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INTUITIONISTIC FUZZY STABILITY OF JENSEN TYPE MAPPING

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Abstract. In this paper we prove result for Jensen type mapping in the setting of intuitionistic fuzzy normed spaces. We generalize a Hyers-Ulam stability result in the framework of classical normed spaces.

1. Introduction

In 1940 and in 1964 S.M. Ulam [13] proposed the famous Ulam stability problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" For very general functional equations, the concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable. In 1941 D.H. Hyers [6] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951 D.G. Bourgin [2]

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was the second author to treat the Ulam stability problem for additive mappings. In 1978 P.M. Gruber [4] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [14] used a stability property of the functional equation

$$f(x-y)+f(x+y)=2f(x) \tag{1.1}$$

to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. In 2003–2006 J.M. Rassias and M.J. Rassias [8, 9] and J.M. Rassias [7] solved the above Ulam problem for Jensen and Jensen type mappings. In this paper we consider the stability of Jensen type mapping in the setting of intuitionistic fuzzy normed spaces.

2. Preliminaries

In this section, using the idea of intuitionistic fuzzy metric spaces introduced by Park [10] and Saadati-Park [11, 12] we define the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t–representable (see [5]).

Lemma 2.1. ([3]) *Consider the set L^* and the order relation $\leq_{L^*}$ defined by:*

$$L^*=\{(x_1,x_2):(x_1,x_2)\in[0,1]^2 \text{ and } x_1+x_2\leq 1\},$$

$$(x_1,x_2)\leq_{L^*}(y_1,y_2)\Longleftrightarrow x_1\leq y_1,\ x_2\geq y_2,\quad \forall(x_1,x_2),(y_1,y_2)\in L^*.$$

Then $(L^*,\leq_{L^*})$ is a complete lattice.

Definition 2.2. ([1]) An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in a universal set U is an object $\mathcal{A}_{\zeta,\eta}=\{(\zeta_{\mathcal{A}}(u),\eta_{\mathcal{A}}(u))|u\in U\}$, where, for all $u\in U$, $\zeta_{\mathcal{A}}(u)\in[0,1]$ and $\eta_{\mathcal{A}}(u)\in[0,1]$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_{\mathcal{A}}(u)+\eta_{\mathcal{A}}(u)\leq 1$.

We denote its units by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. Classically, a *triangular norm* $*=T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T:[0,1]^2\longrightarrow[0,1]$ satisfying $T(1,x)=1*x=x$ for all $x\in[0,1]$. A *triangular conorm* $S=\diamond$ is defined as an increasing, commutative, associative mapping $S:[0,1]^2\longrightarrow[0,1]$ satisfying $S(0,x)=0\diamond x=x$ for all $x\in[0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 2.3. ([3]) A *triangular norm* (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);

(b) $(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);

(c) $(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);

(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then $\mathcal{T}$ is said to be a *continuous t–norm*.

Definition 2.4. ([3]) A continuous t–norm $\mathcal{T}$ on L^* is said to be *continuous t–representable* if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ are continuous t–representable.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \cdots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \cdots, x^{(n)}), x^{(n+1)}), \quad \forall n \geq 2,\ x^{(i)} \in L^*.$$

Definition 2.5. A *negator* on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an *involutive negator*. A *negator* on $[0, 1]$ is a decreasing mapping $N : [0, 1] \longrightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the *standard negator* on $[0, 1]$ defined by

$$N_s(x) = 1 - x, \quad \forall x \in [0, 1].$$

Definition 2.6. Let μ and ν be membership and non-membership degree of an intuitionistic fuzzy set from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed* *space* (briefly IFN-space) if X is a vector space, $\mathcal{T}$ is a continuous t–representable and $\mathcal{P}_{\mu,\nu}$ is a mapping $X \times (0,+\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(a) $\mathcal{P}_{\mu,\nu}(x, 0) = 0_{L^*}$;

(b) $\mathcal{P}_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;

(c) $\mathcal{P}_{\mu,\nu}(\alpha x, t) = \mathcal{P}_{\mu,\nu}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;

(d) $\mathcal{P}_{\mu,\nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic fuzzy norm*. Here,

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)).$$

Example 2.7. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big), \quad \forall t \in \mathbf{R}^+.$$

Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFN-space.

Definition 2.8. (1) A sequence $\{x_n\}$ in an IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall n, m \geq n_0,$$

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be *convergent* to a point $x \in X$ (denoted by $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for every $t > 0$.

(3) An IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x \in X$.

3. The stability result

Theorem 3.1. *Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IFN-space, $\varphi : X \times X \longrightarrow Z$ be a function such that for some $0 < \alpha < 2$,*

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x, 2x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x, x), t) \quad (x, \in X, t > 0) \tag{3.1}$$

and $\lim_{n\to\infty}\mathcal{P}'_{\mu,\nu}(\varphi(2^nx,2^ny),2^nt)=1_{L^*}$ *for all* $x,y\in X$ *and* $t>0$. *Let* $(Y,\mathcal{P}_{\mu,\nu},\mathbf{M})$ *be a complete IFN-space. If* $f:X\to Y$ *is a mapping such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(x+y)-f(x-y)-2f(y),t)\\ \geq_{L^*}\;&\mathcal{P}'_{\mu,\nu}(\varphi(x,y),t)\quad(x,y\in X,t>0)\end{aligned}\tag{3.2}$$

and $f(0)=0$. *Then there exists a unique additive mapping* $A:X\to Y$ *such that*

$$\mathcal{P}_{\mu,\nu}(f(x)-A(x),t)\geq_{L^*}\mathcal{P}'_{\mu,\nu}(\varphi(x,y),(2-\alpha)t)).\tag{3.3}$$

Proof. Putting $y=x$ in (3.2) we get

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2x)}{2}-f(x),t\right)\geq_{L^*}\mathcal{P}'_{\mu,\nu}(\varphi(x,x),2t)\quad(x\in X,t>0).\tag{3.4}$$

Replacing x by 2^nx in (3.4), and using (3.1) we obtain

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n},t\right)\;&\geq_{L^*}\;\mathcal{P}'_{\mu,\nu}(\varphi(2^nx,2^nx),2\times 2^nt)\\ &\geq_{L^*}\;\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{2\times 2^n}{\alpha^n}\right).\end{aligned}\tag{3.5}$$

Since $\frac{f(2^nx)}{2^n}-f(x)=\sum_{k=0}^{n-1}(\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^kx)}{2^k})$, by (3.5) we have

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^nx)}{2^n}-f(x),t\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}\right)\geq_{L^*}\mathbf{M}_{k=0}^{n-1}\left(\mathcal{P}'_{\mu,\nu}(\varphi(x,x),t)\right)=\mathcal{P}'_{\mu,\nu}(\varphi(x,x),t),$$

that is

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^nx)}{2^n}-f(x),t\right)\geq_{L^*}\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right).\tag{3.6}$$

By replacing x with 2^mx in (3.6) we observe that:

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^mx)}{2^m},t\right)\geq\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{2\times 2^k}}\right).\tag{3.7}$$

Then $\{\frac{f(2^nx)}{2^n}\}$ is a Cauchy sequence in $(Y,\mathcal{P}_{\mu,\nu},\mathbf{M})$. Since $(Y,\mathcal{P}_{\mu,\nu},\mathbf{M})$ is a complete IFN-space this sequence convergent to some point $A(x)\in Y$. Fix $x\in X$ and put $m=0$ in (3.7) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^nx)}{2^n}-f(x),t\right)\geq_{L^*}\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right),\tag{3.8}$$

and so for every $\delta>0$ we have that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(A(x)-f(x),t+\delta) &\geq_{L^*} \mathbf{M}\left(\mathcal{P}_{\mu,\nu}\left(A(x)-\frac{f(2^nx)}{2^n},\delta\right),\mathcal{P}_{\mu,\nu}\left(f(x)-\frac{f(2^nx)}{2^n},t\right)\right) \quad (3.9)\\ &\geq_{L^*} \mathbf{M}\left(\mathcal{P}_{\mu,\nu}\left(A(x)-\frac{f(2^nx)}{2^n},\delta\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right)\right).\end{aligned}$$

Taking the limit as $n\longrightarrow\infty$ and using (3.9) we get

$$\mathcal{P}_{\mu,\nu}(A(x)-f(x),t+\delta)\geq_{L^*}\mathcal{P}'_{\mu,\nu}(\varphi(x,x),t(2-\alpha)). \quad (3.10)$$

Since δ was arbitrary, by taking $\delta\to 0$ in (3.10) we get

$$\mathcal{P}_{\mu,\nu}(A(x)-f(x),t)\geq_{L^*}\mathcal{P}'_{\mu,\nu}(\varphi(x,x),t(2-\alpha)).$$

Replacing x,y by $2^nx,2^ny$ in (3.2) to get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}\Big(\frac{f(2^n(x+y))}{2^n}+\frac{f(2^n(x-y))}{2^n}-\frac{2f(2^ny)}{2^n},t\Big)\\ \geq_{L^*}\ &\mathcal{P}'_{\mu,\nu}(\varphi(2^nx,2^ny),2^nt), \quad (3.11)\end{aligned}$$

for all $x,y\in X$ and for all $t>0$. Since $\lim_{n\longrightarrow\infty}\mathcal{P}'_{\mu,\nu}(\varphi(2^nx,2^ny),2^nt)=1_{L^*}$ we conclude that A fulfills (1.1). To Prove the uniqueness of the additive function A, assume that there exists an additive function $A':X\longrightarrow Y$ which satisfies (3.3). Fix $x\in X$. Clearly $A(2^nx)=2^nA(x)$ and $A'(2^nx)=2^nA(x)$ for all $n\in\mathbb{N}$. It follows from (3.3) that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(A(x)-A'(x),t) &= \mathcal{P}_{\mu,\nu}\left(\frac{A(2^nx)}{2^n}-\frac{A'(2^nx)}{2^n},t\right)\\ &\geq_{L^*} \mathbf{M}\left\{\mathcal{P}_{\mu,\nu}\left(\frac{A(2^nx)}{2^n}-\frac{f(2^nx)}{2^n},\frac{t}{2}\right),\mathcal{P}_{\mu,\nu}\left(\frac{A'(2^nx)}{2^n}-\frac{f(2^nx)}{2^n},\frac{t}{2}\right)\right\}\\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(2^nx,2^nx),2^n(2-\alpha)\frac{t}{2}\right)\\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{2^n(2-\alpha)\frac{t}{2}}{\alpha^n}\right).\end{aligned}$$

Since $\lim_{n\to\infty}\frac{27^n(27-\alpha)t}{2\alpha^n}=\infty$, we get $\lim_{n\to\infty}\mathcal{P}'_{\mu,\nu}(\varphi(x,0),\frac{27^n(27-\alpha)t}{2\alpha^n})=1_{L^*}$. Therefore $\mathcal{P}_{\mu,\nu}(A(x)-A'(x),t)=1$ for all $t>0$, whence $A(x)=A'(x)$. □

Corollary 3.2. *Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IFN-space, $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IFN-space, p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(f(x+y) + f(x-y) - 2f(y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^q)z_0, t), \tag{3.12}$$

$x, y \in X$, $t > 0$, $f(0) = 0$ and $p, q < 1$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, (2 - 2^p)t)). \tag{3.13}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 2^p$. □

Corollary 3.3. *Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IFN-space, $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IFN-space and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(f(x+y) + f(x-y) - 2f(y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varepsilon z_0, t) \tag{3.14}$$

$x, y \in X$, $t > 0$, $f(0) = 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varepsilon z_0, t). \tag{3.15}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$. □

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