J. Nonlinear Sci. Appl. 1 (2008), no. 2, 56–60

THE JOURNAL OF NONLINEAR SCIENCES AND APPLICATIONS
http://www.tjnsa.com

SOME REMARK ON THE NONEXISTENCE OF POSITIVE SOLUTIONS FOR SOME a, P-LAPLACIAN SYSTEMS

M. ALIMOHAMMADY[1*] AND M. KOOZEGAR[2]

ABSTRACT. This paper deals with nonexistence result for positive solution in $C^1(\overline{\Omega})$ to the following reaction-diffusion system

$$\begin{cases} -\triangle_{a,p} u = & a_1 v^{p-1} - b_1 v^{\gamma-1} - c, \quad x \in \Omega, \\ -\triangle_{a,p} v = & a_1 u^{p-1} - b_1 u^{\gamma-1} - c, \quad x \in \Omega, \\ \quad u = 0 = v & \quad , \quad x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\triangle_{a,p}$ denotes the a,p-Laplacian operator defined by $\triangle_{a,p} z = div(a \mid \nabla z \mid^{p-2} \nabla z)$; $p > 1$, $\gamma(> p)$; a_1, b_1 and c are positive constant, Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary and $a(x) \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$ for all $x \in \Omega$.

1. INTRODUCTION AND PRELIMINARIES

In this note, we first consider a nonexistence result for positive solution in $C^1(\overline{\Omega})$ to the following reaction-diffusion system

$$\begin{cases} -\triangle_{a,p} u = & a_1 v^{p-1} - b_1 v^{\gamma-1} - c, \quad x \in \Omega, \\ -\triangle_{a,p} v = & a_1 u^{p-1} - b_1 u^{\gamma-1} - c, \quad x \in \Omega, \\ \quad u = 0 = v & \quad , \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_{a,p}$ denotes the p,a-Laplacian operator defined by $\triangle_{a,p} z = div(a \mid \nabla z \mid^{p-2} \nabla z)$; $p > 1$, $\gamma(> p)$; a_1, b_1 and c are positive constant, Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary and $a(x) \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$

Date: Received: 10 August 2008; Accepted: 10 September 2008.
* Corresponding author.
2000 *Mathematics Subject Classification.* Primary 35P20, Secondary 35J70; 35BJ20.
Key words and phrases. p, a-Laplacian; nonexistence; positive solution; reaction-diffusion systems.

for all $x \in \Omega$.
We first show that if $a_1 \leq \lambda_1$, where λ_1 is the first eigenvalue of $-\triangle_{p,a}$ with Dirichlet boundary condition, then (1.1) has no positive solutions. next we consider the system

$$\begin{cases} -\triangle_{a,p} u = & \lambda f(v), \qquad x \in \Omega, \\ -\triangle_{a,p} v = & \mu g(u), \qquad x \in \Omega, \\ \qquad u = & 0 = v, \qquad x \in \partial\Omega. \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 1)$, $\partial\Omega$ is its smooth boundary and λ, μ are positive parameters. Let $f, g : [0, \infty) \to R$ be continuous and assume that there exist positive numbers K_i and M_i, $i = 1, 2$ such that

$$f(v) \geq K_1 v^{p-1} + M_1 \ (v \geq 0), \tag{1.3}$$

and

$$g(u) \geq K_2 u^{p-1} + M_2 \ (u \geq 0), \tag{1.4}$$

and $a : \Omega \to \mathbb{R}^+$ satisfy certain conditions.
We discus a non existence result when $\lambda\mu$ is large.

Definition 1.1. A pair of nonnegative functions (u, v) in $\mathrm{W}_0^{1,\mathrm{p}}(\Omega) \times \mathrm{W}_0^{1,\mathrm{p}}(\Omega)$ are called a weak solution of (1.2) if they satisfies

$$\int_\Omega a \mid \nabla u \mid^{p-2} \nabla u \nabla w dx = \int_\Omega [\lambda f(v)] w dx,$$

and

$$\int_\Omega a \mid \nabla v \mid^{p-2} \nabla v \nabla w dx = \int_\Omega [\mu g(u)] w dx,$$

for all test function $w \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega)$.

In the case $a = 1$ when $p = 2$, system (1.2) studied by Dalmaso [5]. For existence results of positive solutions for (1.2) see [3, 4, 6]. For corresponding result in the single equations case see [2, 8] for (1.1) and [7] for (1.2), (for the case $a = 1$) and in [1] Afrouzi and Rasouli studied the system (1.1), (1.2) for the case $a = 1$.

2. Non-existence results

In this section we state our main results. To prove the non-existence results we use some estimates on the first eigenvalue of $-\triangle_{p,a}$ with Dirichlet boundary conditions.

Theorem 2.1. *let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $a_1 \leq \lambda_1$, (1.1) has no positive solution.*

Proof. On the contrary, there exists a positive solution (u, v) of (1.1). Since for any $\mathrm{w} \in \mathrm{C}_0^\infty(\Omega)$ we have

$$\int_\Omega a \mid \nabla u \mid^{p-2} \nabla u \nabla w dx = \int_\Omega [a_1 v^{p-1} - b_1 v^{\gamma-1} - c] w dx,$$

so

$$\int_\Omega a \mid \nabla u \mid^p dx = \int_\Omega [a_1 v^{p-1} - b_1 v^{\gamma-1} - c] u dx,$$

and since $b_1 > 0$, then

$$\int_\Omega a \mid \nabla u \mid^p dx \leq \int_\Omega [a_1 v^{p-1} - c] u dx, \tag{2.1}$$

But

$$\int_\Omega a \mid \nabla u \mid^p dx \geq \lambda_1 \int_\Omega \mid u \mid^p dx = \lambda_1 \int_\Omega u^p dx \tag{2.2}$$

since $\lambda_1 = \inf_{z \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega)} \frac{\int_\Omega a|\nabla z|^p dx}{\int_\Omega |z|^p dx}$ is the first eigenvalue of $-\triangle_{a,p}$ with Dirichlet boundary condition. Combining (2.1), (2.2), we obtain

$$\lambda_1 \int_\Omega u^p dx \leq a_1 \int_\Omega u v^{p-1} dx - c \int_\Omega u dx. \tag{2.3}$$

Similarly, we obtain

$$\lambda_1 \int_\Omega v^p dx \leq a_1 \int_\Omega v u^{p-1} dx - c \int_\Omega v dx. \tag{2.4}$$

But recall that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ if $\frac{1}{p} + \frac{1}{q} = 1$. Taking $a = u, b = v^{p-1}$ and $a = v, b = u^{p-1}$, we see respectively that $uv^{p-1} \leq \frac{u^p}{p} + \frac{v^p}{q}$ and $vu^{p-1} \leq \frac{v^p}{p} + \frac{u^p}{q}$. Thus adding (2.3) and (2.4),

$$\begin{aligned}
&\lambda_1 \int_\Omega u^p dx + \lambda_1 \int_\Omega v^p dx \\
&\leq a_1 \int_\Omega u v^{p-1} dx + a_1 \int_\Omega v u^{p-1} dx - c \int_\Omega (u+v) dx \\
&< a_1 \int_\Omega u v^{p-1} dx + a_1 \int_\Omega v u^{p-1} dx \\
&\leq a_1 \int_\Omega [\frac{u^p}{p} + \frac{v^p}{q}] dx + a_1 \int_\Omega [\frac{v^p}{p} + \frac{u^p}{q}] dx \\
&= a_1 (\frac{1}{p} + \frac{1}{q}) \int_\Omega u^p dx + a_1 (\frac{1}{p} + \frac{1}{q}) \int_\Omega v^p dx \\
&= a_1 \int_\Omega [u^p + v^p] dx.
\end{aligned}$$

This implies that

$$(\lambda_1 - a_1) \int_\Omega [u^p + v^p] dx < 0,$$

which is contradiction if $a_1 \leq \lambda_1$. Thus (1.1) has no positive solutions for $a_1 \leq \lambda_1$.

Now we consider the system (1.2) and we would establish the following.

Theorem 2.2. *Let (1.3)-(1.4) hold. Then the system (1.2) has no positive solution if* $\lambda\mu > \frac{\lambda_1^2}{K_1 K_2}$.

Proof. Suppose $u > 0$ and $v > 0$ be $C^1(\overline{\Omega})$ functions such that (u, v) is a solution of (1.2). We proceed our proof by arriving to a contradiction. Multiplying the first equation in (1.2) by a positive eigenfunction say ϕ_1 corresponding to λ_1, we obtain

$$-\int_\Omega \triangle_{a,p} u \phi_1 dx = \int_\Omega \lambda f(v)\phi_1 dx,$$

and hence using (1.3),

$$-\int_\Omega \triangle_{a,p} u \phi_1 dx \geq \int_\Omega \lambda(K_1 v^{p-1} + M_1)\phi_1 dx.$$

That is

$$\int_\Omega u^{p-1}\lambda_1 \phi_1 dx \geq \int_\Omega \lambda(K_1 v^{p-1} + M_1)\phi_1 dx. \tag{2.5}$$

Similarly using the second equation in (1.2) and (1.4),

$$\int_\Omega v^{p-1}\lambda_1 \phi_1 dx \geq \int_\Omega \mu(K_2 u^{p-1} + M_2)\phi_1 dx. \tag{2.6}$$

Combining (2.5) and (2.6),

$$\int_\Omega [\lambda_1 - (\lambda\mu)\frac{K_1K_2}{\lambda_1}]v^{p-1}\phi_1 dx \geq \int_\Omega \mu[K_2 u^{p-1} + M_2 + \lambda\frac{M_1K_2}{\lambda_1} - K_2 u^{p-1}]\phi_1 dx$$
$$= \int_\Omega \mu[\lambda\frac{K_2M_1}{\lambda_1} + M_2]\phi_1 dx \tag{2.7}$$

This clearly require $\lambda\mu \leq \frac{\lambda_1^2}{K_1K_2}$. This completes the proof.

References

1. G. A. Afrouzi, S. H. Rasouli, *A remark on the nonexistence of positive solutions for some p-Laplacian systems* , Global J. Pure. Appl. Math. **1/2**(2005), 197–201. 1
2. G. A. Afrouzi, S. H. Rasouli, *Population models involving the p-Laplacian with indefinite weight and constant yeild harvesting.* , chaos, Solutions and Fractals, **31**(2007), 404–408. 1
3. L. Boccardo , D. G. Figueiredo, *Some remarks on a system of quasilinear elliptic equations.* , Nonl. Diff. Eqns. Appl. **9**(2002), 231–240. 1
4. P. Clement, J. Fleckinger, E. Mitidieri , F. de Thelin, *Existence of positive solutions for a nonvariational quasilinear elliptic system.*, J. Diff. Eqns. **166**(2000), 455–477. 1
5. R. Dalmasso, *Existence and uniqueness of positive solutions of a semilinear elliptic system.*, Nonl. Anal. **39**(2000), 559–568. 1

6. A. Djellit, S. Tas, *On some nonlinear elliptic systems.*, Nonl. Anal. **59**(2004), 675–706. 1
7. D. D. Hai, *On a cllas of sublinear quasilinear elliptic problems.* Proc. Amer. Math. Soc. **131**(2003), 2409–2414. 1
8. S. Oruganti , J. Shi, R. Shivaji, *Logistic equation with the p-Laplacian and constant yeild harvesting.*, Applied. Anal. **9**(2004), 723–727. 1

[1] Islamic Azad University, branch Noor, Iran
E-mail address: m.kaleji@yahoo.com

[2] Department of Mathematics, University of Mazandaran, Babolsar 47416 – 1468, Iran.
E-mail address: amohsen@umz.ac.ir