$

If $x \in X$ there exists $i \in I$ such that $x \in E_i$, then $S(x) > 0$. Furthermore, there exists n such that $\mu(E_n) \neq 0$ and

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{(1+\mu(E_n))(1+w(\sigma(x)))} < \sum_{n=1}^{\infty} 2^{-n} < 1$$

So, $0 < S < 1$. Now we should show $S \in L_{1,w}$. Hence

$$\int_X S(x)w(\sigma(x))d\mu(x) = \int_X (\sum_{n=1}^{\infty} S_n(x))w(\sigma(x))d\mu(x) =$$

$$\sum_{n=1}^{\infty} \int_X S_n(x)w(\sigma(x))d\mu(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \int_{E_i} S_i(x)w(\sigma(x))d\mu(x) =$$

$$\sum_{n=1}^{\infty} \int_{E_n} S_n(x)w(\sigma(x))d\mu(x) = \sum_{n=1}^{\infty} \int_{E_n} \frac{2^{-n}}{(1+\mu(E_n))} \frac{w(\sigma(x))}{(1+w(\sigma(x)))} d\mu(x) \leq$$

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{1+\mu(E_n)} \int_{E_n} w(\sigma(x))d\mu(x) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$

Then $S(x) \in L_{1,w}$. □

Acknowledgements: We would like to thank Dr M. S. Asgari in Science and Research Branch, Islamic Azad University and Dr R. Ahmadi in Tabriz University for their useful guidances and our freinds in Tabriz University. Thanks are also due to the anonymous referees, whose remarks led to a significant improvement of the quality of the manuscript.

References

1. O. Christensen, *An Introduction to Frame and Riesz Bases*, Birkhauser,Boston,2003.
2. S. Dahlke, G. Steidl and G. Teschke, Coorbit spaces and Banach frames on homogeneous spaces with applications to the sphere, Advances in Computational Mathematics 21 (2004) 147–180.
3. S. Dahlke, G. Steidl and G. Teschke, Weighted Coorbit Spaces and Banach Frames on Homogeneous Spaces, J. Fourier Anal. Appl. 10 (2004) 507–539.
4. S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl and G. Teschke, *Generalized Coorbit Theory, Banach Frames and the Relation to α-Modulation Spaces*, Bericht Nr. 2005-6, Philis-University of Marburg.
5. H.G. Feichtinger, K. Gröchenig, A unified approach to atomic decompositions via integrable group representation, In: Proc. Conference on Functions, Spaces and Applications, Lund, 1986, Springer Lect. Notes Math., 1302 (1988), 52–73.
6. H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representation and their atomic decompositions I, J. Funct. Anal., 86 (1989), 307-340.
7. H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representation and their atomic decompositions II, Monatsh. Math., 108 (1989), 129–148.
8. W. Rudin , *Functional Analysis*, Mc Graw Hill Book company, NewYork, 1991.

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