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POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract. Existence of positive solution for a class of singular boundary value problems of the type

$$\begin{aligned} -x''(t) &= f(t, x(t), x'(t)), \quad t \in (0, 1) \\ x(0) &= 0,\ x(1) = 0, \end{aligned}$$

is established. The nonlinearity $f \in C((0, 1) \times (0, \infty) \times (-\infty, \infty), (-\infty, \infty))$ is allowed to change sign and is singular at $t = 0$, $t = 1$ and/or $x = 0$. An example is included to show the applicability of our result.

1. Introduction and preliminaries

Singular boundary value problems arise in various fields of Mathematics and Physics such as nuclear physics, boundary layer theory, nonlinear optics, gas dynamics, etc, [1, 5, 9, 12, 14, 17, 18, 19]. For more details on singular BVPs and recent developments, we refer the readers to the recent monograph by R. P. Agarwal and D. O' Regan [4] and [6, 8, 9, 15].

In this paper, we consider a class of second order singular boundary value problems of the type

$$\begin{aligned} -x''(t) &= f(t, x(t), x'(t)),\ t \in (0, 1), \\ x(0) &= 0,\ x(1) = 0, \end{aligned} \tag{1.1}$$

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where $f \in C((0,1) \times (0,\infty) \times (-\infty,\infty), (-\infty,\infty))$ and may be singular at $t=0$, $t=1$ and/or $x=0$ and is allowed to change sign. We establish existence of positive solutions for the BVP (1.1) under a weaker hypothesis on f. Recently, existence of positive solutions for singular boundary value problems, in the case when the nonlinearity f is independent of the derivative term, has been studied by many authors, [3, 10, 11, 13, 21]. In these papers, the nonlinearity is assumed to be non-negative and either be sublinear or superlinear. Hence, the results of these papers would be applicable to a limited class of boundary values problems. Moreover, most of nonlinear problems from the applied sciences, the nonlinearity explicitly depends on the derivative term, for example, the differential equation

$$x''(t) = -\lambda\Big(\frac{1-t^2}{x(t)}\Big)' - \frac{t}{x(t)},\ t \in [0,1),$$

together with some suitable boundary conditions, that explicitly depends on the derivative, arises in the boundary layer theory in fluid mechanics [19]. Further, the nonlinearity f does not satisfy the sublinear and superlinear conditions in most cases and may change sign. Hence, the study of boundary value problems without the above mentioned restrictions is of great importance.

Recently, existence theory for positive solutions of two point boundary value problems without the first derivative term is studied in [16, 19, 20]. Inspired by the above papers, the aim of the present paper is to improve and generalize the result studied in [20] to the case when the nonlinearity f explicitly depends on the derivative term x'. We study the problem under much weaker hypothesis on f. We include an example to show the applicability of our result.

Throughout this paper, we assume that the following condition holds.
(A_1) there exist $k \in C((0,1),(0,\infty))$ and a decreasing $F \in C((0,\infty),(0,\infty))$ such that

$$\int_0^1 t(1-t)k(t)dt < \infty \text{ and } \int_0^\infty \frac{du}{F(u)} = \infty.$$

The condition $\int_0^\infty \frac{du}{F(u)} = \infty$ implies that we can choose $R > 1$ such that

$$\int_1^R \frac{du}{F(u)} > \max\left\{\int_0^{1/2} sk(s)ds, \int_{1/2}^1 (1-s)k(s)ds\right\}. \tag{1.2}$$

For fixed $n \in \{3,4,5,\dots\}$, let $M = \max\{F(\frac{1}{n})k(t) : t \in [\frac{1}{n}, 1-\frac{1}{n}]\}$ and choose $C > \sqrt{2MR}$.

For $u \in C[0,1]$ we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$ and for $u \in C^1[0,1]$, we write $\|u\|_1 = \max\{\|x\|, \frac{\gamma}{3}\|x'\|\}$ where $\gamma = \frac{n-2}{n}$ for $n \geq 3$. Clearly, $C^1[0,1]$ with the norm $\|.\|_1$ is a Banach space.

The only condition we are imposing on the nonlinearity f is the following:
(A_2)

$$0 \leq f(t,x(t),x'(t)) \leq k(t)F(x(t)) \text{ on } (0,1) \times (0,R] \times [-C,C].$$

For fixed $n \in \{3, 4, 5, ...\}$, consider the BVPs

$$\begin{aligned} -x''(t) &= f(t, x(t), x'(t)),\ t \in [\frac{1}{n}, 1-\frac{1}{n}], \\ x(\frac{1}{n}) &= \frac{1}{n},\ x(1-\frac{1}{n}) = \frac{1}{n}. \end{aligned} \tag{1.3}$$

We write (1.3) as an equivalent integral equation

$$x(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} G_n(t,s) f(s, x(s), x'(s)) ds,\ t \in [\frac{1}{n}, 1-\frac{1}{n}], \tag{1.4}$$

where

$$G_n(t,s) = \frac{n}{n-2} \begin{cases} (s-\frac{1}{n})(1-\frac{1}{n}-t), & \text{if } \frac{1}{n} \le s < t \le 1-\frac{1}{n} \\ (t-\frac{1}{n})(1-\frac{1}{n}-s), & \text{if } \frac{1}{n} \le t \le s \le 1-\frac{1}{n}, \end{cases}$$

is the Green's function for the corresponding homogeneous problem

$$\begin{aligned} -x''(t) &= 0,\ t \in [\frac{1}{n}, 1-\frac{1}{n}] \\ x(\frac{1}{n}) &= 0,\ x(1-\frac{1}{n}) = 0. \end{aligned} \tag{1.5}$$

Notice that $G_n(t,s) \ge 0$ on $(\frac{1}{n}, 1-\frac{1}{n}) \times (\frac{1}{n}, 1-\frac{1}{n})$ and $G_n(t,s) \le G_n(s,s)$, $t \in (0,1)$. Moreover,

$$\max_{t\in[0,1]} \int_{1/n}^{1-1/n} G_n(t,s) ds = \int_{1/n}^{1-1/n} G_n(s,s) ds = \frac{\gamma^2}{6},$$

$$\max_{t\in[0,1]} |\int_{1/n}^{1-1/n} \frac{\partial G_n}{\partial t}(t,s) ds| = \int_{1/n}^{1-1/n} |\frac{\partial G_n}{\partial t}(t,s)| ds = \frac{\gamma}{2}.$$

2. Main results

Theorem 2.1. *Assume that (A_1) and (A_2) hold. Then boundary value problem (1.3) has a solution $x \in C^1[0,1]$ such that $\frac{1}{n} \le x(t) < R$ and $|x'(t)| < C$ for $t \in [\frac{1}{n}, 1-\frac{1}{n}]$.*

Proof. Define retractions $q : (-\infty, \infty) \to [-C, C]$ by $q(v) = \max\{-C, \min\{v, C\}\}$ and $p : (-\infty, \infty) \to [\frac{1}{n}, R]$ by $p_n(x(t)) = \max\{\frac{1}{n}, \min\{x(t), R\}\}$. Clearly, q, p are continuous and $q(v) = v$ for $|v| \le C$, $p(v) = v$ for $\frac{1}{n} \le v \le R$.

Consider the modified BVP

$$\begin{aligned} -x''(t) &= F_n(t, x(t), x'(t)),\ t \in [\frac{1}{n}, 1-\frac{1}{n}], \\ x(\frac{1}{n}) &= \frac{1}{n},\ x(1-\frac{1}{n}) = \frac{1}{n}, \end{aligned} \tag{2.1}$$

where $F_n(t, x(t), x'(t)) = f(t, p_n(x(t)), q(x'(t)))$. Clearly, F_n is continuous, bounded and nonnegative on $[\frac{1}{n}, 1-\frac{1}{n}] \times \mathbb{R} \times \mathbb{R}$. Further, any solution $x \in C^1[0,1]$ of (2.1) such that

$$\frac{1}{n} \le x(t) < R,\ |x'(t)| < C,\ t \in [\frac{1}{n}, 1-\frac{1}{n}], \tag{2.2}$$

is a solution of (1.3). Obviously, $x(t) \ge \frac{1}{n}$ on $[\frac{1}{n}, 1-\frac{1}{n}]$ as $F_n \ge 0$.

We write the BVP (2.1) as an equivalent integral equation of the type

$$x(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} G_n(t,s)F_n(s,x(s),x'(s))ds \tag{2.3}$$

and define an operator $T_n : C^1[0,1] \to C^1[0,1]$ by

$$(T_n x)(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} G_n(t,s)F_n(s,x(s),x'(s))ds. \tag{2.4}$$

By a solution of (2.1) we mean a solution of the operator equation $(I - T_n)(x) = 0$, that is, a fixed point of T_n. We show that T_n has a fixed point $x \in C^1[0,1]$. Clearly, T_n is continuous and completely continuous as F_n is continuous and bounded.

Choose $\bar{R} > \max\{R, \frac{1}{3} + \frac{M_1\gamma^2}{6}\}$, where

$$M_1 = \max\{f(t,x,x') : t \in [\frac{1}{n}, 1 - \frac{1}{n}],\ x \in [\frac{1}{n}, R],\ x \in [-C, C]\}$$

and define an open, bounded and convex set

$$\Omega_{\bar{R}} = \{x \in C^1[0,1] : \|x\|_1 < \bar{R}\}.$$

For $x \in \overline{\Omega}_{\bar{R}}$, we have

$$\begin{aligned}\|T_n x\| &\le \frac{1}{n} + \max_{t\in[0,1]} \left| \int_{1/n}^{1-1/n} G_n(t,s)F_n(s,x(s),x'(s))ds \right| \\ &\le \frac{1}{3} + M_1 \int_{1/n}^{1-1/n} |G_n(s,s)|ds = \frac{1}{3} + \frac{M_1\gamma^2}{6},\end{aligned}$$

$$\|(T_n x)'\| \le \max_{t\in[0,1]} \left| \int_{1/n}^{1-1/n} \frac{\partial G_n}{\partial t}(t,s)F_n(s,x(s),x'(s))ds \right| \le \frac{M_1\gamma}{2}.$$

It follows that

$$\|T_n x\|_1 = \max\{\|T_n x\|, \frac{\gamma}{3}\|(T_n x)'\|\} \le \frac{1}{3} + \frac{M_1\gamma^2}{6} < \bar{R} \text{ for every } x \in \overline{\Omega}_{\bar{R}}.$$

Hence, $T_n(\overline{\Omega}_{\bar{R}}) \subset \Omega_{\bar{R}}$. Consequently, by Schauder's fixed point theorem, the BVP (2.1) has a solution in $\Omega_{\bar{R}}$.

Now, we show that any solution x of (2.1) must satisfies (2.2). Firstly, we show that $x < R$ on $[\frac{1}{n}, 1 - \frac{1}{n}]$. Assume that this is not true and $x(t) \ge R$ for some $t \in [\frac{1}{n}, 1 - \frac{1}{n}]$. Let

$$\xi = \min\{t \in [\frac{1}{n}, 1 - \frac{1}{n}] : x(t) = R\}.$$

We discuss different cases:

Case 1: If $\xi \le 1/2$, since $x(\frac{1}{n}) = \frac{1}{n} < R$, there exist subintervals, say $[\xi_{2i-1}, \xi_{2i}] \subseteq [\frac{1}{n}, \xi]$, $i = 1,2,3,\cdots,m$ such that

(1): $\xi_1 = \frac{1}{n}$, $\xi_{2m} = \xi$, $\xi_{2i-1} < \xi_{2i}$ for $i = 1,2,3,\cdots,m$,

(2) $\xi_{2i} \le \xi_{2i+1}$, $x(\xi_{2i}) = x(\xi_{2i+1})$ and $x'(\xi_{2i}) = 0$ for $i = 1,2,3,\cdots,m-1$,

(3) $x'(t) \ge 0$ for $t \in [\xi_{2i-1}, \xi_{2i}]$, $i = 1,2,3,\cdots,m$.

For $t \in [\xi_{2i-1}, \xi_{2i}]$, $i = 1, 2, 3, \cdots, m$, using (A_2) and the fact that $x(t) \in [\frac{1}{n}, R]$, we have

$$-x''(t) = F_n(t, x(t), x'(t)) = f(t, p_n(x(t)), q(x'(t))) \le k(t)F(x(t)),\ t \in [\xi_{2i-1}, \xi_{2i}]. \tag{2.5}$$

Integrating (2.5) from t to ξ_{2i}, using (2) and the decreasing property of F, we obtain

$$x'(t) \le F(x(t)) \int_t^{\xi_{2i}} k(s)ds,\ t \in [\xi_{2i-1}, \xi_{2i}],\ i = 1, 2, 3, \cdots, m,$$

which implies that

$$\frac{x'(t)}{F(x(t))} \le \int_t^{\xi_{2i}} k(s)ds,\ t \in [\xi_{2i-1}, \xi_{2i}],\ i = 1, 2, 3, \cdots, m. \tag{2.6}$$

Integrating (2.6) from ξ_{2i-1} to ξ_{2i}, we have

$$\int_{\xi_{2i-1}}^{\xi_{2i}} \frac{x'(t)}{F(x(t))} dt \le \int_{\xi_{2i-1}}^{\xi_{2i}} \int_t^{\xi_{2i}} k(s)dsdt,$$

which can be written as

$$\int_{x(\xi_{2i-1})}^{x(\xi_{2i})} \frac{du}{F(u)} \le \int_{\xi_{2i-1}}^{\xi_{2i}} sk(s)ds,\ i = 1, 2, 3, \cdots, m. \tag{2.7}$$

Summing from $i = 1$ to m and using (2) $(x(\xi_{2i}) = x(\xi_{2i+1}))$, we obtain

$$\int_{1/n}^{R} \frac{du}{F(u)} \le \int_0^{1/2} sk(s)ds.$$

Letting $n \to \infty$, we have

$$\int_0^R \frac{du}{F(u)} \le \int_0^{1/2} sk(s)ds,$$

a contradiction to (1.2).

Case 2: Let $\xi \ge 1/2$ and $\eta = \max\{t \in [\frac{1}{2}, 1 - \frac{1}{n}] : x(t) = R\}$. Since $x(1 - \frac{1}{n}) = \frac{1}{n} < R$, there exist subintervals $[\eta_{2i}, \eta_{2i-1}] \subseteq [\frac{1}{2}, 1 - \frac{1}{n}]$, $i = 1, 2, 3, \cdots, m'$ such that

(4) $\eta_1 = 1 - \frac{1}{n}$, $\eta_{2m'} = \eta$, $\eta_{2i} < \eta_{2i-1}$ for $i = 1, 2, 3, \cdots, m'$,
(5) $\eta_{2i+1} \le \eta_{2i}$, $x(\eta_{2i}) = x(\eta_{2i+1})$, $x'(\eta_{2i}) = 0$ for $i = 1, 2, 3, \cdots, m' - 1$ and
(6) $x'(t) \le 0$ for $t \in [\eta_{2i}, \eta_{2i-1}]$, $i = 1, 2, 3, \cdots, m'$.

Integrating (2.5) from η_{2i} to t, using (5), the decreasing property of F and then integrating from η_{2i} to η_{2i-1}, we obtain

$$\int_{x(\eta_{2i-1})}^{x(\eta_{2i})} \frac{du}{F(u)} \le \int_{\eta_{2i}}^{\eta_{2i-1}} (1 - s)k(s)ds,\ i = 1, 2, 3, \cdots, m'. \tag{2.8}$$

Summing (2.8) from $i = 1$ to $i = m'$ and using (5) $(x(\eta_{2i}) = x(\eta_{2i+1}))$, we obtain

$$\int_{1/n}^{R} \frac{du}{F(u)} \le \int_{1/2}^{1} (1 - s)k(s)ds.$$

Letting limit $n \to \infty$, we get

$$\int_0^R \frac{du}{F(u)} \le \int_{1/2}^1 (1-s)k(s)ds$$

which is a contradiction to (1.2).

Now we show that any solution $x(t)$ of (2.1) must satisfies $|x'(t)| \le C$, $t \in [\frac{1}{n}, 1-\frac{1}{n}]$. From the boundary conditions, $x(\frac{1}{n}) = \frac{1}{n}$ and $x(1-\frac{1}{n}) = \frac{1}{n}$, it follows that there exist $p \in (\frac{1}{n}, 1-\frac{1}{n})$ such that $x'(p) = 0$. Suppose there exist $t_0 \in (\frac{1}{n}, 1-\frac{1}{n})$ such that $x'(t_0) > C$. As in the first part of this theorem, choose $[\xi_{2i-1}, \xi_{2i}] \subseteq [\frac{1}{n}, 1-\frac{1}{n}]$ such that

$$x'(t) \ge 0 \text{ on } [\xi_{2i-1}, \xi_{2i}] \text{ and } x'(\xi_{2i}) = 0,\ i = 1, 2, 3, \cdots.$$

Hence, there exist some i_0 such that $t_0 \in [\xi_{2i_0-1}, \xi_{2i_0}]$. Let

$$C_1 = \max\{x'(t) : t \in [\xi_{2i_0-1}, \xi_{2i_0}]\} = x'(\xi^*).$$

Clearly, $C_1 \ge C$ and in view of (A_2), we have

$$-x''(t) = f(t, x(t), q(x'(t))) \le k(t)F(x(t)) \le M.$$

Hence,

$$-x'(t)x''(t) \le Mx'(t),\ t \in [\xi_{2i_0-1}, \xi_{2i_0}].$$

Integrating from ξ^* to ξ_{2i_0}, we have

$$-\int_{\xi^*}^{\xi_{2i_0}} x'(t)x''(t)dt \le M \int_{\xi^*}^{\xi_{2i_0}} x'(t)dt,$$

implies that

$$\int_0^{C_1} vdv \le MR \Rightarrow C_1 \le \sqrt{2MR},$$

which contradict the definition of C. Hence, $x'(t) \le C,\ t \in [\frac{1}{n}, 1-\frac{1}{n}]$.

Similarly, we can show that $x'(t) \ge -C,\ t \in [\frac{1}{n}, 1-\frac{1}{n}]$. □

Theorem 2.2. *Assume that* (A_1) *and* (A_2) *hold. Then, the boundary value problem* (1.1) *has a positive solution* x.

Proof. By Theorem 2.1, any solution x_n of (1.3) satisfies

$$\frac{1}{n} \le x_n(t) \le R,\ |x_n'(t)| \le C \text{ for } t \in [\frac{1}{n}, 1-\frac{1}{n}],\ n = 3, 4, 5, ...$$

Hence, for each $h \in (0, 1/2)$, there exist a natural number $m \in \{3, 4, 5, \cdots\}$ such that $x_n(t) > 0$ for all $t \in [h, 1-h]$ and $n \ge m$.

Consider the integral equation,

$$\begin{aligned} x_n(t) = &\frac{x_n(1-h) - x_n(h)}{1-2h}(t-h) + x_n(h) + \int_h^t \frac{(s-h)(1-h-t)}{1-2h} f(s, x_n(s), x_n'(s))ds \\ &+ \int_t^{1-h} \frac{(t-h)(1-h-s)}{1-2h} f(s, x_n(s), x_n'(s))ds,\ t \in [h, 1-h]. \end{aligned} \tag{2.9}$$

Differentiating with respect to t, we obtain

$$\begin{aligned} x_n'(t) =&\frac{x_n(1-h)-x_n(h)}{1-2h} - \frac{1}{1-2h}\int_h^t (s-h)f(s,x_n(s),x_n'(s))ds \\ &+ \frac{1}{1-2h}\int_t^{1-h}(1-h-s)f(s,x_n(s),x_n'(s))ds. \end{aligned} \tag{2.10}$$

For any t_1, $t_2 \in [h, 1-h]$, we have

$$\begin{aligned} |x_n'(t_2)-x_n'(t_1)| =&\Big|\frac{-1}{1-2h}\int_h^{t_2}(s-h)f(s,x_n(s),x_n'(s))ds \\ &+ \frac{1}{1-2h}\int_{t_2}^{1-h}(1-h-s)f(s,x_n(s),x_n'(s))ds \\ &+ \frac{1}{1-2h}\int_h^{t_1}(s-h)f(s,x_n(s),x_n'(s))ds \\ &- \frac{1}{1-2h}\int_{t_1}^{1-h}(1-h-s)f(s,x_n(s),x_n'(s))ds\Big| \\ =&\frac{1}{1-2h}\Big|\int_{t_1}^{t_2}(s-h)f(s,x_n(s),x_n'(s))ds + \int_{t_1}^{t_2}(1-h-s)f(s,x_n(s),x_n'(s))ds\Big| \\ \le& L|t_2-t_1|, \end{aligned}$$

where $L = \max\{f(t,u,v) : (t,u,v) \in [h, 1-h] \times [\frac{1}{n}, R] \times [-C, C]\}$. Thus the sequences $\{x_n\}$ and $\{x_n'\}$ are uniformly bounded and equicontinuous. By Arzelà-Ascoli theorem, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging uniformly on $[h, 1-h]$ such that

$$\begin{aligned} \lim_{n_k\to\infty} x_{n_k}(t) = x(t), \\ \lim_{n_k\to\infty} x_{n_k}'(t) = x'(t), \end{aligned}$$

where $x \in C^1[0,1]$. Taking $\lim_{h\to 0}$, we have

$$\lim_{n_k\to\infty} x_{n_k}(t) = x(t) \text{ on } (0,1).$$

Further, $x > 0$ on $(0,1)$. Letting $\lim_{n_k\to\infty}$, (2.12) and (2.10) yield

$$\begin{aligned} x(t) =&\frac{x(1-h)-x(h)}{1-2h}(t-h) + x(h) + \frac{1}{1-2h}\int_h^t (s-h)(1-h-t)f(s,x(s),x'(s))ds \\ &+ \frac{1}{1-2h}\int_t^{1-h}(t-h)(1-h-s)f(s,x(s),x'(s))ds, \end{aligned}$$

$$\begin{aligned} x'(t) = \frac{x(1-h)-x(h)}{1-2h} &- \frac{1}{1-2h}\int_h^t (s-h)f(s,x(s),x'(s))ds \\ &+\frac{1}{1-2h}\int_t^{1-h}(1-h-s)f(s,x(s),x'(s))ds. \end{aligned}$$

Hence,

$$x''(t) = -\frac{1}{1-2h}(t-h)f(t,x(t),x'(t)) - \frac{1}{1-2h}(1-h-t)f(t,x(t),x'(t)) = -f(t,x(t),x'(t))$$

$\Rightarrow$

$$-x''(t) = f(t,x(t),x'(t)), t \in (0,1) \tag{2.11}$$

which implies that x satisfies the differential equation (1.1). Moreover,

$$x(0) = \lim_{n_k\to\infty} x\Big(\frac{1}{n_k}\Big) = \lim_{n_k\to\infty} x_{n_k}\Big(\frac{1}{n_k}\Big) = \lim_{n_k\to\infty} \frac{1}{n_k} = 0,$$

and

$$x(1) = \lim_{n_k\to\infty} x\Big(1-\frac{1}{n_k}\Big) = \lim_{n_k\to\infty} x_{n_k}\Big(1-\frac{1}{n_k}\Big) = \lim_{n_k\to\infty} \frac{1}{n_k} = 0,$$

which implies that x also satisfies the boundary conditions and hence is a solution of (1.1). □

Example 2.3. Consider the boundary value problem

$$\begin{aligned} -x''(t) &= \frac{x'(t)+5}{t(1-t)(x(t))^2};\ t \in (0,1) \\ x(0) &= 0,\ x(1) = 0. \end{aligned} \tag{2.12}$$

Choose $k(t) = \frac{1}{t(1-t)}$ and $F(u) = \frac{9}{u^2}$. Clearly F is decreasing and $\int_0^\infty \frac{du}{F(u)} = \infty$. Since

$$\int_0^{\frac{1}{2}} tk(t)dt = \ln 2,\ \int_{\frac{1}{2}}^1 (1-t)k(t)dt = \ln 2.$$

Hence, from the relation $\int_1^R \frac{du}{F(u)} > \ln 2$, we have $R > (1+27\ln 2)^3$. Also,

$$M = \max\{F(\frac{1}{n})k(t) : t \in [\frac{1}{n}, 1-\frac{1}{n}]\} = \max\{\frac{9n^2}{t(1-t)} : t \in [\frac{1}{n}, 1-\frac{1}{n}]\} = 9.$$

Hence $C = 3$. Moreover,

$$0 \le \frac{x'(t)+5}{t^\mu(1-t)^\nu(x(t))^2} = f(t,x(t),x'(t)) \le k(t)F(x(t)) \text{ for } x'(t) \in [-3,3], t \in (0,1).$$

By Theorem 2.2, the problem (2.12) has a solution x such that

$$0 < x(t) \le (1+27\ln 2)^3,\ |x'(t)| \le 3,\ t \in (0,1).$$

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