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PRESERVING HOMOLOGY

MICHAEL BARR

Abstract. We raise the question of saying what it means for a functor between abelian categories to preserve homology. We give a kind of answer and explore the reasons it is unsatisfactory in general (although fine for left or right exact functors).

1. Introduction

As far as I know the question of what does it mean for a functor to preserve homology has not been studied, let alone answered. In this note, we repair the first lacuna, although not the second. To my surprise there seems to be no completely satisfactory answer.

What does it mean for a functor to preserve the homology of a differential object? It is clear what it means for a functor to preserve a limit or a colimit, since limits and colimits are defined by universal mapping properties. For example, if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, there is a canonical map $F(A_1 \times A_2) \longrightarrow FA_1 \times FA_2$ and if that map is an isomorphism, then F preserves that product.

In contrast, there is no possibility of defining what it means for a functor to preserve tensor product in a category, since that is not defined by any universal mapping property. What is needed in that case is additional structure, which raises coherence questions and it can get quite complicated. But essentially, the functor requires what is called tensorial strength.

The situation with homology falls between the two cases. Homology is defined as a cokernel of a map to a kernel and we know what it means to preserves kernels and cokernel. A functor that preserves both preserves homology. In this paper we analyze what is actually required for a particular functor to preserve the homology of a particular differential object. However homology can also be defined as a kernel of a map to a cokernel and one would hope that that the conditions for preserving the two versions are the same. Alas, that turns out not to be the case.

What we do here is show that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories and (C, d) is a differential object (which could, of course, be graded) there is a span $HF(C, d) \longleftarrow \Phi(C, d) \longrightarrow FH(C, d)$, but not generally a natural map in either direction. One could say that F preserves the homology if both legs of the span are isomorphisms. But a differential object in $\mathcal{A}$ is also a differential object in $\mathcal{A}^{\mathrm{op}}$ with

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exactly the same homology and it thus came as a considerable surprise that the homology could be preserved in the sense just described, while the cohomology (the homology in the dual space) is not. This raises the question of whether there is a different, self dual, notion of preservation.

We also show an additive functor preserves the homology of all differential objects if and only if it is exact and find necessary and sufficient conditions on a differential object that its homology is preserved by all additive functors. At least that condition is self dual.

All categories considered in this paper are abelian and all functors are additive.

1.1. Notation. Let (C,d) be a differential object. We will denote by $Z(C,d)$ and $Z'(C,d)$ the kernel and cokernel of d, by $B(C,d)$ the image of d, (which is also the coimage of d), by $H(C,d)$ the cokernel of the inclusion $B(C,d) \longrightarrow Z(C,d)$ and by $H'(C,d)$ the kernel of the induced map $Z'(C,d) \longrightarrow B'(C,d)$. Our notation for arrows will be as in the following exact sequences:

$$0 \longrightarrow Z(C,d) \xrightarrow{z} C \xrightarrow{b} B(C,d) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow B(C,d) \xrightarrow{b'} C \xrightarrow{z'} Z'(C,d) \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow B(C,d) \xrightarrow{s} Z(C,d) \xrightarrow{p} H(C,d) \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow H'(C,d) \xrightarrow{p'} Z'(C,d) \xrightarrow{s'} B(C,d) \longrightarrow 0 \tag{4}$$

These satisfy the following equations:

$$b = s' \circ z' \qquad b' = z \circ s \qquad d = b' \circ b \qquad z' \circ z = p' \circ p$$

Only the last equation is not obvious. But each side takes an element of Z to its image in $Z' = C/B$. It will be convenient to denote by t the map $s \circ b : C \longrightarrow Z(C,d)$ as well as $t' = b' \circ s' : Z'(C,d) \longrightarrow C$. In addition, we will use the convention that if we have a map f, then f^ℓ will denote a left inverse, if any exists, and similarly f^r will denote a right inverse, if any exists.

If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor, then (FC, Fd) is a differential object in $\mathcal{B}$ and we will denote it by $F(C,d)$. In addition, we will use $ZF(C,d)$, $BF(C,d)$, and $HF(C,d)$, as well as the primed forms to denote the corresponding objects of $\mathcal{B}$. We will use zF, bF, sF, and pF, as well as their primed forms, to denote the arrows that correspond. On the other hand, $FZ(C,d)$, $FB(C,d)$, $FH(C,d)$, Fz, Fb, Fs, and Fp, as well as all the primed forms, denote the values of F on the corresponding objects and arrows.

2. Preservation of homology

Generally when we ask about functors preserving some structure, there will be a map in one direction or other depending on when the functor is applied. Often this comes about because the structure is characterized by a universal mapping property. However homology, generally defined as a cokernel of a kernel cannot be so characterized. So before we can ask whether a functor preserves homology, we have to say what preservation means. We will give a definition here that seems at first to be satisfactory, but turns out not to be for the reasons just described.

The exactness of $0 \longrightarrow ZF(C,d) \xrightarrow{zF} FC \xrightarrow{Fd} FC$ implies the existence of a map $u : FZ(C,d) \longrightarrow ZF(C,d)$ such that $zF \circ u = Fz$. In the diagram

$$\begin{array}{ccccccc} FC & \xrightarrow{Ft} & FZ(C,d) & \xrightarrow{Fp} & FH(C,d) & \longrightarrow & 0 \\ {\scriptstyle =}\Big\downarrow & & \Big\downarrow{\scriptstyle u} & & & & \\ FC & \xrightarrow[tF]{} & ZF(C,d) & \xrightarrow[pF]{} & HF(C,d) & \longrightarrow & 0 \end{array} \tag{5}$$

the bottom row is exact, but the top is not in general. From this diagram, it is clear that if F preserves the left exactness of (1), u is an isomorphism and then the universal mapping property of the cokernel implies the existence of a natural map $HF(C,d) \longrightarrow FH(C,d)$. If, on the other hand, F preserves the right exactness of (1), then the top row is a cokernel and there is a natural map $FH(C,d) \longrightarrow HF(C,d)$. In general, there is no map in either direction and it appears that the best we can do is a span. Let $q : FZ(C,d) \longrightarrow \Phi(C,d)$ be a cokernel of Ft. Then there are unique maps $r : \Phi(C,d) \longrightarrow FH(C,d)$ and $\ell : \Phi(C,d) \longrightarrow HF(C,d)$ such that $r \circ q = Fp$ and $\ell \circ q = pF \circ u$. Thus the span is

$$\begin{array}{ccc} & \Phi & \\ {\scriptstyle \ell}\swarrow & & \searrow{\scriptstyle r} \\ HF(C,d) & & FH(C,d) \end{array}$$

It is clear that when Fp is a cokernel of Ft, then we can take $\Phi(C,d) = FH(C,d)$ and then $\ell : FH(C,d) \longrightarrow HF(C,d)$ is the map described above. It is also evident that when u is an isomorphism, we can take $\Phi(C,d) = HF(C,d)$ and then $r : HF(C,d) \longrightarrow FH(C,d)$ is the map described above. In general, neither exactness condition is satisfied and the best we can get is a span.

2.1. Example. Suppose $f : A' \longrightarrow A$ is a map in $\mathcal{A}$ and we let $C = A' \oplus A$ and d have the matrix $\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}$. Then one easily calculates that $H(C,d) = \ker f \oplus \operatorname{cok} f$, $FH(C,d) = F(\ker f) \oplus F(\operatorname{cok} f)$, $HF(C,d) = \ker Ff \oplus \operatorname{cok} Ff$ and $\Phi = F(\ker f) \oplus \operatorname{cok} Ff$. The span is

$$\begin{array}{ccc} & F(\ker f) \oplus \operatorname{cok} Ff & \\ {\scriptstyle \ell}\swarrow & & \searrow{\scriptstyle r} \\ \ker Ff \oplus \operatorname{cok} Ff & & F(\ker f) \oplus F(\operatorname{cok} f) \end{array}$$

The maps ℓ and r are those induced by the universal mapping properties of kernels and cokernels.

So far, this looks like a convincing way of answering the original question. Homology is preserved when ℓ and r are isomorphisms. The problem comes from the fact that homology can equally well be defined in the dual category. In fact, there is a completely symmetric way of doing this. The sequence

$$0 \longrightarrow Z \xrightarrow{z} C \xrightarrow{d} C \xrightarrow{z'} Z' \longrightarrow 0$$

is exact. Then the image of $z' \circ z : Z \longrightarrow Z'$ is $H(C,d) = H'(C,d)$. Of course, the image of a map in an abelian category is the same as its coimage (that is the main thing needed for a category to be abelian) so this definition is self-dual. Everything can be dualized and a cospan Φ' can be constructed to give one big diagram:

$$\begin{array}{ccccccccccc}
FC & \xrightarrow{Ft} & FZ(C,d) & \xrightarrow{Fp} & & FH(C,d) & & \xrightarrow{Fp'} & FZ'(C,d) & \xrightarrow{Ft'} & FC \\
\Big\downarrow{=} & & \Big\downarrow{u} & \searrow^{q} & & {}^{r}\nearrow \quad \searrow^{r'} & & \nearrow^{q'} & \Big\uparrow{u'} & & \Big\uparrow{=} \\
 & & & & \Phi(C,d) & & \Phi'(C,d) & & & & \\
 & & & & & {}_{\ell}\searrow \quad \nearrow_{\ell'} & & & & & \\
FC & \xrightarrow{tF} & ZF(C,d) & \xrightarrow{pF} & & HF(C,d) & & \xrightarrow{p'F} & Z'F(C,d) & \xrightarrow{Ft'} & FC
\end{array} \tag{6}$$

What we would like is that $r \circ \ell^{-1} = r'^{-1} \circ \ell'$ as relations $HF(C,d) \dashrightarrow FH(C,d)$ and that $\ell \circ r^{-1} = \ell'^{-1} \circ r' : FH(C,d) \dashrightarrow HF(C,d)$. The two equalities are equivalent, both being true if and only if

$$\Phi \xrightarrow{\begin{pmatrix} \ell \\ r \end{pmatrix}} FH(C,d) \oplus HF(C,d) \xrightarrow{\begin{pmatrix} \ell' & -r' \end{pmatrix}} \Phi'$$

is exact.

2.1. Revisited. When we calculate the span and cospan in the example, we get

$$\begin{array}{ccccc}
 & & F(\ker f) \oplus \operatorname{cok} Ff & & \\
 & {}^{\ell}\swarrow & & \searrow^{r} & \\
\ker Ff \oplus \operatorname{cok} Ff & & & & F(\ker f) \oplus F(\operatorname{cok} f) \\
 & {}_{r'}\searrow & & \swarrow_{\ell'} & \\
 & & \ker Ff \oplus F(\operatorname{cok} f) & &
\end{array} \tag{7}$$

which not only commutes, but is both a pullback and pushout and the sequence (7) is exact with zeroes at both ends.

Unfortunately, this example is not characteristic of the general case. We do have some desirable properties.

2.2. Theorem. *The diagram*

$$
\begin{array}{ccccc}
 & & \Phi(C,d) & & \\
 & \ell \swarrow & & \searrow r & \\
HF(C,d) & & & & FH(C,d) \\
 & \ell' \searrow & & \swarrow r' & \\
 & & \Phi'(C,d) & &
\end{array}
$$

commutes.

Proof. We begin by showing that

$$
\begin{array}{ccc}
FZ(C,d) & \xrightarrow{Fz' \circ Fz} & FZ'(C,d) \\
{\scriptstyle u}\downarrow & & \uparrow{\scriptstyle u'} \\
ZF(C,d) & \xrightarrow[z'F \circ zF]{} & Z'F(C,d)
\end{array}
$$

commutes. In fact $u' \circ z'F \circ zF \circ u = Fz' \circ Fz$ from the universal properties that defined u and u'. An instance of $z' \circ z = p' \circ p$ (see 1.1) is $z'F \circ zF = p'F \circ pF$, while the result of applying F to the same equation is $Fz' \circ Fz = Fp' \circ Fp$. Now we have

$$q' \circ r' \circ r \circ q = Fp' \circ Fp = Fz' \circ Fz = u' \circ z'F \circ zF \circ u = u' \circ p'F \circ Fp \circ u = q' \circ \ell' \circ \ell \circ q$$

But q' is monic and q is epic, so they can be cancelled to produce $r' \circ r = \ell' \circ \ell$. ∎

An immediate consequence is

2.3. Corollary. *If ℓ and r' are isomorphisms, then $r \circ \ell^{-1} = r'^{-1} \circ \ell' : HF(C,d) \longrightarrow FH(C,d)$ and if r and ℓ' are isomorphisms then $\ell \circ r^{-1} = \ell'^{-1} \circ r : FH(C,d) \longrightarrow HF(C,d)$.*

2.4. Theorem. *For a fixed differential object (C,d), the map r is an isomorphism if and only if $FC \xrightarrow{Ft} FZ(C,d) \xrightarrow{Fp} FH(C,d) \longrightarrow 0$ is exact; ℓ is an isomorphism if and only if $0 \longrightarrow FZ \xrightarrow{Fz} FC \xrightarrow{Fd} FC$ is exact.*

Proof. The map r is the unique map for which

$$
\begin{array}{ccccccc}
FC & \xrightarrow{Ft} & FZ(C,d) & \xrightarrow{Fp} & \Phi(C,d) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle r} & & \\
FC & \xrightarrow[Ft]{} & FZ(C,d) & \xrightarrow[Fp]{} & FH(C,d) & \longrightarrow & 0
\end{array}
$$

commutes. Since the top line is exact, r is an isomorphism if and only if the bottom line is. As for ℓ, it is defined so that

$$\begin{array}{ccccccc}
FC & \xrightarrow{Ft} & FZ(C,d) & \xrightarrow{Fp} & \Phi(C,d) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \ell} & & \\
FC & \xrightarrow[tF]{} & ZF(C,d) & \xrightarrow[pF]{} & HF(C,d) & \longrightarrow & 0
\end{array}$$

commutes. We wish to show that ℓ is an isomorphism if and only if u is. Since $tF = sF \circ bF$ we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & ZF(C,d) & & & & & & \\
 & & {\scriptstyle zF}\downarrow & & & & & & \\
 & & FC & \xrightarrow{Ft} & FZ(C,d) & \xrightarrow{Fp} & \Phi(C,d) & \longrightarrow & 0 \\
 & & {\scriptstyle bF}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \ell} & & \\
0 & \longrightarrow & BF(C,d) & \xrightarrow[sF]{} & ZF(C,d) & \xrightarrow[pF]{} & HF(C,d) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

in which both rows and the column are exact. The snake lemma implies that

$$ZF(C,d) \longrightarrow \ker u \longrightarrow \ker \ell \longrightarrow 0 \longrightarrow \operatorname{cok} u \longrightarrow \operatorname{cok} \ell \longrightarrow 0$$

is exact. It is immediate that if u is an isomorphism, so is ℓ and that if ℓ is an isomorphism, then u is epic. We finish by showing that u is also monic in that case. To show this, it suffices to show that the map $ZF(C,d) \longrightarrow \ker u$ is 0 or, equivalently, that $Ft \circ zF = 0$. Since u is epic, it suffices to show that $Ft \circ zF \circ u = Ft \circ Fz = 0$ which follows immediately from $t \circ z = 0$. ■

Dualizing, we obtain

2.5. Theorem. *For a fixed differential object* (C,d), *the map* r' *is an isomorphism if and only if* $0 \longrightarrow FH(C,d) \xrightarrow{Fp'} FZ'(C,d) \xrightarrow{Ft'} FC$ *is exact;* ℓ *is an isomorphism if and only if* $FC \xrightarrow{Fd} FC \xrightarrow{Fz'} FZ' \longrightarrow 0$ *is exact.*

In the following, the "if" is a consequence of the above, while the "only if" follows immediately from the example 2.1

2.6. Theorem. *The maps ℓ and r' are isomorphisms for every (C, d) if and only if F is left exact; r and ℓ' are isomorphisms for every (C, d) if and only if F is right exact.*

The previous results can be summarized as follows.

2.7. Theorem. *Let (C, d) be a fixed differential object. In the diagram*

$$
\begin{array}{ccccccccc}
 & & C & \xrightarrow{=} & C & & & & \\
 & & \downarrow t & & \downarrow d & & & & \\
0 & \longrightarrow & Z(C,d) & \xrightarrow{z} & C & \xrightarrow{d} & C & & (\ell) \\
 & & \downarrow p & & \downarrow t' & & \downarrow = & & \\
0 & \longrightarrow & H(C,d) & \xrightarrow[p']{} & Z'(C,d) & \xrightarrow[z']{} & C & & (r') \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & \\
 & & (r) & & (\ell') & & & &
\end{array}
$$

F preserves the exactness of the sequences labelled (ℓ), (r), (ℓ), and (r') if and only if the maps ℓ, r, ℓ', and r', resp., are isomorphisms.

3. Absolutely preserved homology

We can ask what properties of a differential object forces every additive functor to preserve its homology. If H is an arbitrary object of $\mathcal{A}$, the complex $(H, 0)$ obviously has this property. We will show in this section that, up to homotopy, this is the only example. We begin with:

3.1. Theorem. *Let (C, d) be a differential object. Then ℓ and r (and dually, ℓ' and r')are isomorphisms for all additive functors to an abelian category if and only if the sequence*

$$0 \longrightarrow Z(C,d) \xrightarrow{z} C \xrightarrow{d} C \xrightarrow{z'} Z'(C,d) \longrightarrow 0$$

is contractible.

Proof. Contractibility implies that both sequences

$$0 \longrightarrow Z(C,d) \longrightarrow C \longrightarrow B(C,d) \longrightarrow 0$$

and

$$0 \longrightarrow B(C,d) \longrightarrow C \longrightarrow Z'(C,d) \longrightarrow 0$$

split. But if $B(C,d) \longrightarrow C$ is split monic, so is $B(C,d) \longrightarrow Z(C,d)$ and then the sequence

$$0 \longrightarrow B(C,d) \longrightarrow Z(C,d) \longrightarrow H(C,d) \longrightarrow 0$$

is split, from which the preservation under all functors is clear. Conversely, suppose that r is an isomorphism for every left exact functor F. In the diagram,

$$\begin{array}{ccccccc}
 & FC & \longrightarrow & FZ(C,d) & \xrightarrow{q} & \Phi(C,d) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle Ft} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle r} & \\
0 \longrightarrow & FB(C,d) & \longrightarrow & FZ(C,d) & \xrightarrow[Fp]{} & FH(C,d) &
\end{array}$$

we first observe that q and r are epic and therefore so is Fp. The snake lemma then gives us the exact sequence

$$\ker Ft \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{cok} Ft \longrightarrow 0 \longrightarrow 0$$

so that Ft is also epic and hence both sequences

$$0 \longrightarrow FZ(C,d) \longrightarrow FC \longrightarrow FB(C,d) \longrightarrow 0$$

and

$$0 \longrightarrow FB(C,d) \longrightarrow FZ(C,d) \longrightarrow FH(C,d) \longrightarrow 0$$

are exact. First let $F = \operatorname{Hom}(B(C,d), -)$ to show that the first one splits and then let $F = \operatorname{Hom}(H(C,d))$ to show that the second one does. Now apply the same argument on the dual to show the same things for

$$0 \longrightarrow B(C,d) \longrightarrow C \longrightarrow Z'(C,d) \longrightarrow 0$$

and

$$0 \longrightarrow H(C,d) \longrightarrow Z'(C,d) \longrightarrow B(C,d) \longrightarrow 0$$

We will demonstrate the existence of the contracting homotopy after the next theorem.

3.2. Theorem. *Let* (C,d) *have the property that any additive functor preserves its homology. Then* (C,d) *is homotopic to* $(H(C,d),0)$ *(and conversely).*

Proof. We use the preceding result. The following sequences are split exact and the splittings are denoted as shown:

$$0 \longrightarrow Z(C,d) \underset{z}{\overset{z^\ell}{\rightleftarrows}} C \underset{b}{\overset{b^r}{\rightleftarrows}} B \longrightarrow 0$$

$$0 \longrightarrow B(C,d) \underset{s}{\overset{s^\ell}{\rightleftarrows}} Z(C,d) \underset{p}{\overset{p^r}{\rightleftarrows}} H(C,d) \longrightarrow 0$$

This means that $z^\ell\circ z$, $s^\ell\circ s$, $b\circ b^r$, and $p\circ p^r$ are all identities and, in addition, $z\circ z^\ell + b^r\circ b = 1$ and $s\circ s^\ell + p\circ p^r = 1$. We define $f = p\circ z^\ell : C \longrightarrow H(C,d)$ and $g = z\circ p^r : H(C,d) \longrightarrow C$. We begin by showing that f and g are maps between (C,d) and $(H(C,d),0)$. We have $f\circ d = p\circ z^\ell\circ b'\circ b = p\circ z^\ell\circ z\circ s\circ b = p\circ s\circ b = 0 = d\circ f$. We also have $d\circ g = d\circ z\circ p^r = 0 = g\circ d$. We also calculate that $f\circ g = p\circ z^\ell\circ z\circ p^r = p\circ p^r = 1$. We will show that if we define $h = b^r\circ s^\ell\circ z^\ell : C \longrightarrow C$, then $1 - g\circ f = h\circ d + d\circ h$. We calculate that

$$\begin{aligned} 1 - g\circ f &= 1 - z\circ p^r\circ p\circ z^\ell = 1 - z\circ(1 - s\circ s^\ell)\circ z^\ell \\ &= 1 - z\circ z^\ell + z\circ s\circ s^\ell\circ z^\ell = b^r\circ b + z\circ s\circ s^l\circ z^\ell \end{aligned}$$

Then

$$h\circ d = b^r\circ s^\ell\circ z^\ell\circ z\circ t = b^r\circ s^\ell\circ s\circ b = b^r\circ b$$

and

$$d\circ h = v\circ b\circ b^r\circ s^\ell\circ z^\ell = z\circ s\circ s^\ell\circ z^l$$

so that $h\circ d + d\circ h = 1 - g\circ f$ which establishes the homotopy.

As for the converse, if (C,d) is homotopic to $(H(C,d),0)$, then this is preserved by any functor and the previous theorem provides the required splittings. ∎

The proof of Theorem 3.1 will be complete when we have proved the following, which should have been done in [Barr, 2002, Section 3.2], but unaccountably was not.

3.3. Lemma. *Suppose $f : (C,d) \longrightarrow (C',d)$ and $g : (C',d) \longrightarrow (C,d)$ are homotopy inverse to each other. Then (C,d) is contractible if and only if (C',d) is.*

Proof. What we will actually show is that if (C,d) is contractible and $f\circ g$ is homotopic to the identity, then (C',d) is contractible. In fact, suppose $h : C \longrightarrow C$ and $s : C' \longrightarrow C'$ are such that $d\circ h + h\circ d = 1$ and $d\circ s + s\circ d = 1 - f\circ g$. Let $k = s + f\circ h\circ g$. Then

$$\begin{aligned} d\circ k + k\circ d &= d\circ(s + f\circ h\circ g) + (s + f\circ h\circ g)\circ d = d\circ s + s\circ d + d\circ f\circ h\circ g + f\circ h\circ g\circ d \\ &= 1 - f\circ g + f\circ d\circ h\circ g + f\circ h\circ d\circ g = 1 - f\circ g + f\circ(d\circ h + h\circ d)\circ g \\ &= 1 - f\circ g + g\circ f = 1 \end{aligned}$$

so that (C',d) is contractible. ∎

4. Conclusion

We begin with an example that shows that preservation of homology (at least as defined here) is not always the same as preservation of cohomology, even though the objects are the same.

4.1. Example. Here is an example to illustrate the use of Theorem 2.7, as well as illustrate the main problem. Take as domain and range category, the category $\mathcal{A}b$ of abelian groups (although it would be the same if we used finite abelian groups or even finite abelian 2-groups). We will denote by $\mathbf{Z}_n$ the cyclic group of order n. We let $(C, d) = (\mathbf{Z}_8, 4)$, which is to say that the boundary is multiplication by 4. Then $Z(C, d) = \mathbf{Z}_4$, $B(C, d) = \mathbf{Z}_2$ and $H(C, d) = \mathbf{Z}_2$. We define $F : \mathcal{A}b \longrightarrow \mathcal{A}b$ as $\mathbf{Z}_2 \otimes \mathrm{Hom}(\mathbf{Z}_4, -)$. Thus F is the composite of a right exact functor and a left exact one. For any non-zero cyclic 2-group A, it is obvious that $F(A) = \mathbf{Z}_2$ and hence $FH(C, d) \cong HF(C, d)$ as an abstract group. But the game is in what F does to arrows. By writing $F = F_2 \circ F_1$ where $F_1 = \mathrm{Hom}(\mathbf{Z}_4, -)$ and $F_2 = \mathbf{Z}_2 \otimes -$ we can readily compute the following values, in which $\twoheadrightarrow$ and $\rightarrowtail$ denote an epimorphism and monomorphism, resp. In fact, there is more than one epimoprhism $\mathbf{Z}_8 \longrightarrow \mathbf{Z}_4$, but any two are related by an automorphism of the domain or codomain.

$$F(\mathbf{Z}_8 \twoheadrightarrow \mathbf{Z}_4) = F_2(\mathbf{Z}_4 \xrightarrow{2} \mathbf{Z}_4) = \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2$$

$$F(\mathbf{Z}_4 \twoheadrightarrow \mathbf{Z}_2) = F_2(\mathbf{Z}_4 \twoheadrightarrow \mathbf{Z}_2) = \mathbf{Z}_2 \xrightarrow{1} \mathbf{Z}_2$$

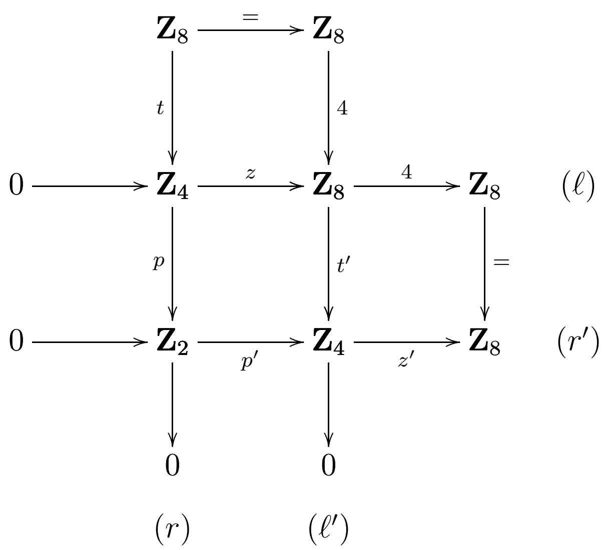
$$F(\mathbf{Z}_2 \rightarrowtail \mathbf{Z}_4) = F_2(\mathbf{Z}_2 \rightarrowtail \mathbf{Z}_4) = \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2$$

$$F(\mathbf{Z}_4 \rightarrowtail \mathbf{Z}_8) = F_2(\mathbf{Z}_4 \xrightarrow{1} \mathbf{Z}_4) = \mathbf{Z}_2 \xrightarrow{1} \mathbf{Z}_2$$

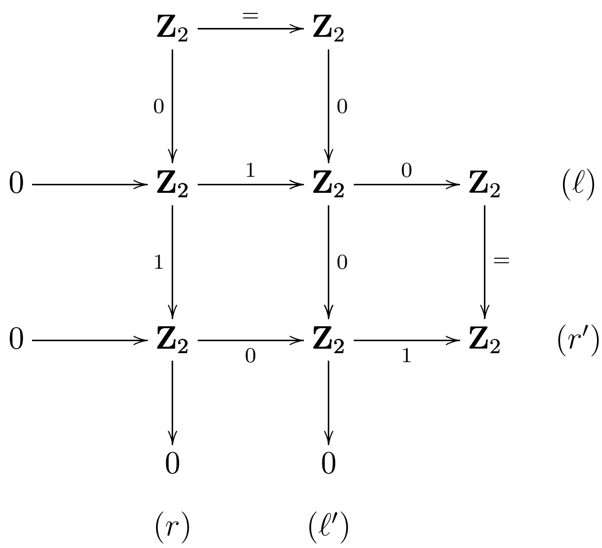
Putting these together, we can, for example, calculate that

$$F(\mathbf{Z}_8 \twoheadrightarrow \mathbf{Z}_2 \rightarrowtail \mathbf{Z}_8) = \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2 = \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2$$

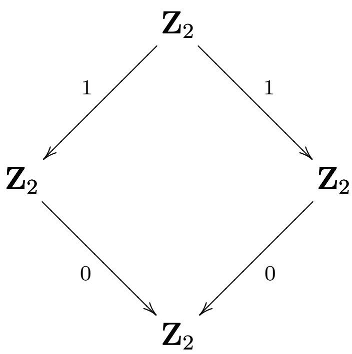
so that $F(C, d) = (\mathbf{Z}_2, 0)$. The diagram of the theorem above becomes



In this diagram, t is the map $\mathbf{Z}_4 \twoheadrightarrow \mathbf{Z}_2 \rightarrowtail \mathbf{Z}_4$, z' is the map $\mathbf{Z}_4 \twoheadrightarrow \mathbf{Z}_2 \rightarrowtail \mathbf{Z}_8$ and the others are forced by exactness. When we apply F, then using the rules above, we get



and we can see by inspection that the sequences labelled (ℓ) and (r) are exact, while those labelled (ℓ') and (r') are not. Thus the bispan is



and hence the two relations are certainly different.

Incidentally, if you replace F by G defined by $G(A) = \mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z}_4 \otimes A)$, the situation is reversed: ℓ and r are 0, while ℓ' and r' are the identity.

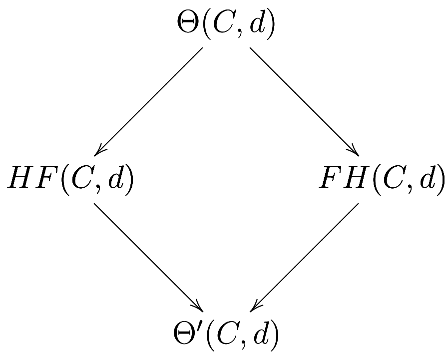
4.2. Conclusion. There are three possible conclusions to choose among.

1. Simply accept the fact that, although homology and cohomology of the same differential object are isomorphic, their preservation conditions are distinct.

2. Require that in order to say that F preserves homology, all of ℓ, r, ℓ', and r' be preserved (if any three are, so is the fourth).

3. Search for a more convincing criterion.

At this point, it is not clear which one to choose. Here is one idea that did not work. Let $\Psi(C,d)$ be defined as a coequalizer of the map $Fd : FC \longrightarrow FC$. The universal mapping property results in maps $\Psi(C,d) \longrightarrow FB(C,d)$ and $\Psi(C,d) \longrightarrow BF(C,d)$. Now let $\Theta(C,d)$ be a coequalizer of the composite $\Psi \longrightarrow FB(C,d) \longrightarrow FC$. This and its dual leads to a bispan



Unfortunately, when this is applied to 4.1, exactly the same problem arises: the upper two arrows are identities and the lower two are 0. In fact, $\Theta = \Phi$ both in that example and in 2.1. Thus it is not obvious whether Θ is actually different from Φ, but the question does not seem worth pursuing.

Reference

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Department of Mathematics and Statistics, McGill University
805 Sherbrooke St. W. Montreal, QC, H3A 2K6
Email: `mbarr@barrs.org`

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