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# COPOWER OBJECTS AND THEIR APPLICATIONS TO FINITENESS IN TOPOI

TOBY KENNEY

Abstract. In this paper, we examine a new approach to topos theory – rather than considering subobjects, look at quotients. This leads to the notion of a copower object, which is the object of quotients of a given object. We study some properties of copower objects, many of which are similar to the properties of a power objects. Given enough categorical structure (i.e. in a pretopos) it is possible to get power objects from copower objects, and vice versa.

We then examine some new definitions of finiteness arising from the notion of a copower object. We will see that the most naturally occurring such notions are equivalent to the standard notions, $K$-finiteness (at least for well-pointed objects) and $\widetilde{K}$-finiteness, but that this new way of looking at them gives new information, and in fact gives rise to another notion of finiteness, which is related to the classical notion of an amorphous set — i.e. an infinite set that is not the disjoint union of two infinite sets.

Finally, We look briefly at two similar notions: potency objects and per objects.

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## 1. Introduction and Background

The standard definition of an (elementary) topos is based on power objects. A power object is the object that classifies subobjects of a fixed object. A natural question to ask is whether the definition in terms of power objects is the only reasonable way to approach elementary topos theory, or whether things like an object that classifies the quotient objects of a fixed object can be used to define an elementary topos. This would mean that in proofs in a general topos, the use of power objects could be replaced by these copower objects. However, it seems likely that while there are many things that are better understood using the standard power object based methods, the fact that copower objects have the same expressive power means that there should also be some things that can be better understood in terms of copower objects, so hopefully a study of copower objects and the best ways to use them will lead to new insights in a variety of areas.

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In Section 2, we show that while copower objects do not produce as simple a definition of a topos, for a pretopos (a coherent category in which all equivalence relations occur as kernel pairs and in which there are disjoint coproducts), the existence of copower objects is equivalent to the existence of power objects. We also prove some simple properties of copower objects that are similar to useful properties of power objects. To express some of the results in this section in full generality, we will need the notion of a quasitopos:

1.1. Definition. [J. Penon] *A* weak subobject classifier *is the codomain of a morphism from 1 of which every strong monomorphism (morphism that doesn't factor through an epi) is a pullback along a unique morphism. A* quasitopos *is a locally cartesian closed cocartesian (i.e. having all finite colimits) category with a weak subobject classifier.*

As any topos is a quasitopos, the reader unfamiliar with quasitopoi may simply read "topos" instead of "quasitopos" wherever it occurs.

Also, for Proposition 2.12, we will need to recall the definitions of the closed subtopos complementary to a given subterminal object $U$ of $\mathcal{E}$, denoted, $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, and an internally widespread subterminal object.

The objects of $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$ (called $c(U)$-sheaves) are objects $X$ of $\mathcal{E}$ such that $X\times U \cong U$. For example, if $\mathcal{E}$ is the topos of sheaves on a topological space or locale $X$, then a subterminal object $U$ in $\mathcal{E}$ will correspond to an open subspace or sublocale of $X$. The subtopos $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$ will then be equivalent to the topos of sheaves on the complement of $U$, with the topology that a subset of this complement is open if its union with $U$ is open in $X$.

A subterminal object $U$ is called *internally widespread* if the subtopos $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$ is boolean — i.e. all its subobject lattices have complements. For example, in sheaves on a 3-point topological space $\{0, 1, 2\}$ whose open sets are exactly those that contain 0, the subterminal object corresponding to $\{0\}$ is internally widespread, since the topology induced on its complement is discrete.

If the reader is unfamiliar with these concepts, Proposition 2.12 is not important in the rest of the paper. I believe that there should be an easier proof of the result, not involving power objects.

In Section 3, we examine the partial order on copower objects. We show that they are in fact lattices. In the topos of sets, these lattices have already been studied. In this context they are called partition lattices, with copower objects being viewed as sets of equivalence relations, rather than quotients. (The order on them is by inclusion of equivalence relations, which is the opposite of the order we consider in this section, since smaller equivalence relations lead to larger quotients). One reason for their importance is a theorem of P. M. Whitman [7] that any lattice can be embedded in a partition lattice.

Whitman's proof of this result is fairly long and complicated. The main ideas are as follows:

1. For each element $x$ in the lattice that we wish to embed, we take some set $S_x$ with elements $l(x)$ and $r(x)$, so that we will embed our lattice in the partition lattice of the disjoint union of the $S_x$, in such a way that the equivalence relation corresponding to $y$ will relate $l(x)$ and $r(x)$ if and only if $x \leqslant y$. This will ensure that the map is injective.

2. The problem with this is that we might have $y_1$ and $y_2$ such that $x \leqslant y_1 \vee y_2$ but neither $x \leqslant y_1$ nor $x \leqslant y_2$. Therefore, we need to do something to force the join of the equivalence relations corresponding to $y_1$ and $y_2$ to relate $l(x)$ and $r(x)$. To achieve this, we introduce some extra points $p_1(y_1, y_2), \ldots, p_5(y_1, y_2)$ to $S_x$, for every pair of points $y_1, y_2$ such that $x \leqslant y_1 \vee y_2$, with the idea that the equivalence relation corresponding to $y_1$ should relate $l(x)$ to $p_1(y_1, y_2)$, $p_2(y_1, y_2)$ to $p_3(y_1, y_2)$, and $p_4(y_1, y_2)$ to $p_5(y_1, y_2)$, while the equivalence relation corresponding to $y_2$ should relate $p_1(y_1, y_2)$ to $p_2(y_1, y_2)$, $p_3(y_1, y_2)$ to $p_4(y_1, y_2)$ and $p_5(y_1, y_2)$ to $r(x)$. (Introducing 5 new points rather than just one is necessary in order to get both joins and meets to work.)

3. This still leaves the problem that if $y_1 \leqslant z_1 \vee z_2$, while neither $y_1 \leqslant z_1$ nor $y_1 \leqslant z_2$ holds, we need to ensure that the join of the equivalence relations corresponding to $z_1$ and $z_2$ should relate $l(x)$ to $p_1(y_1, y_2)$, and the other pairs of elements that we required to be related above. To achieve this, we introduce more points between them – indeed we eventually want to introduce a copy of $S_{y_1}$ between them. We do this iteratively; at each stage, adding a copy of what we currently have in $S_{y_1}$ between them. Then we take the union of all the iterations.

Each element of $S_x$, apart from $r(x)$ can be viewed as a list of pairs of elements of the lattice, with a natural number from 1 to 5, indicating to which of the $p_i$ the element corresponds. ($l(x)$ corresponds to the empty list.) This construction can be implemented in any topos with a natural numbers object (see Example 1.7 for a recap of the definition of a natural numbers object). Therefore, Whitman's theorem should hold in any such topos.

It has also been proved in the topos of finite sets by P. Pudlák and J. Tůma [5]. It would be interesting to determine whether Whitman's theorem actually holds in any topos, but I suspect that this would be hard to prove, since the finite case was an open problem for 30 years, and a proof for a general topos would almost certainly give a new proof of the finite case.

1.2. Remark. For distributive lattices, things are much easier: If $L$ is distributive, then the morphism $L \xrightarrow{f} QL$, sending $a$ to $R_a$ given by $bR_ac \Leftrightarrow (b \vee a = c \vee a)$ is a lattice embedding, so any distributive lattice embeds in its own partition lattice in an arbitrary topos.

Another relevant result: in [3], G. Grätzer and E. T. Schmidt show that every compactly generated lattice in $\underline{\mathcal{S}et}$ is the lattice of global sections of a copower object in $[M, \underline{\mathcal{S}et}]$ for some monoid $M$. (They use the terms *unary algebra* for a functor in $[M, \underline{\mathcal{S}et}]$, and *lattice of congruences* for the lattice of global sections of its copower object.) Again, this proof requires a natural number object, but other than that, should be able to be done in an arbitrary topos.

In Section 4 we look at one possible application of copower objects in the study of finiteness in topoi. Classically, there are many different definitions of finiteness, all of which give rise to the same concept. Constructively, these definitions give rise to different types of finiteness. In some cases, two different definitions of finiteness are only equivalent if the axiom of choice holds. These different definitions of finiteness are useful in different cases – some results hold for one definition of finiteness, while other results will hold for another.

We introduce a new definition of finiteness, based on copower objects, which we call $K^*$-finiteness, because the way in which it is defined is similar to the definition of $\widetilde{K}$-finiteness (recalled below). We then show that this definition is equivalent to $\widetilde{K}$-finiteness, so can be used to help us study $\widetilde{K}$-finiteness.

Here is a recap of some well-known notions of finiteness, and some of their properties that we will need.

1.3. Definition. *Given an object $X$ in a topos, $KX$ is the sub-join-semilattice of $PX$ generated by $\{\} : X \rightarrowtail PX$. $X$ is* Kuratowski finite *(or $K$-finite for short) if $1 \xrightarrow{\ulcorner X \urcorner} PX$ factors through $KX$.*

It is well known that:

1.4. Lemma. *$KA$ is the object of $K$-finite subobjects of $A$, i.e. given $A' \overset{m}{\rightarrowtail} A$, $A'$ is $K$-finite if and only if $\ulcorner m \urcorner : 1 \rightarrowtail PA$ factors through $KA \rightarrowtail PA$.*

$K$-finiteness corresponds to the following induction principle: "If all singleton subobjects of $X$ have property $\Phi$, where $\Phi$ is expressible in the internal language of the topos, and the class of objects with property $\Phi$ is closed under union, then all $K$-finite subobjects of $X$ have property $\Phi$."

$K$-finiteness is closed under quotients, products and unions, but not under subobjects – $K1 = 2$, so all $K$-finite subterminal objects are complemented. To fix this problem, the notion of $\widetilde{K}$-finiteness was invented. This is similar to $K$-finiteness, but starts with the partial map classifier $\widetilde{X}$ of $X$, instead of the subobject $X \xrightarrow{\{\}} PX$. Recall that $\widetilde{X}$ is an object with a canonical monomorphism from $X$, such that morphisms $A \xrightarrow{f} \widetilde{X}$ correspond bijectively to partial morphisms from $A$ to $X$. i.e. for every morphism $A' \xrightarrow{f} X$ from a subobject $A' \rightarrowtail A$ to $X$, there is a unique morphism $A \xrightarrow{f'} \widetilde{X}$ such that

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, tail] & X \arrow[d, tail] \\ A \arrow[r, "f'"] & \widetilde{X} \end{tikzcd}$$

is a pullback. $\widetilde{X}$ can be embedded in $PX$ as the object of subsingleton subobjects of $X$, i.e. the downset in $PX$ generated by $X \xrightarrow{\{\}} PX$.

1.5. Definition. *Given an object $X$ in a topos, $\widetilde{K}X$ is the sub-join-semilattice of $PX$ generated by $\widetilde{X} \rightarrowtail PX$. $X$ is $\widetilde{K}$-finite if $1 \xrightarrow{\ulcorner X \urcorner} PX$ factors through $\widetilde{K}X$.*

$\widetilde{K}$-finiteness is closed under subobjects, quotients, products and unions. It corresponds to the induction principle: "If all subsingleton subobjects of $X$ have property $\Phi$, where $\Phi$

is expressible in the internal language of the topos, and the class of objects with property $\Phi$ is closed under union, then all $\widetilde{K}$-finite subobjects of $X$ have property $\Phi$."

To prove the result that $\widetilde{K}$-finiteness and $K^*$-finiteness are equivalent, we will need to use the theory of numerals, due to P. Freyd [2].

1.6. Definition. *A* pre-numeral *is a poset with a constant 0 and a unary operation s which is order preserving and inflationary.*

*A* numeral *is a minimal pre-numeral i.e. a pre-numeral whose only subobject that contains 0 and is closed under application of s is itself.*

This allows induction over numerals – if $\Phi(0)$ holds and $\Phi(x) \Rightarrow \Phi(sx)$ holds for all $x$, then $\Phi(x)$ holds for all $x$.

1.7. Examples. (i) A natural numbers object is an object $N$ with morphisms $1 \xrightarrow{0} N$ and $N \xrightarrow{s} N$, such that for any other object $X$ with morphisms $1 \xrightarrow{x} X$ and $X \xrightarrow{f} X$, there is a unique morphism $N \xrightarrow{g} X$ such that

$$\begin{tikzcd} 1 \arrow[r, "s"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, "g"] & N \arrow[d, "g"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

commutes i.e. $N$ is an initial prenumeral. Any natural numbers object $N$ is a numeral, since given a subobject $N' \rightarrowtail N$ that contains 0 and is closed under the action of $s$, there is a unique morphism $N \xrightarrow{g} N'$ as above, and the composite $N \xrightarrow{g} N' \rightarrowtail N$ commutes with 0 and $s$, and is therefore the identity on $N$. Thus, $N'$ must be the whole of $N$. If it exists, $N$ is clearly therefore an initial numeral. Conversely, if there is an initial numeral, then it will be a natural numbers object, as any pre-numeral contains a numeral, and any morphism from the initial numeral to it is contained in this numeral.

(ii) If $n$ is a natural number (in the set theoretic sense), then the coproduct of $n$ copies of 1 is a numeral, with 0 picking out the first copy of 1, and the successor map $s$ sending the $k$th copy of 1 to the $(k+1)$th copy of 1 for $k < n$, and sending the $n$th copy to itself.

(iii) In the Sierpinski topos (the functor category $[2, \underline{\mathcal{S}et}]$, where 2 is the category with two objects $A$ and $B$ and one non-identity morphism $A \xrightarrow{f} B$) the functor $F$ with $F(A) = m$, $F(B) = n$ where $m$ and $n$ represent $m$ and $n$ element sets for natural numbers $m \geqslant n$ with $F(f)(i) = \min(i, n)$ is a numeral; 0 picks out the least element of $m$ and $n$, and $s$ increases the elements of $m$ and $n$ by 1 if they are not the top element, and fixes the top element.

(iv) Given a reflexive relation $R$ on $X$, and a subobject $X_0$ of $X$, form a pre-numeral on $PX$ by letting 0 be $X_0$ and letting $s$ send a subobject $X'$ of $X$ to the subobject of elements of $X$ related to something in $X'$. Restricting to the numeral $n$, contained in this pre-numeral, there is a relation, $\phi$ between $n$ and $X$, given by $\phi(0, x)$ if $x \in X_0$ and $\phi(si, x)$ if $(\exists y : X)(\phi(i, y) \wedge (yRx))$. The image of $n$ under this relation is then the $R$-closure of $X_0$.

1.8. Definition. *When the relation $R$ in Example 1.7(iv) is the relation "differ by a singleton" on $PX$, i.e. the relation*

$$(X_1 R X_2) \Leftrightarrow (\exists x : X)(X_1 \cup \{x\} = X_2)$$

*the transitive closure of the singleton containing the empty subset of $X$ is just $KX$. In this case, the numeral produced in forming the transitive closure is called $num_X$.*

In this paper, we use the notation $K_n$-finite for an object $X$ that is $K$-finite and has $num_X \leqslant n$. We also define $\widetilde{num}_X$ in the same way as $num_X$, but using the relation "differ by a partial singleton" – i.e. the construction of $\widetilde{K}X$ as a transitive closure.

1.9. Examples. (i) In $\underline{\mathcal{S}et}$, if $X$ is an $n$ element set for some natural number $n$, then $X$ is $K_n$-finite.

(ii) In the Sierpinski topos, any functor $F$ with $F(A) = m$ and $F(B) = n$ (for $m \geqslant n$ natural numbers) is $K_G$-finite, where $G$ is the numeral with $G(A) = m$ and $G(B) = n$ (defined in Example 1.7(iii)).

We then observe that the definition of $K^*$-finiteness leads to another definition, which we call $K^{**}$-finiteness. This is not even classically equivalent to other definitions of finiteness without the axiom of choice – classically, it is related to the notion of an amorphous set:

1.10. Definition. *A set is* amorphous *if it is infinite but not the disjoint union of two infinite sets.*

1.11. Examples. (i) If $G$ is the group of permutations of an infinite set $X$, given the topology that sets containing the pointwise stabilizers of finite subsets of $X$ are open, then there is a Fraenkel-Mostowski model of set theory that is equivalent as a topos to the topos of continuous $G$-sets for this topology. In this topos, $X$ with the obvious $G$-action (which is continuous) is an amorphous set, since a subset of $X$ in the original model of set theory will only give a subobject of $X \times U$ for some subterminal object $U$ here if its stabilizer is an open subgroup of $G$, and this only occurs if the set is either finite or has finite complement.

Many of the properties of amorphous sets are studied in [6]. In any partition of an amorphous set with infinitely many pieces, all the pieces must be finite, and all but finitely many of the pieces must be the same size. This leads to the following definition.

1.12. Definition. *An amorphous set $X$ is* bounded amorphous *if there is a natural number $n$, such that in any partition of $X$, all but finitely many of the pieces have size at most $n$. $X$ is* strictly amorphous *if this $n$ is 1.*

At the end of Section 4, we show that in a general topos, $K^{**}$-finite objects are exactly some suitably constructive version of bounded amorphous objects with an extra property which we call usual finiteness, which is necessary to compensate for the possibility that the subobject classifier might not be $K$-finite. The proof of this result is fairly technical, and so the reader might wish to move to Section 5 after Remark 4.10. In the proof, we use a well-known theorem from combinatorics:

1.13. Theorem. [F. P. Ramsey] *For any two natural numbers $m$ and $n$, there is an $N$ such that whenever the edges of a complete graph on $N$ vertices is coloured with two colours, red and blue, say, there is either a set of $m$ vertices with all edges between them coloured red, or a set of $n$ vertices with all edges between them coloured blue.*

The form we shall actually need is the following.

1.14. Theorem. *Given a complete graph on a set $X$ of vertices, and a two-colouring of the edges of $X$, if there is a finite bound on the size of monochromatic subsets of $X$, then there is a finite bound on the size of $X$.*

This version of the theorem is constructive, and so true in any topos (there is an infinite version also called "Ramsey's theorem" which is not constructive).

In Section 5, we briefly consider some alternative approaches to topos theory that might require further study, such as the object of all retracts of $X$, or the object of all subquotients of $X$. The former does not lead to a definition of a topos, but might be of use for studying weaker categories, while the latter does define a topos, and so might be of use in topos theory.

## 2. Copower Objects

In this section, we investigate a new approach to topos theory, based on quotients rather than subobjects. We show that in a pretopos (an effective regular, positive coherent category), the existence of copower objects is equivalent to the existence of power objects, and show how in this case, we can construct exponentials directly from copower objects. We show that the assignment of copower objects can be made functorial in either a contravariant or a covariant way. (We will see in Section 3 that these functors are internally adjoint with respect to the canonical lattice structure on copower objects.)

2.1. Definition. *Given an object $A$ in a cartesian category $\mathcal{C}$, a* copower object *for $A$ is an object $QA$ equipped with a cover (a morphism that doesn't factor through any non-identity monos) $QA \times A \xrightarrow{q} RA$ through which the projection $QA \times A \xrightarrow{\pi_1} QA$ factors, such that if $X \times A \xrightarrow{x} Y$ is any other cover through which the projection $\pi_1$ factors, then there is an unique morphism $X \xrightarrow{\llcorner x \lrcorner} QA$ such that in the following diagram, where the vertical composites are projections, both small squares are pullbacks:*

$$\begin{tikzcd}
X \times A \arrow[r, "\llcorner x \lrcorner \times 1_A"] \arrow[d, "x"'] & QA \times A \arrow[d, "q"] \\
Y \arrow[r] \arrow[d] & RA \arrow[d] \\
X \arrow[r, "\llcorner x \lrcorner"'] & QA
\end{tikzcd}$$

2.2. Remark. Of course, the top square will be a pullback whenever the bottom one is and the top square commutes, as the whole rectangle is always a pullback.

2.3. Remark. It would be nice to have a generic quotient, as well as a copower object for each object. Unfortunately, this does not occur in any non-degenerate topos, or indeed in any category with disjoint coproducts, since the functor sending an object to the set of its quotients doesn't send coproducts to products, and so cannot be representable.

2.4. Examples. (i) In $\underline{\mathcal{S}et}$, $QA$ is just the set of all isomorphism classes of surjections with domain $A$, or equivalently the set of all equivalence relations on $A$. $RA$ is the set of ordered pairs consisting of an equivalence relation on $A$ and an equivalence class for this relation, or equivalently pairs consisting of a surjection with domain $A$ and an element of the codomain.

(ii) In a poset with a top element, the top element is a copower object for every element, since the only quotients are the identities.

(iii) In $[\mathcal{C}, \underline{\mathcal{S}et}]$, copower objects are given by the Yoneda lemma – for a functor $F$, and an object $X$ of $\mathcal{C}$,

$$QF(X) \cong [\mathcal{C}, \underline{\mathcal{S}et}](\mathcal{C}(X, \_), QF) \cong \{\text{quotients of } \mathcal{C}(X, \_) \times F \text{ over } \mathcal{C}(X, \_)\}$$

2.5. Lemma. *(i) In a regular category, the assignment $A \mapsto QA$ can be made into a contravariant functor $Q$. This functor sends covers to monos, and in a category with monic pushouts of monos, it sends monos to covers.*

*(ii) In a regular category with pullback-stable pushouts of covers (e.g. a quasitopos), $A \mapsto QA$ can be made into a covariant functor $\exists$. This functor preserves covers, and if monos are stable under pushout then it preserves them also.*

Proof. (i) Let $A \xrightarrow{f} B$ be a morphism. We need to determine the morphism $QB \xrightarrow{Qf} QA$. In $\underline{\mathcal{S}et}$, it sends a quotient $q$ of $B$ to the cover part of the composite $qf$, or in terms of equivalence relations, it takes the pullback along $f$.

For a general topos, form the composite $QB \times A \xrightarrow{1\times f} QB \times B \xrightarrow{q} RB$. Let its cover-mono factorization be $QB \times A \xrightarrow{c} I \rightarrowtail^{i} RB$. $\pi_1$ factors through $c$, so take $QB \xrightarrow{Qf} QA$ to be $\llcorner c \lrcorner$. It is a straightforward diagram chase to verify that this is indeed a functor.

If $f$ is a cover then for any $X \xrightarrow{\llcorner q \lrcorner} QA$, $Qf \llcorner q \lrcorner = \llcorner q(f \times 1_X) \lrcorner$, so that whenever $Qf \llcorner q \lrcorner = Qf \llcorner q' \lrcorner$, $qf = q'f$. $f$ is epi, so this can only happen if $q = q'$. Therefore, $Qf$ is mono.

If $f$ is mono and there are pushouts that preserve monos then the pushout of any quotient $q$ of $X \times A$ along $1_X \times f$ gives a quotient $q'$ of $X \times B$ such that $Qf \llcorner q' \lrcorner = \llcorner q \lrcorner$. Therefore, $Qf$ is a cover.

(ii) Let $A \xrightarrow{f} B$ be a morphism. We need to determine the morphism $QA \xrightarrow{\exists f} QB$. In $\underline{\mathcal{S}et}$, it sends a quotient $q$ of $A$ to its pushout along $f$, or in terms of equivalence relations, $\exists f(R)$ relates elements $a$ and $b$ of $B$ if and only if $a = f(c)$ and $b = f(d)$ for some elements $c$ and $d$ of $A$ that are related by $R$.

Covers are preserved under pushouts. Given $A \xrightarrow{f} B$, form the pushout:

$$\begin{tikzcd}
QA \times A \arrow[r, "1\times f"] \arrow[d, "q"'] & QA \times B \arrow[d, "x"] \\
RA \arrow[r] & R_f
\end{tikzcd}$$

$\pi_1$ clearly factors through $x$. We set $\exists f = \llcorner x \lrcorner$. It remains to check that composition works. Given $A \xrightarrow{f} B \xrightarrow{g} C$, there is a diagram:

$$\begin{tikzcd}
QA \times B \arrow[r, "\exists f \times 1_B"] \arrow[d] & QB \times B \arrow[r, "1_{QB}\times g"] \arrow[d, "q"] & QB \times C \arrow[d] \\
R_f \arrow[r] \arrow[d] & RB \arrow[r] \arrow[d] & R_g \arrow[dl] \\
QA \arrow[r, "\exists f"] & QB &
\end{tikzcd}$$

where both left-hand squares are pullbacks. To show that $\exists(gf) = \exists g \exists f$, it is sufficient to show that in the following diagram:

$$\begin{tikzcd}
QA \times B \arrow[r, "1_{QA}\times g"] \arrow[d] & QA \times C \arrow[r, "\exists f \times 1_C"] \arrow[d] & QB \times C \arrow[d] \\
R_f \arrow[r] \arrow[dr] & R_{gf} \arrow[r] \arrow[d] & R_g \arrow[d] \\
 & QA \arrow[r, "\exists f"] & QB
\end{tikzcd}$$

the right-hand squares are pullbacks, and the left-hand one is a pushout. However, this follows trivially from the fact that $\exists f^*$ preserves pushouts of covers.

If $f$ is a cover then for any quotient $q$ of $X \times B$ the pushout of $q(1_X \times f)$ along $1_X \times f$ is just $q$, so $\exists f$ is a cover.

If $f$ is mono, and monos are stable under pushout, then any quotient $q$ of $X \times A$ is the cover part of the diagonal of the pushout square

$$\begin{tikzcd}
X \times A \arrow[r, tail, "1_X\times f"] \arrow[d, "q"] & X \times B \arrow[d, "r"] \\
Q \arrow[r, tail] & R
\end{tikzcd}$$

so $Qf\exists f$ is the identity, making $\exists f$ mono. ■

2.6. Proposition. *In a quasitopos $\mathcal{E}$, all objects have copower objects.*

Proof. Given an object $A$, the copower object $QA$ is just the object of (strong) equivalence relations on $A$, i.e. the object

$$\{R : P(A \times A) | (\forall a, b, c : A)((a,a) \in R \wedge (a,b) \in R \Rightarrow (b,a) \in R \wedge \\ (((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R)\}$$

The object $RA$ is then the coequalizer of the equivalence relation on $QA \times A$ that relates $(R, a)$ to $(S, b)$ if and only if $R = S$ and $(a, b) \in R$.

Given an arbitrary cover $X \times A \xrightarrow{x} Y$ through which the projection factors, $\llcorner x \lrcorner$ is the factorization of the name of the relation

$$\{(m, a, b) : X \times A \times A | x(m, a) = x(m, b)\}$$

(which is a morphism $X \longrightarrow P(A \times A)$) through $QA \rightarrowtail P(A \times A)$. Let the pullback of $RA$ along $\llcorner x \lrcorner$ be $Y'$, and let the morphism $(\llcorner x \lrcorner)^*(q)$ be $X \times A \xrightarrow{x'} Y'$. Its kernel pair is the pullback of the kernel pair of $q$, which is the relation on $X \times A$ that relates $(m, a)$ to $(n, b)$ if and only if $m = n$ and $x(m, a) = x(m, b)$. This relation is the kernel pair of $x$. Therefore, $Y' = Y$ and $x' = x$.

Let $f$ be any morphism with $f^*(q) = x$. Then the composite

$$X \xrightarrow{f} QA \rightarrowtail P(A \times A)$$

names the kernel pair of $x$, so $f = \llcorner x \lrcorner$ as $QA \rightarrowtail P(A \times A)$ is monic. ∎

There is an analogue of the Beck-Chevalley condition for copower objects in a quasitopos.

2.7. Lemma. *In a quasitopos, let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

*be a pushout. Then*

$$\begin{tikzcd} QB \arrow[r, "Qf"] \arrow[d, "\exists h"] & QA \arrow[d, "\exists g"] \\ QD \arrow[r, "Qk"] & QC \end{tikzcd}$$

*commutes.*

Proof. Let $QB \times D \longrightarrow Y$ be the pushout of $QB \times B \longrightarrow RB$ along $1_{QB} \times h$. $Qk\exists h$ corresponds to the quotient $QB \times C \longrightarrow Z$, where $Z \rightarrowtail Y$ is the image of $QB \times C \xrightarrow{1_{QB} \times k} QB \times D \longrightarrow Y$. On the other hand, if $I \rightarrowtail RB$ is the image of

$$QB \times A \xrightarrow{1_{QB} \times f} QB \times B \longrightarrow RB$$

then $\exists g Qf$ corresponds to the pushout $Z'$ of $QB \times A \longrightarrow I$ along $1_{QB} \times g$. To show that these are the same, it is merely necessary to show that the morphism from $Z'$ to $Y$ is mono. Consider the following diagram:

$$\begin{tikzcd}
QB \times A \arrow[rr] \arrow[dr] \arrow[dd] & & QB \times B \arrow[dr] \arrow[dd] & \\
& QB \times C \arrow[rr] \arrow[dd] & & QB \times D \arrow[dd] \\
I \arrow[rr, tail] \arrow[dr] & & RB \arrow[dr] & \\
& Z' \arrow[rr] & & Y
\end{tikzcd}$$

The left-hand, right-hand and top squares are pushouts, so the bottom square is also a pushout, making $Z' \longrightarrow Y$ monic. ∎

This shows that in a quasitopos $Q$ sends covers to split monos and $\exists$ sends covers to split epis. The proof of Lemma 2.5 has already shown that in a topos $Q$ sends monos to split epis and $\exists$ sends monos to split monos.

2.8. THEOREM. *A pretopos in which all objects have copower objects is a topos.*

PROOF. The idea here is that subobjects of $A$ correspond bijectively to equivalence relations on $A \amalg 1$ whose only non-trivial equivalence class is the equivalence class containing 1. The object of these equivalence relations is a retract of the object of all equivalence relations on $A$, since any equivalence relation can be restricted to just the equivalence class containing 1. To obtain the power object $PA$, we just need to construct the composite $Q(A \amalg 1) \longrightarrow PA \rightarrowtail Q(A \amalg 1)$, then we can take its cover-mono factorization.

First we construct the pullback:

$$\begin{tikzcd}
P \arrow[r] \arrow[dd, tail, "p"] & Q(A \amalg 1) \arrow[d, tail, "1\times\nu_2"] \\
& Q(A \amalg 1) \times (A \amalg 1) \arrow[d] \\
Q(A \amalg 1) \times (A \amalg 1) \arrow[r] & R(A \amalg 1)
\end{tikzcd}$$

The right vertical morphism is monic because it is split by $R(A \amalg 1) \longrightarrow Q(A \amalg 1)$.

The idea here is that $P$ should contain the equivalence class of 1 for each equivalence relation on $A \amalg 1$. We will next restrict to the equivalence relation $q'$ that relates two elements $(q, a)$ and $(q, b)$ if and only if $q$ relates both $a$ and $b$ to 1. The morphism $\llcorner q' \lrcorner$ will then send two quotients of $A$ to the same thing if and only if they relate the same subset of $A$ to 1, so its image will be isomorphic to the power object of $A$.

To do this, construct the intersection $m$ in $\mathrm{Sub}(Q(A \amalg 1) \times (A \amalg 1) \times (A \amalg 1))$ of $p_{12}$ and $p_{13}$, where $p_{12}$ is just the Cartesian product of the subobject $p$ with $A \amalg 1$, and $p_{13}$ is the composite of this with the isomorphism that swaps the two copies of $A \amalg 1$ – i.e. in $\underline{\mathcal{S}et}$, $p_{12}$ is the set of triples $(q, a, b)$ such that $(q, a) \in p$, while $p_{13}$ is the set of triples such that $(q, b) \in p$, and thus, $m$ is the set of triples $(Q, a, b)$ such that both $(q, a) \in p$ and

$(q,b) \in p$. Let $Q(A \amalg 1) \times (A \amalg 1) \xrightarrow{q'} R'(A \amalg 1)$ be the coequalizer of $\pi_{12}m$ and $\pi_{13}m$, where $\pi_{12}$ and $\pi_{13}$ are the projections. This coequalizer exists because $(\pi_{12}m, \pi_{13}m)$ is an equivalence relation, and is therefore effective. It is a map over $Q(A \amalg 1)$ because $\pi_1$ has the same composite with $\pi_{12}$ and $\pi_{13}$. $Q(A \amalg 1) \xrightarrow{c} PA \rightarrowtail{i} Q(A \amalg 1)$ is the cover-image factorization of $\llcorner q' \lrcorner$.

The universal relation, $\in \rightarrowtail PA \times A$ occurs as the pullback of the relation

$$S \rightarrowtail Q(A \amalg 1) \times (A \amalg 1) \times (A \amalg 1)$$

along $PA \times A \cong PA \times A \times 1 \xrightarrow{i \times \nu_1 \times \nu_2} Q(A \amalg 1) \times (A \amalg 1) \times (A \amalg 1)$, where $S$ is the kernel pair of $Q(A \amalg 1) \times (A \amalg 1) \xrightarrow{q} R(A \amalg 1)$ and $\nu_1$ and $\nu_2$ are the coprojections.

It remains to show that this is indeed a power object for $A$. Given an arbitrary relation $T \overset{m}{\rightarrowtail} X \times A$, let $m' = (m_{12} \cap m_{13}) \cup (1_X \times \Delta_{A\amalg 1})$, where $m_{12}$ is the subobject:

$$(T \amalg X) \times (A \amalg 1) \xrightarrow{(m\amalg 1)\times 1} (X \times A \amalg X) \times (A \amalg 1) \cong X \times (A \amalg 1) \times (A \amalg 1)$$

and $m_{13}$ is similar, but with the order of the copies of $A \amalg 1$ swapped. The idea is that $m'$ should relate $(x,a)$ to $(x,1)$ if and only if $(x,a) \in m$ (consider the case $X = 1$, where $m_{12}$ is just $(m \amalg 1) \times (A \amalg 1)$, i.e. it relates the elements of $m \amalg 1$ to all elements of $A$, while $m_{13}$ relates elements of $A$ to elements of $m \amalg 1$, so that two elements of $A$ are related by $m_{12} \cap m_{13}$ if and only if they are both in $m \amalg 1$).

Let $X \times (A \amalg 1) \xrightarrow{x} Y$ be the coequalizer of $\pi_{12}m'$ and $\pi_{13}m'$. $(\pi_{12}m', \pi_{13}m')$ is an equivalence relation, so the coequalizer exists. The composite $X \xrightarrow{\llcorner x \lrcorner} Q(A \amalg 1) \xrightarrow{c} PA$ is then the name of $T$.

The pullback of $R'(A \amalg 1)$ along $\llcorner x \lrcorner$ is $Y$, so the pullback of its kernel pair is $T'$, by effectiveness, and $\llcorner q' \lrcorner \llcorner x \lrcorner = \llcorner x \lrcorner$. Therefore, the pullback of $\in$ along

$$X \times A \xrightarrow{\llcorner x \lrcorner \times 1_A} Q(A \amalg 1) \times A \xrightarrow{c \times 1_A} PA \times A$$

is the pullback of $T'$ along $X \times A \xrightarrow{1_X \times \nu_1 \times \nu_2} X \times (A \amalg 1) \times (A \amalg 1)$, which is $T$.

On the other hand, if the pullback of $\in$ along $X \times A \xrightarrow{f \times 1} PA \times A$ is $T$, then $if = \llcorner y \lrcorner$ for some $X \times (A \amalg 1) \xrightarrow{y} Y'$ for which the pullback of $Z \overset{z}{\rightarrowtail} X \times (A \amalg 1) \times (A \amalg 1)$ along $1_X \times \nu_1 \times \nu_2$ is $T$, where $(\pi_{12}z, \pi_{13}z)$ is the kernel pair of $y$. Also, $if = icif$ factors through $\llcorner q' \lrcorner$, meaning that $Z$ only relates $(v,a)$ and $(v,b)$ if it also relates $(v,a)$ and $(v,1)$. This forces $y = x$, since $y$ is the coequalizer of its kernel pair, and hence, $f = c\llcorner x \lrcorner$. ∎

The construction of exponentials from copower objects can be done directly by considering the cographs:

2.9. Theorem. *In a topos $\mathcal{E}$, the exponential $A^B$ is the intersection of the pullbacks:*

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, tail] & 1 \arrow[d, "\llcorner 1_A \lrcorner"] \\ Q(A \amalg B) \arrow[r, "Q\nu_1"] & QA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P_2 \arrow[r] \arrow[d, tail] & 1 \arrow[d, "\llcorner 1 \amalg B \lrcorner"] \\ Q(A \amalg B) \arrow[r, "\exists g"] & Q(1 \amalg B) \end{tikzcd}$$

*in* $\mathrm{Sub}(Q(A \amalg B))$*, where* $g$ *is the morphism* $A \amalg B \xrightarrow{g} 1 \amalg B$ *induced by* $A \longrightarrow 1$ *and* $1_B$*, and* $1 \amalg B$ *is used to refer to the morphism* $1 \amalg B \longrightarrow 1$.

PROOF. The idea here is that given a quotient $q$ of $A \amalg B$, $q$ is the cograph of a morphism $B \xrightarrow{f} A$ if and only if $q\nu_1 \cong 1_A$. Since a topos is balanced, it suffices to show that $q\nu_1$ is monic and epic. However, $q$ is in $P_1$ if and only if $q\nu_1$ is monic, since the cover part of the cover-mono factorization of $q\nu_1$ is the identity. On the other hand, $q$ is in $P_2$ if and only if $q\nu_1$ is epic, since

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\nu_1"] & 1 \arrow[d, "\nu_1"] \\ A \amalg B \arrow[r] & 1 \amalg B \end{tikzcd}$$

is a pushout, so the pushout of $q\nu_1$ along $A$ is the identity on 1, so $q\nu_1$ must be a cover.

More formally, let $f$ be the morphism $X \times A \rightarrowtail X \times (A \amalg B) \xrightarrow{x} Y$. We need to prove:

*(i) The morphism* $X \xrightarrow{\llcorner x \lrcorner} Q(A \amalg B)$ *factors through* $P_1$ *if and only if* $f$ *is monic.*

*(ii)* $\llcorner x \lrcorner$ *factors through* $P_2$ *if and only if* $f$ *is epic.*

(i): Let $Q(A \amalg B) \times A \longrightarrow I \rightarrowtail R(A \amalg B)$ be the cover-mono factorization of $q\nu_1$. The pullback of $I$ along $\llcorner x \lrcorner$ is $X \times A$ if and only if $Q\nu_1 \llcorner x \lrcorner$ factors through $\llcorner 1_A \lrcorner$, or equivalently, if and only if $\llcorner x \lrcorner$ factors through $P_1$. The pullback of $I$ along $\llcorner x \lrcorner$ is the image of $f$, so saying that it is $X \times A$ is the same as saying that $f$ is monic.

(ii): Let

$$\begin{tikzcd} Q(A \amalg B) \times (A \amalg B) \arrow[r] \arrow[d] & Q(A \amalg B) \times (1 \amalg B) \arrow[d, "y"] \\ R(A \amalg B) \arrow[r] & W \end{tikzcd}$$

be a pushout. Then $\exists_g \llcorner x \lrcorner$ factors through $\llcorner 1 \amalg B \lrcorner$ if and only if $\llcorner x \lrcorner^*(Y) = X$. We therefore want to show that $\llcorner x \lrcorner^*(W) = X$ if and only if $f$ is epic.

$$\begin{tikzcd} Q(A \amalg B) \times A \arrow[r] \arrow[d] & Q(A \amalg B) \arrow[d, tail, "y'"] \\ R(A \amalg B) \arrow[r] & W \end{tikzcd}$$

Is also a pushout. (The right-hand arrow is monic because it has a left inverse.) Let $Q(A \amalg B) \times A \longrightarrow Z \rightarrowtail R(A \amalg B)$ be the cover-mono factorization. Then

$$\begin{tikzcd} Z \arrow[r] \arrow[d, tail] & Q(A \amalg B) \arrow[d, tail, "y'"] \\ R(A \amalg B) \arrow[r] & W \end{tikzcd}$$

is a pushout, since pushouts preserve cover-mono factorizations, and therefore a pullback. Let $X \xrightarrow{h} W$ be the pullback of $\llcorner x \lrcorner$ along the left inverse of $y'$ (the domain is $X$ because the left-hand edge in this pullback is an isomorphism). $h$ factors through $y'$, since the pullback of $y'$ along it is the identity. Therefore, in the following diagram, all squares are pullbacks.

$$\begin{tikzcd} X \times A \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\cong"] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "h"] & X \arrow[d, "\llcorner x \lrcorner"] \\ Q(A \amalg B) \times A \arrow[r] & Z \arrow[r, tail] & R(A \amalg B) \arrow[r] & W \arrow[r] & Q(A \amalg B) \end{tikzcd}$$

Therefore, $f$ is epi.

Having proved (i) and (ii), morphisms $C \longrightarrow P_1 \cap P_2$ correspond bijectively to quotients $C \times (A \amalg B) \xrightarrow{c} C \times A$ over $C$, whose restriction to $C \times A$ is the identity. Each such morphism is uniquely determined by its restriction to $C \times B$, and the first projection is fixed, so the morphism is uniquely determined by the second projection $C \times B \longrightarrow A$, i.e. $P_1 \cap P_2$ is the exponential $A^B$. ∎

2.10. Remark. The above proof did require $\mathcal{E}$ to be a topos, since in general, the pushout of a mono won't also be a pullback, so the above does not show that categories with copower objects are cartesian closed in any weaker cases.

2.11. Proposition. *Any slice of a cocartesian regular category with copower objects has copower objects.*

Proof. Given $A \xrightarrow{f} B$ in $\mathcal{C}/B$, Let $A \xrightarrow{f'} B' \overset{f''}{\rightarrowtail} B$ be the cover-mono factorization of $f$. The copower object $Qf$ is given by the pullback:

$$\begin{tikzcd} Qf \arrow[r] \arrow[d, tail] & \leqslant \arrow[d, tail] \\ B \times QA \arrow[r, "\llcorner g \lrcorner \times 1"] & QA \times QA \end{tikzcd}$$

where $g$ is the pushout of $f'$ along the graph of $f$, i.e. the quotient that identifies $(a, b)$ and $(a', b')$ if and only if $b = b'$ and $f(a) = f(a') = b$, and $\leqslant$ is the order relation on $QA$, where quotients are ordered by factorization. (This order on $QA$ will be studied in more detail in Section 3.) $Qf$ can be thought of as the object of pairs $(b, R)$ where $b$ is an element of $B$ and $R$ is a quotient of $f^{-1}(\{b\})$.

Given an object $C \xrightarrow{h} B$ of $\mathcal{C}/B$, morphisms from $h$ to $Qf$ in $\mathcal{C}/B$ correspond to morphisms from $C$ to $B \times QA$ whose second projection names a quotient through which $h^*(g)$ factors, i.e. to quotients of $h \times f$ in $\mathcal{C}/B$. ∎

2.12. Proposition. *In a topos $\mathcal{E}$, an object $A$ satisfies $A \cong QA$ if and only if there is a pushout:*

$$\begin{tikzcd} U \arrow[r, tail] \arrow[d, tail] & 1 \arrow[d, tail] \\ 1 \arrow[r, tail] & A \end{tikzcd}$$

*where $U$ is an internally widespread subterminal object in $\mathcal{E}$.*

Proof. Given a pushout diagram as above, $A$ is a $c(U)$-sheaf, and $A \cong QA$ clearly holds in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, so it holds in $\mathcal{E}$, since any quotient of $A$ is also a $c(U)$-sheaf.

Conversely, let $A \cong QA$. There are two canonical morphisms $1 \underset{\perp}{\overset{\top}{\rightrightarrows}} A$, where $\top$ corresponds to the identity on $A$, and $\perp$ to the map from $A$ to its support (which is 1, as $A$ admits a morphism from 1). Let $U \rightarrowtail 1$ be the equalizer of $\top$ and $\perp$, and let $A \xrightarrow{c} C$ be their coequalizer. Clearly, $U$ is also the equalizer of $\top$ and $\llcorner c \lrcorner$. Let $V \rightarrowtail 1$ be the equalizer of $\perp$ and $\llcorner c \lrcorner$.

As elements of $A$ are quotients of $A$, they are partially ordered by factorization, i.e. $c_1 \leqslant c_2$ if and only if $c_1$ factors through $c_2$, so that $\perp$ is a bottom element. Note that this is the reverse of the inclusion ordering on the corresponding equivalence relations. Now for any $a \in A$, we have the equivalence relation $R$, given by

$$(bRb') \Leftrightarrow ((b \leqslant a) \wedge (b' \leqslant a)) \vee (b = b')$$

This gives a morphism $A \xrightarrow{f} A$. $f$ is monic because $A$ has a bottom element, so if $f(a) = f(b)$ then both $f(a)$ and $f(b)$ relate $a$ to $\perp$, so $a \leqslant b$, and similarly $b \leqslant a$, making $b = a$, as required.

Define an equivalence relation $S$ by

$$aSb \Leftrightarrow (a = b) \vee ((\llcorner c \lrcorner \leqslant a) \wedge (\llcorner c \lrcorner \leqslant b))$$

$\llcorner S \lrcorner$ is in the image of $f$ if and only if $c = \perp$, which is an assertion whose truth value is $V$. Let $V \overset{m}{\rightarrowtail} A$ be the pullback of $\llcorner S \lrcorner$ along $f$. There is a morphism $P^m A \xrightarrow{g} A$, where $P^m A$ is the object of subobjects of $A$ containing $m$, given by $x \mapsto R_x$, where

$$aR_x b \Leftrightarrow (a = b) \vee (a \in (\exists_f(x) \cup \{\llcorner S \lrcorner\}) \wedge b \in (\exists_f(x) \cup \{\llcorner S \lrcorner\}))$$

Now $g(x) = g(x')$ if and only if $\exists_f(x) \cup \{\llcorner S \lrcorner\} = \exists_f(x') \cup \{\llcorner S \lrcorner\}$, which occurs if and only if $x \cup m = x' \cup m$. Therefore, in $\mathcal{S}\mathbf{h}_{c(V)}(\mathcal{E})$, $g$ becomes a mono $PA \rightarrowtail A$, meaning that $\mathcal{S}\mathbf{h}_{c(V)}(\mathcal{E})$ is degenerate, i.e. that $V = 1$.

Thus $A$ is a quotient of 2. In $\mathcal{S}\mathbf{h}_{c(U)}(E)$, $\top$ and $\perp$ are disjoint, so that $A = 2$. Therefore, in this topos $2 \cong Q2 \cong \Omega$. Therefore, $\mathcal{S}\mathbf{h}_{c(U)}(E)$ is boolean. Hence $U$ is internally widespread and $A$ is the pushout of $U \rightarrowtail 1$ along itself. ■

## 3. Lattice Structure of Copower Objects

Recall that there is a canonical way of inducing a partial order on a copower object by factorization as in Proposition 2.12 – a quotient $X \xrightarrow{q} Y$ is less than or equal to another quotient $X \xrightarrow{q'} Y'$ if $q$ factors through $q'$. In this section, we study this partial order in more detail, and show that in a topos any copower object is a complete lattice, and the canonical functors $Q$ and $\exists$ send any morphism to an adjoint pair of order preserving lattice homomorphisms.

3.1. Proposition. *In a quasitopos, $QA$ is an internal lattice for any $A$.*

Proof. Let $X \times A \xrightarrow{y} Y \xrightarrow{y'} X$ and $X \times A \xrightarrow{z} Z \xrightarrow{z'} X$ be two quotients. The least upper bound of $x$ and $y$ is the image of

$$X \times A \xrightarrow{1\times\Delta_A} X \times A \times A \longrightarrow P$$

where $P$ is the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Z \arrow[d, "z'"] \\ Y \arrow[r, "y'"] & X \end{tikzcd}$$

since any upper bound factors through $P$ and is a quotient of $X$. Therefore, we seek a quotient $q'$ of $QA \times QA \times A$ over $QA \times QA$ whose pullback along $(\llcorner x \lrcorner, \llcorner y \lrcorner)$ is this image. Then $QA \times QA \xrightarrow{\vee} QA$ will just be $\llcorner q' \lrcorner$.

The pullback of

$$QA \times QA \times A \xrightarrow{1_{QA\times QA}\times\Delta_A} QA \times QA \times A \times A \xrightarrow{q\times q} RA \times RA$$

along $(\llcorner x \lrcorner, \llcorner y \lrcorner)$ is

$$X \times A \xrightarrow{1\times\Delta_A} X \times A \times A \longrightarrow P$$

so $q'$ is just the cover part of the cover-mono factorization of $(q \times q)(1_{QA\times QA} \times \Delta_A)$, where $q$ is the generic cover $QA \times A \longrightarrow RA$.

The greatest lower bound of $x$ and $y$ is clearly the pushout

$$\begin{tikzcd} X \times A \arrow[r, "y"] \arrow[d, "z"] & Y \arrow[d] \\ Z \arrow[r] & S' \end{tikzcd}$$

So we seek a quotient of $QA \times QA \times A$ over $QA \times QA$ whose pullback along $(\llcorner x \lrcorner, \llcorner y \lrcorner)$ is the diagonal of the above pushout. The diagonal $s$ of the pushout

$$\begin{tikzcd} QA \times QA \times A \arrow[r] \arrow[d] & QA \times RA \arrow[d] \\ RA \times QA \arrow[r] & S \end{tikzcd}$$

clearly has this property. We therefore obtain the binary meet $\wedge$ as $\llcorner s \lrcorner$. ∎

3.2. Proposition. *In a topos, $QA$ is an internal complete lattice for any object $A$.*

Proof. Given a subobject $S \overset{s}{\rightarrowtail} X \times QA$, the projection to $QA$ is $\llcorner t \lrcorner$ for some quotient $S \times A \xrightarrow{t} T$. The meet of $S$ is the pushout

$$\begin{tikzcd} S \times A \arrow[r] \arrow[d, "t"] & X \times A \arrow[d, "z"] \\ T \arrow[r] & Z \end{tikzcd}$$

because for any quotient $r : QA$ in $S$, any $a$ and $b$ related by $r$, will also be related by $z$, as $(r, a)$ and $(r, b)$ are related by $t$. The morphism $PQA \xrightarrow{\bigwedge} QA$ must therefore be $\llcorner y \lrcorner$ for some quotient $PQA \times A \xrightarrow{y} Y \longrightarrow PQA$, such that for any subobject $S \rightarrowtail X \times QA$, the following squares are both pullbacks:

$$\begin{tikzcd}
X \times A \arrow[r, "\ulcorner S \urcorner \times 1"] \arrow[d, "z"] & PQA \times A \arrow[d, "y"] \\
Z \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r, "\ulcorner S \urcorner"] & PQA
\end{tikzcd}$$

where $z$ is the meet of the elements of $S$ as described above.

Let $\in_{QA} \times A \longrightarrow I \rightarrowtail PQA \times RA$ be the cover-mono factorization of

$$\in_{QA} \times A \rightarrowtail PQA \times QA \times A \longrightarrow PQA \times RA$$

This means that $\in_{QA} \times A \longrightarrow I$ corresponds to the morphism $\in_{QA} \rightarrowtail PQA \times QA \xrightarrow{\pi_2} QA$. We will show that the morphism $y$ we require is the pushout:

$$\begin{tikzcd}
\in_{QA} \times A \arrow[r, tail] \arrow[d] & PQA \times QA \times A \arrow[r, "\pi_{13}"] & PQA \times A \arrow[d, "y"] \\
I \arrow[rr] & & Y
\end{tikzcd}$$

Since $T \longrightarrow X \times RA$ is a pullback of $S \rightarrowtail X \times QA$, it is mono. Also, in the following diagram, both squares are pullbacks.

$$\begin{tikzcd}
S \times A \arrow[r] \arrow[d, tail] & \in_{QA} \times A \arrow[d, tail] \\
X \times QA \times A \arrow[r, "\ulcorner S \urcorner \times 1"] \arrow[d] & PQA \times QA \times A \arrow[d] \\
X \times RA \arrow[r, "\ulcorner S \urcorner \times 1"] & PQA \times RA
\end{tikzcd}$$

Therefore, in the following cube, the left, right, top and bottom faces are pullbacks (the bottom face is a pullback because cover-mono factorizations are pullback-stable).

$$\begin{tikzcd}
S \times A \arrow[rr] \arrow[dr] \arrow[dd] & & X \times A \arrow[dr] \arrow[dd] & \\
& \in_{QA} \times A \arrow[rr, crossing over] & & PQA \times A \arrow[dd] \\
T \arrow[rr, tail] \arrow[dr] & & Z \arrow[dr] & \\
& I \arrow[rr, tail] \arrow[from=uu, crossing over] & & Y
\end{tikzcd}$$

The front face is a pushout, and pullback along $Z \longrightarrow Y$ preserves pushouts because it has a right adjoint, so the back square is also a pushout.

The existence of infinitary meets ensures the existence of infinitary joins. However, it is enlightening to construct the joins explicitly. Given $S$ and $S \xrightarrow{t} T$ as above, the join of $S$ must be a quotient $X \times A \xrightarrow{z'} Z' \longrightarrow Z$ such that for any quotient $X \times A \xrightarrow{y} Y \longrightarrow X$, the pullback of $y$ along $S \xrightarrow{\pi_1 s} X$ factors through $t$ if and only if $z'$ factors through $y$, since the pullback factoring through $t$ means that the quotient is $\geqslant$ all quotients in $S$. This means that $z'$ must be $\Pi_{\pi_1 s}(t)$, where $\Pi_{\pi_1 s}$ is the functor right adjoint to pullback along $\pi_1 s$.

We therefore need to get $\Pi_{\pi_1 s}(t)$ as the pullback along $\ulcorner S \urcorner$ of some

$$PQA \times A \xrightarrow{w} W \xrightarrow{w'} PQA$$

for any $S$. The required object $W$ is just $\Pi_{\in_{QA} \longrightarrow QA}(I)$, since pullback preserves $\Pi$-functors and

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \in_{QA} \arrow[d] \\ X \arrow[r, "\ulcorner S \urcorner"] & PQA \end{tikzcd}$$

is a pullback. The infinitary join $\bigvee$ is then just $\llcorner w \lrcorner$. ∎

3.3. Proposition. *For any $A \xrightarrow{f} B$, $Qf \dashv \exists f$ as order-preserving morphisms. Thus, $Qf$ is a join-homomorphism and $\exists f$ is a meet-homomorphism, and they are complete in a topos.*

Proof. Note that $\exists f Qf$ corresponds to the morphism $QB \times B \longrightarrow P$ in the following diagram, where the top left square is a pushout:

$$\begin{tikzcd} QB \times A \arrow[r, "1 \times f"] \arrow[d] & QB \times B \arrow[d] \arrow[ddr] & \\ I \arrow[r, tail] \arrow[drr, tail] & P \arrow[dr] & \\ & & RB \end{tikzcd}$$

so for any $x : QB$, $x \leqslant \exists f Qf(x)$.

$Qf \exists f$ corresponds to the morphism $QA \times A \longrightarrow I'$ in the following diagram, where the top square is a pushout.

$$\begin{tikzcd}
QA \times A \arrow[r, "1\times f"] \arrow[d] & QA \times B \arrow[d] \\
RA \arrow[r] \arrow[d] & P \\
I' \arrow[ur, tail] &
\end{tikzcd}$$

$RA \longrightarrow P$ factors through $I' \rightarrowtail P$ because it is the image of $QA \times A \longrightarrow P$. Thus for any $x : QA$, $Qf\exists f(x) \leqslant x$. Therefore, $Qf \dashv \exists f$ as order-preserving morphisms. ■

3.4. Proposition. *(i) If $f$ is mono, then $\exists f$ is a lattice homomorphism, and in a topos preserves all inhabited meets.*

*(ii) If $f$ is a cover, then $Qf$ is a lattice homomorphism, and in a topos preserves all inhabited joins.*

Proof. (i) When $f$ is mono, we will show that $\exists f Qf = \_ \vee c$, so $QA$ is order-isomorphic, and therefore complete-lattice-isomorphic to the sublattice of $QB$ consisting of quotients that are $\geqslant c$. The embedding of this lattice into $QB$ is a lattice homomorphism that preserves inhabited joins, so $\exists f$ will also preserve all inhabited joins.

Let

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d] & B \arrow[d, "c"] \\
\sigma(A) \arrow[r, tail] & C
\end{tikzcd}$$

be a pushout, and let the cover-mono factorization of $QB \times A \overset{1\times f}{\rightarrowtail} QB \times B \xrightarrow{q} RB$ be $QB \times A \longrightarrow I \rightarrowtail RB$ (Here $\sigma(A)$ denotes the support of $A$, i.e. the image of the unique morphism $A \longrightarrow 1$). Consider the following diagram in which both squares are pushouts (and therefore also pullbacks).

$$\begin{tikzcd}
QB \times A \arrow[r, tail, "1\times f"] \arrow[d] & QB \times B \arrow[d, "g"] \\
I' \arrow[r, tail, "f"] \arrow[d] & I' \arrow[d] \\
\sigma(A) \arrow[r, tail] & C
\end{tikzcd}$$

$q$ and $c$ both factor through $g$, so $q \vee c$ also factors through $g$. This gives a diagram:

$$\begin{tikzcd}
I \arrow[r, tail, "f'"] \arrow[dr, tail] \arrow[dd] & I' \arrow[d, "x"] \\
& C \vee RB \arrow[d] \\
\sigma(A) \arrow[r, tail] & C
\end{tikzcd}$$

The kernel pair of $x$ is contained in the kernel pair of $I' \longrightarrow C$, which is $(f' \times f') \cup \Delta$ as a relation on $I' \times I'$. Let $S$ be the kernel pair of $x$ as a relation on $I' \times I'$. Since $I \rightarrowtail C \vee RB$ is monic, the intersection of $S$ with $f' \times f'$ is the diagonal, so $S$ is contained in the diagonal, meaning that $x$ is an iso.

(ii) When $f$ is a cover, we want to show that $\bigwedge \exists Qf = Qf \bigwedge$ when restricted to $P^+QB$. First we consider $\bigwedge \exists Qf$. It will be $\llcorner z \lrcorner$ for some $P^+QB \times A \xrightarrow{z} Z \longrightarrow P^+QB$. This occurs as a pullback:

$$\begin{tikzcd}
P^+QB \times A \arrow[r, "\exists Qf \times 1"] \arrow[d, "z"] & P^+QA \times A \arrow[d, "y"] \\
Z \arrow[r] \arrow[d] & Y \arrow[d] \\
P^+QB \arrow[r, "\exists Qf"] & P^+QA
\end{tikzcd}$$

where $y$ is as in Proposition 3.2.

There is a commutative diagram:

$$\begin{tikzcd}
QB \times A \arrow[r, "1\times f"] \arrow[d, tail, "Qf\times 1"] & QB \times B \arrow[r] & RB \arrow[r] \arrow[d, tail] & QB \arrow[d, tail, "Qf"] \\
QA \times A \arrow[rr] & & RA \arrow[r] & QA
\end{tikzcd}$$

where both squares are pullbacks. Therefore, the pullback of

$$\in_{QA} \times A \longrightarrow P^+QA \times QA \times A \longrightarrow P^+QA \times RA$$

along $\exists Qf \times 1_{RA}$ is

$$\in_{QB} \times A \xrightarrow{1\times f} \in_{QB} \times B \longrightarrow P^+QB \times RB \rightarrowtail P^+QB \times RA$$

Let the cover-mono factorization of this pullback be

$$\in_{QB} \times A \longrightarrow L \rightarrowtail P^+QB \times RA$$

Then $z$ is the pushout of $\in_{QB} \times A \longrightarrow L$ along $\in_{QB} \times A \longrightarrow P^+QB \times A$.

Now consider $Qf \bigwedge$. This corresponds to the right-hand composite in the following diagram, where the bottom square is a pushout and $\in_{QB} \times B \longrightarrow J \rightarrowtail P^+QB \times RB$ is the cover-mono factorization of $\in_{QB} \times B \longrightarrow P^+QB \times RB$:

$$\begin{tikzcd}
\in_{QB} \times A \arrow[r] \arrow[d, "1\times f"] & P^+QB \times A \arrow[d, "1\times f"] \\
\in_{QB} \times B \arrow[r] \arrow[d] & P^+QB \times B \arrow[d] \\
J \arrow[r] & M
\end{tikzcd}$$

However, since $\in_{QB} \longrightarrow P^+QB$ is a cover, the top square is also a pushout, and therefore, so is the whole rectangle. Also, $J \cong L$, as they are both the image of

$$\in_{QB} \times A \xrightarrow{1\times f} \in_{QB} \times B \longrightarrow P^+QB \times RB \rightarrowtail P^+QB \times RA$$

Thus, the quotients of $P^+QB \times A$ corresponding to $\bigwedge \exists Qf$ and $Qf \bigwedge$ are equal, so $\bigwedge \exists Qf = Qf \bigwedge$. ∎

## 4. Finiteness

In this section, we look at an application of copower objects to the study of finiteness. In a topos, the many different classical definitions of finiteness are no longer equivalent, so we have a number of different definitions which can be applied in different situations. We use copower object based ideas to produce a new definition, which we call $K^*$-finiteness. We then show that this is equivalent to $\widetilde{K}$-finiteness – therefore giving a new way of studying $\widetilde{K}$-finiteness.

We then observe that the definition of $K^*$-finiteness gives rise to a new definition of finiteness, which we call $K^{**}$-finiteness. We then characterise $K^{**}$-finiteness in terms of a constructive version of amorphousness, and a concept which we call usual finiteness.

Given an object $X$ in a topos $\mathcal{E}$ and elements $a, b$ of $\widetilde{X}$, we have a relation $R_{ab}$ that relates only the members of $a$ and $b$, i.e. two elements $c$ and $d$ are related if and only if $(c = d) \vee (c \in a \wedge d \in b) \vee (c \in b \wedge d \in a)$ holds. Now define a relation $R$ on the copower object $QX$, by setting $xRy$ if $(\exists a, b : \widetilde{X})(x \wedge R_{ab} = y \wedge R_{ab})$. (Recall that the order on $QX$ is by factorization – i.e. the opposite of inclusion of equivalence relations.) Let $\overline{R}$ be the transitive closure of $R$.

4.1. Definition. *Let $K^*X$ be the quotient of $QX$ corresponding to $\overline{R}$ above. $X$ is $K^*$-*finite *if $K^*X \cong 1$.*

4.2. Lemma. *Subterminal objects are $K^*$-finite.*

Proof. If $U$ is subterminal then $QU \cong 1$, so $K^*U \cong 1$. ∎

4.3. Lemma. *(i) Subobjects of $K^*$-finite objects are $K^*$-finite.*
*(ii) Quotients of $K^*$-finite objects are $K^*$-finite.*

Proof. (i) Let $X$ be $K^*$-finite and let $Y \overset{m}{\rightarrowtail} X$ be a subobject of $X$. $Qm(R_{ab}) = R_{m^*(a)m^*(b)}$, so the diagram:

$$\begin{tikzcd} QX \arrow[r, "Qm"] \arrow[d] & QY \arrow[d] \\ K^*X \cong 1 \arrow[r] & K^*Y \end{tikzcd}$$

commutes. Therefore, $K^*Y \cong 1$.

(ii) Let $X$ be $K^*$-finite, and let $X \xrightarrow{c} Y$ be a quotient. $\exists c(R_{ab}) = R_{c(a)c(b)}$, so the diagram:

$$\begin{tikzcd} QX \arrow[r, "\exists c"] \arrow[d] & QY \arrow[d] \\ K^*X \cong 1 \arrow[r] & K^*Y \end{tikzcd}$$

commutes. Therefore, $K^*Y \cong 1$. ∎

4.4. Remark. This proof in fact gives that if $X'$ is a subquotient of $X$, then $K^*X'$ is a quotient of $K^*X$.

4.5. Lemma. *The disjoint union of two $K^*$-finite objects is $K^*$-finite.*

Proof. Let $X$ and $Y$ be $K^*$-finite. There are embeddings of $QX$ and $QY$ into $Q(X \amalg Y)$ given by $q \mapsto q \amalg 1_Y$ and $q' \mapsto (X \longrightarrow \sigma(X)) \amalg q'$ respectively, where $\sigma(X)$ denotes the support of $X$, i.e. the image of the unique morphism $X \longrightarrow 1$. These embeddings both preserve $R$, and are both quotiented to 1 in $K^*(X \amalg Y)$, so $1_{X \amalg Y} \overline{R}(\sigma(X) \amalg \sigma(Y))$. But $(\sigma(X) \amalg \sigma(Y)) R \sigma(X \amalg Y)$, so $K^*(X \amalg Y) \cong 1$. ∎

4.6. Corollary. *All $\widetilde{K}$-finite objects are $K^*$-finite.*

Proof. This is immediate from the preceding lemmas. ∎

For the other direction we will need the following:

4.7. Lemma. ["outduction"] *Let $\Phi(x)$ be a formula with one free variable $x$, expressible in the internal language of a topos. Let $(n, 0, s)$ be a numeral in which $\Phi(sx) \Rightarrow \Phi(x)$ holds for all $x \geqslant l$, and suppose $(\exists y : n)(y \geqslant l \wedge \Phi(y))$ holds. Then $\Phi(l)$ holds.*

Proof. Let $\Psi(x) = (\Phi(x) \Rightarrow \Phi(l))$. Clearly $\Psi(l)$ holds, and $\Psi(x) \Rightarrow \Psi(sx)$ holds for all $x : n$ with $x \geqslant l$. Therefore, $(\forall x : n)(x \geqslant l \Rightarrow (\Phi(x) \Rightarrow \Phi(l)))$, and also, $(\exists y : n)(y \geqslant l \wedge \Phi(y))$. Therefore, $\Phi(l)$ holds. ∎

4.8. Lemma. *If $X \xrightarrow{y} Y$ and $X \xrightarrow{z} Z$ are quotients of $X$, and $yRz$, then $Y$ is $\widetilde{K}$-finite if and only if $Z$ is.*

Proof. It is sufficient to prove the result when $(\exists a, b : \widetilde{X})(z = y \wedge R_{ab})$ and $Z$ is $\widetilde{K}$-finite, since taking $z' = y \wedge R_{ab} = z \wedge R_{ab}$ reduces the general case to this one. We know that $z'$ is $\widetilde{K}$-finite as it is a quotient of both $y$ and $z$, one of which is $\widetilde{K}$-finite.

The proof proceeds by induction on $Z$. If $Z$ is a subsingleton, then $Y$ is a quotient of the union of $\{a\}$ and $\{b\}$, so it is $\widetilde{K}$-finite. Now suppose $Z = U \cup V$, where the hypothesis holds for $Z = U$ and $Z = V$. Let $Y \xrightarrow{f} Z$ be the factorization of $z$ through $y$. Let $X' = z^*(U)$, $X'' = z^*(V)$. Let $R'_{ab}$ be $R_{(a \cap X')(b \cap X')}$ in $QX'$, and let $R''_{ab}$ be $R_{(a \cap X'')(b \cap X'')}$ in $QX''$. Then $Y = f^*(U) \cup f^*(V)$, $U = f^*(U) \wedge R'_{ab}$ and $V = f^*(V) \wedge R''_{ab}$, so by the inductive hypothesis, $f^*(U)$ and $f^*(V)$ are $\widetilde{K}$-finite. Therefore, $Y$ is also $\widetilde{K}$-finite. ∎

Putting this all together gives:

4.9. Theorem. *In a topos, $\mathcal{E}$, an object $X$ is $K^*$-finite if and only if it is $\widetilde{K}$-finite.*

Proof. One direction is just Corollary 4.6.

For the converse, suppose $X$ is $K^*$-finite. Define a numeral $(n, 0, s)$ in $PQX$ by letting 0 be $\{1_X\}$, and letting

$$sx = \{z : QX | (\exists y : QX)((y \in x) \wedge (yRz))\}$$

Let $\Phi(x)$ be the assertion $(\exists y : QX)(y \in x \wedge \widetilde{K}_y)$. By Lemma 4.8, $\Phi(sx) \Rightarrow \Phi(x)$. Because $X$ is $K^*$-finite, $1_X \overline{R}(X \longrightarrow \sigma(X))$, so

$$(\exists x : n)((X \longrightarrow \sigma(X)) \in x)$$

Therefore, $(\exists x : n)(\Phi(x))$, so by outduction $\Phi(0)$ holds, i.e. $X$ is $\widetilde{K}$-finite. ■

4.10. Remark. If $R$ were taken to be $(\exists a, b : X)(x \wedge R_{ab} = y \wedge R_{ab})$, and $X$ were required to have complemented support, then by the same argument as above, the resulting notion of finiteness would be $K$-finiteness. The only tricky part in this modification is proving the modified version of Lemma 4.8. If $z = y \wedge R_{ab}$ and $f$ is the factorisation of $z$ through $y$, as in the proof above, and $f(a) = f(b) = a'$, then for any $K$-finite subobject $Z'$ of $Z$, the preimage $f^*(Z' \cup \{a'\})$ is $K$-finite. This is shown by induction on $Z'$ – It is obvious that this property is preserved under unions, so it is sufficient to prove the result when $Z'$ is a singleton. If $X'$=$z^*(Z' \cup \{a'\})$ then in $QX'$, $z' = y' \wedge R_{ab}$. Therefore, it is sufficient to prove the case where $Z$ is a singleton $\{x\}$. In this case, the following deduction is valid:

$$\begin{aligned}
&(\forall x_1, x_2 \in f^*(x))((x_1 = x_2) \vee (x_2 = a \vee x_2 = b)) \\
&(\forall x_1 \in f^*(x))(f^*(x) \cup \{a, b\} = \{x_1\} \cup \{a, b\}) \\
&(\forall x_1 \in f^*(x))(K_{f^*(x) \cup \{a,b\}}) \\
&K_{f^*(x) \cup \{a,b\}}
\end{aligned}$$

Therefore, $f^*(Z' \cup \{a'\})$ is $K$-finite whenever $Z'$ is, so in particular, $Y$ is $K$-finite whenever $Z$ is.

So $K^*$-finiteness gives another way of looking at $\widetilde{K}$-finiteness. This other way does give some genuinely new information. It is well known that if $\widetilde{K}X$ is $\widetilde{K}$-finite then so is $X$. The same is not true for $K^*X$.

4.11. Example. In the topos $[\mathcal{C}, \underline{\mathcal{S}et}]$, where $\mathcal{C}$ is the 2-element monoid whose non-identity element is idempotent, let $X$ be the representable functor. Then $K^*X = 2$, so $X$ is not $K^*$-finite, but $K^*X$ is.

This leads to the following definition:

4.12. Definition. *An object $X$ is* $K^{**}$-finite *if $K^*X$ is $K^*$-finite. More generally, $X$ is* $K^{n*}$-finite *for a standard natural number $n \geqslant 2$ if $K^*X$ is $K^{(n-1)*}$-finite, where $K^{1*}$-finiteness is just $K^*$-finiteness.*

This definition is weaker than the normal definition of finiteness in $\underline{\mathcal{S}et}$ without AC – any bounded amorphous set is $K^{**}$-finite. To study $K^{**}$-finiteness more thoroughly, it will be necessary to look at the analogue of amorphous sets in general topoi. Classically the definition of an amorphous set is non-constructive – it assumes that the set has complements. The following is a more constructive definition.

4.13. Definition. *An object $X$, in a topos, is* amorphous *if whenever $Y$ and $Z$ are subobjects of $X$, with $Y \cap Z$ $\widetilde{K}$-finite, either $Y$ is $\widetilde{K}$-finite, or $Z$ is $\widetilde{K}$-finite.*

*It is* bounded amorphous *if it is amorphous and there is a $j : \widetilde{num}_X$, such that for any quotient of $X$ with finite fibres, all but finitely many of the fibres $\{x \in X | f(x) = y\}$ are $\widetilde{K}_j$-finite. In this case, $j$ is called a* bound *for $X$.*

Two subobjects $X'$ and $X''$ of $X$ will be said to *differ finitely* if there is a $\widetilde{K}$-finite subobject $Y$ of $X$, such that $Y \cup X' = Y \cup X''$. We will use $P^*X$ to denote the quotient of $PX$ by the relation

$$X'RX'' \Leftrightarrow (\exists Y : \widetilde{K}X)(Y \cup X' = Y \cup X'')$$

For an amorphous object $X$, the term *almost all* elements of $X$ will sometimes be used to refer to subobjects of $X$ that differ only finitely from $X$.

The collection of amorphous objects in a topos is clearly closed under subobjects, but it is not closed under quotients – Let $\mathcal{C}$ be the category

$$\begin{tikzcd} A \arrow[dr] & \\ & C \\ B \arrow[ur] & \end{tikzcd}$$

and let $X$ be an amorphous set. Then in $[\mathcal{C}, \underline{\mathcal{S}et}]$ the functor sending $A$, $B$ and $C$ all to $X$ and all morphisms to the identity on $X$ is amorphous. However, the quotient that sends $A$ and $B$ to $X$ and $C$ to 1 is not amorphous.

In $\underline{\mathcal{S}et}$ (even without AC) $K^{**}$-finite sets are finite unions of bounded amorphous sets (this will follow from the results later in this section). However, this is because if a set has the property that whenever it is partitioned into 3 disjoint parts one is finite, then the set must be a union of 2 amorphous sets, and similarly for sets with larger bounds on the number of infinite parts into which they can be partitioned. However, in a topos, this does not apply. For example, let $X$ be an amorphous set and consider $X \times \mathbb{R} \longrightarrow \mathbb{R}$ in LH/$\mathbb{R}$, where LH is the category of locales and local homeomorphisms between them (the topos LH/$\mathbb{R}$ is equivalent to the topos of sheaves on the real numbers). For any 2 $\widetilde{K}$-finitely-intersecting subobjects $A$ and $B$ of $X \times \mathbb{R} \longrightarrow \mathbb{R}$, if $A$ has an infinite fibre at $x$, then taking a decreasing sequence $x_n \searrow x$, let $A_n$ be the set of elements of $X$ that are in all fibres of $A$ above points in the interval $[x, x_n]$. The $A_n$ form an increasing sequence of subsets of $X$ whose union is infinite. Therefore, some $A_n$ must be infinite (otherwise there would be an infinite subsequence with every $A_n \setminus A_{n-1}$ non-empty, and we could partition $X$ into $\bigcup_{n=1}^{\infty} A_{2n+1} \setminus A_{2n}$ and $\bigcup_{n=1}^{\infty} A_{2n} \setminus A_{2n-1}$, both of which are infinite). This means that there must be a finite bound on the fibres of $B$ in an open neighbourhood of $x$. This means that given any 3 $K$-finitely intersecting subobjects of $X \times \mathbb{R} \longrightarrow \mathbb{R}$, and any real number $x$, one of them has finite fibres on some neighbourhood of $x$. However, $X \times \mathbb{R} \longrightarrow \mathbb{R}$ is not amorphous, and all its amorphous subobjects are finite, so it is not a union of amorphous sets. We will therefore need the following definition.

4.14. Definition. *An object $X$ in a topos, is $S$-*weakly amorphous *for some $K$-finite numeral $S$, if for any $S \xrightarrow{f} PX$, with $(\forall y, z : S)(\widetilde{K}_{f(y)\cap f(z)} \vee (y = z))$, there is a $\widetilde{K}$-finite $f(z)$ for some $z \in S$. $X$ is* weakly amorphous *if there is a $K$-finite numeral $S$ for which it is $S$-weakly amorphous.*

*$X$ is* bounded weakly amorphous *if it is weakly amorphous, and there is some $j : \widetilde{num}_X$, such that for any quotient $R$ of $X$, all but finitely many equivalence classes of $R$ are $\widetilde{K}_j$-finite. Again, $j$ is called a* bound *for $X$.*

In fact, weak amorphousness is still not enough, as we would like to know that if we partition our object into subobjects whose intersections are $S'$-weakly amorphous, then one of these subobjects will be $S'$-weakly amorphous. The appropriate definition is therefore:

4.15. Definition. *Let $S$ be a $K$-finite numeral. An object $X$ is $S$-*pseudo-amorphous *if for any $K$-finite numeral $S' < S$, and any $S \xrightarrow{f} PX$ such that for any $i, j : S$ either $f(i) \cap f(j)$ is $S'$-pseudo-amorphous or $i = j$, there is an $i : S$ such that $f(i)$ is $S'$-pseudo-amorphous, where an object is 0-pseudo-amorphous if it is $\widetilde{K}$-finite. $X$ is* pseudo-amorphous *if there is a $K$-finite numeral $S$ for which it is $S$-pseudo-amorphous.*

Clearly any $S$-pseudo-amorphous object is weakly amorphous. I have not yet been able to determine whether pseudo-amorphous objects and weakly amorphous objects are in fact the same thing.

The truth of the assertion "$X$ is $S$-pseudo-amorphous" will sometimes be denoted $S$-$\mathrm{psam}_X$. The definition of *bounded pseudo-amorphous* is analogous to the definitions of bounded amorphous and bounded weakly amorphous.

The notion of pseudo-amorphousness is not sufficient to describe $K^{**}$-finiteness. For example, if $X$ is an amorphous set, then in $[\omega, \underline{\mathcal{S}et}]$, the constantly $X$ functor is bounded weakly amorphous, but it has an infinite number of quotients that are pairwise infinitely different – all the associated sheaves complementary to $U$ for subterminal $U$, i.e. the quotients identifying any pair of elements to extent $U$. The problem is that $\Omega$ is infinite. If $\Omega$ is $K$-finite, then bounded pseudo-amorphousness is sufficient to ensure $K^{**}$-finiteness. To fix this problem, we introduce an extra condition on $X$ which will allow our arguments to work as if $\Omega$ were $K$-finite.

4.16. Definition. *Given an object $X$ in a topos $\mathcal{E}$, define an equivalence relation $R_X$ on $\Omega$, by $UR_XV$ if and only if the associated sheaf of $X \times (U \cup V)$ is $\widetilde{K}$-finite in $\mathcal{S}\mathbf{h}_{c(U\cap V)}(\mathcal{E})$.*

*Let $\Omega_X$ be the quotient of $\Omega$ by the relation $R_X$. Say that $X$ is* usually finite *if $\Omega_X$ is $\widetilde{K}$-finite.*

The remainder of this paper will be devoted to the proof of:

4.17. THEOREM. *An object $X$ is $K^{**}$-finite if and only if it is bounded pseudo-amorphous and usually finite.*

To motivate what follows, we will start by giving a sketch proof:

PROOF (SKETCH). Only if: We will start by dealing with the case where $\Omega$ is $K$-finite. The general result will follow from this without too much difficulty.

Let $j : \widetilde{num}_X$ be a bound for $X$. We will show (Theorem 4.22) that $P^*X$ is $\widetilde{K}$-finite. Therefore, there are only $\widetilde{K}$-finitely many possible collections of infinite equivalence classes up to finite difference. Therefore, if we can show that there are only $\widetilde{K}$-finitely many finitary equivalence relations (equivalence relations all of whose equivalence classes are $\widetilde{K}$-finite) up to finite difference, then $X$ will be $K^{**}$-finite, since given any equivalence relation $R$ on $X$, we can map it to the finitary equivalence relation $R'$ given by $x\ R'\ y$ if and only if $x\ R\ y$ and the $R$-equivalence class $[x]_R$ of $x$ is $\widetilde{K}$-finite. We let $K^*_{\text{fin}}X$ denote the object of finitary equivalence relations of $X$ up to finite difference.

To show that $K^*_{\text{fin}}X$ is $\widetilde{K}$-finite, we will start by forming some partially ordered object $L$ of possible 'sizes' of $\widetilde{K}$-finite subobjects of $X$. This will give us the morphism $K^*_{\text{fin}}X \xrightarrow{f} (L^X)^*$, where $(L^X)^*$ is the quotient of $L^X$ that identifies functions that agree on almost all elements of $X$, sending each equivalence relation $R$ to the function that sends $x : X$ to the 'size' of $[x]_R$ (the $R$-equivalence class of $x$). We will choose $L$ in such a way that $(L^X)^*$ is a $\widetilde{K}$-finite poset with a top element.

We will then use an induction principle on posets of this type (Lemma 4.20) to show that all the fibres of $f$ are $\widetilde{K}$-finite. This is where we will need Lemma 4.19 and Lemma 4.23.

If: There is a monomorphism $P^*X \rightarrowtail K^*X$, sending a subobject $X'$ of $X$ to the equivalence relation that relates two elements of $X$ if they are equal, or they are both in $X'$. Therefore, if $X$ is $K^{**}$-finite then $P^*X$ is $\widetilde{K}$-finite, so (as we will show in Theorem 4.22) $X$ is pseudo-amorphous and usually finite.

We want to find a bound for $X$. We construct a morphism $X \times Q_{\text{fin}}X \xrightarrow{g} U(\widetilde{num}_X)$ sending $(R, x)$ to the object of upper bounds on the size of $[x]_R$. We will show (Lemma 4.25) that any weakly amorphous subobject of a numeral is in fact $\widetilde{K}$-finite, and therefore has an upper bound. This means that for any finitary relation $R$, there is an inhabited object $i_R$ of upper bounds on the size of $[x]_R$. For each element of $K^*_{\text{fin}}X$, we take the union of the $i_R$ for relations on that equivalence class. The intersection of these unions is a $\widetilde{K}$-finite intersection of inhabited upsets of a numeral, and is therefore inhabited. An element of this intersection is a bound for $X$. ∎

4.18. Lemma. *(i) If $X$ is $K_{sn}$-finite, then for any $x : X$, there is a $K_n$-finite $X' \subseteq X$ such that $X = X' \cup \{x\}$.*

*(ii) If $X$ is $\widetilde{K}_{sn}$-finite, then for any $x : X$, there is a $\widetilde{K}_n$-finite $X' \subseteq X$ such that $X = X' \cup \{x\}$.*

Proof. (i) Induction on $X$. If $X$ is a singleton or empty, the result is obvious. If $X = X_1 \cup \{y\}$, $n = sm$ and $\phi(n, X_1)$ (i.e. $X_1$ is $K_n$-finite) then for any $x : X$, $x \in X_1 \vee x = y$. If $x = y$, the result is trivial, so suppose $x \in X_1$. $\phi(sm, X_1)$, so there is an $X_1'$ satisfying $\phi(m, X_1')$ and $X_1' \cup \{x\} = X_1$. But now $Y = X_1' \cup \{y\}$ satisfies $\phi(n, Y)$ and $Y \cup \{x\} = X$.

(ii) The proof is essentially the same as (i). ∎

As mentioned in the sketch proof, we will need a way of measuring the 'size' of an equivalence class in order to get the object $L$ that we use there. We will need this measure of 'size' to distinguish proper subobjects, i.e. if $A' \subseteq A$ are $\widetilde{K}$-finite subobjects of $X$, then we will want $A'$ and $A$ to have the same 'size' only if they are equal. The obvious measure of 'size' – sending a $\widetilde{K}$-finite subobject $A$ of $X$ to the least $i : \widetilde{num}_X$ for which $\phi_X(i, A)$ – won't work, as a proper subobject of $A$ can have the same 'size' as $A$ in this sense (also the least such $i$ may not be defined). One measure of size which does work is the upset $U$ of $\widetilde{num}_{\Omega^X}$ given by $U = \{i : \widetilde{num}_{\Omega^X} | \phi_{\Omega^X}(i, \Omega^A)\}$. We need to check that this distinguishes proper subobjects:

4.19. Lemma. *If $A \rightarrowtail B$ is a subobject of a $\widetilde{K}$-finite object $B$, $\Omega^B$ is $K$-finite, and $(\forall j : \widetilde{num}_{\Omega^B})(\phi_{\Omega^B}(j, \Omega^A) \Rightarrow j = sj)$, then $A \rightarrowtail B$ is an isomorphism.*

Proof. Fix $b : \widetilde{B}$. We will show that $(\forall i : \widetilde{num}_{\Omega^B})(b \in \Omega^A \vee i = si)$. We prove this by outduction on $i$. As $\widetilde{num}_{\Omega^B}$ is $K$-finite, it has a fixed point, which clearly satisfies the hypothesis. Now suppose that $si$ satisfies $(b \in \Omega^A \vee si = ssi)$. Clearly $b \in \Omega^A$ implies $b \in \Omega^A \vee i = si$, so suppose $si = ssi$. Then $\phi_{\Omega^B}(si, \Omega^B)$, so by Lemma 4.18, there is some subobject $C$ of $\Omega^B$ such that $\phi_{\Omega^B}(i, C) \wedge (\{b\} \cup C = \Omega^B)$. Now as $\Omega^A$ is injective, we can pick $a : \Omega^A$ such that $b \in \Omega^A \Rightarrow b = a$. Clearly $(a \in C) \vee (a = b)$ as $C \cup \{b\} = \Omega^B$.

However, if $a \in C$ then $b \in \Omega^A \Rightarrow b \in C$. Since $\Omega^A = (\Omega^A \cap C) \cup (\Omega^A \cap \{b\})$, this means that $\Omega^A \subseteq C$, and therefore, $\phi_{\Omega^B}(i, \Omega^A)$, whence $i = si$.

Thus we have $(b \in \Omega^A) \vee (i = si)$, which completes the outduction, and allows us to deduce that $(b \in \Omega^A) \vee (\bot = s\bot)$. However $\bot = s\bot$ never holds, as $\Omega^B$ has a point. Therefore, we deduce that $b \in \Omega^A$. As $b$ was arbitrary, $\widetilde{B} \subseteq \Omega^A$, and thus $A \cong B$. ∎

This lemma is related to the fact that $\Omega$ is an internal cogenerator (proved by F. Borceux [1]).

The induction principal that we will need to use is the following:

4.20. Lemma. *Given a formula $\Phi(x)$ in the internal logic of the topos and a $\widetilde{K}$-finite partial order $L$ with a top element $\top$, if $(\forall x : L)(((\forall y : L)(y \in \upharpoonleft(x) \Rightarrow \Phi(y))) \Rightarrow \Phi(x))$, where $\upharpoonleft(x) = \{z : L | (x \leqslant z) \wedge (x = z \Rightarrow x = \top)\}$ and $\Phi(\top)$, then $(\forall x : L)(\Phi(x))$.*

Proof. Induction on $L$: if $L = \{\top\}$, then the result is obvious. If $L = L' \cup x$ where $x$ is a subsingleton and $\top \in L'$, and the lemma holds for $L'$, then as for any $x' : x$, $\upharpoonleft(x') \subseteq L'$, we have that $(\forall x' : x)(\forall y : L)(y \in \upharpoonleft(x') \Rightarrow \Phi(y))$, so by hypothesis, $\Phi(x')$. Therefore, for $z : L'$,

$$(\forall y : L)(y \in \upharpoonleft(z) \Rightarrow \Phi(y)) \Leftrightarrow (\forall y : L')(y \in \upharpoonleft(z) \Rightarrow \Phi(y))$$

Hence, $(\forall y : L')(\Phi(y))$, and thus $(\forall y : L)(\Phi(y))$. ∎

4.21. Lemma. *For any $\widetilde{K}$-finite $X$, any monomorphism $\widetilde{num}_X \overset{f}{\rightarrowtail} X$ is a cover.*

Proof. Induction on $\widetilde{num}_X$. If $\widetilde{num}_X = 0$, the result is obvious. Let $n : \widetilde{num}_X$ satisfy $sn = ssn$, and let $m : \widetilde{num}_X$ satisfy $sm = n$. By Lemma 4.18, there is an $X' \subseteq X$ such that $\phi_X(n, X')$ and $X = X' \cup f(sn)$. However, $f(n)$ is a member of $X$, so $(f(n) \in X') \vee (f(n) = f(sn))$. But since $f$ is monic, $(f(n) = f(sn)) \Rightarrow (n = sn)$ which means that $X$ is $\widetilde{K}_n$-finite. Therefore, in either case, $X'' = X' \cup \{y\}$ is $\widetilde{K}_n$-finite. The restriction of $f$ to $\{i : \widetilde{num}_X | i \leqslant n\}$ is therefore a monomorphism into $X'$, and $\{i : \widetilde{num}_X | i \leqslant n\}$ can be made into a numeral isomorphic to $\widetilde{num}_{X'}$. Therefore, by the inductive hypothesis, it is a cover. Thus, $f$ is also a cover. ∎

4.22. Theorem. *For any $X$, $P^*X$ is $\widetilde{K}$-finite if and only if $X$ is pseudo-amorphous and usually finite.*

Proof. If $P^*X$ is $\widetilde{K}$-finite, then the subobject consisting of equivalence classes of subobjects of $X$ of the form $X \times U$ for some subterminal $U$ is also $\widetilde{K}$-finite, but this subobject is isomorphic to the quotient of $\Omega$ by $R_X$, so $X$ is usually finite.

By Lemma 4.21, any morphism $\widetilde{num}_{P^*X} \xrightarrow{f} PX$ whose composite with $PX \longrightarrow P^*X$ is mono covers $P^*X$. If $f$ satisfies

$$(\forall i, j : \widetilde{num}_{P^*X})((i = j) \vee S\text{-psam}_{f(i) \cap f(j)})$$

then the composite must either be mono or send some $i$ to an $S$-pseudo-amorphous object. If it is mono, then it must cover $P^*X$, so in either case, some $i$ must be sent to an $S$-pseudo-amorphous object in $P^*X$. Therefore, $X$ is $\widetilde{num}_{P^*X}$-pseudo-amorphous.

For the converse, it suffices to show that for any $K$-finite numeral $S$, the object $P^*_S X$ of $R$-equivalence classes of subobjects of $X$ that are $S$-pseudo-amorphous is $\widetilde{K}$-finite. $P^*X$ will then be $\widetilde{K}$-finite.

Induct on $S$. If $S = 1$, then the subobjects in question are $\widetilde{K}$-finite and so all equivalent. If $S = S_1 \cup \{s\}$, then, for any $R$-closed and downwards closed collection $A$ of $S_1$-pseudo-amorphous subobjects of $X$, the object of subobjects of $X$ whose $S_1$-pseudo-amorphous subobjects are exactly those in $A$ is $K$-finite. The object of $R$-closed and downwards closed families of $S_1$-pseudo-amorphous subobjects of $X$ is $\widetilde{K}$-finite, as it is a subquotient of ${\Omega_X}^{P^*_{S_1}X}$. Therefore, $P^*_S X$ is $\widetilde{K}$-finite. ∎

In the proof of Theorem 4.17, we will need to consider refinements of given equivalence relations. An equivalence relation $R'$ on $X$ is a refinement of an equivalence relation $R$ if any pair of elements related by $R'$ is also related by $R$, i.e. if and only if $R \leqslant R'$ in the order relation on $QX$. The following lemma will be necessary.

4.23. Lemma. *Let $X$ be a usually finite object, and let $R$ be a finitary relation on $X$, such that for some $j : \widetilde{num}_X$, all equivalence classes of $R$ are $\widetilde{K}_j$-finite. Then $R$ has $K$-finitely many refinements up to finite difference.*

Proof. Let $T$ be the object of refinements of $R$, quotiented by finite difference. There is a morphism $T \times T \xrightarrow{g} P^*X$ sending a pair of relations $(R_1, R_2)$ to the object of elements of $X$ whose $R_1$ and $R_2$ equivalence classes are the same. As almost all equivalence classes of $R$ are $\widetilde{K}_j$-finite, if we quotient each $Q([x]_R)$ (where $[x]_R$ denotes the equivalence class of $x$) by relating two quotients $R_1$ and $R_2$ if there is a sequence $0 = U_0 \leqslant U_1 \leqslant U_2 \leqslant \ldots \leqslant U_n = 1$ of subterminal objects such that in each $\mathcal{S}\mathbf{h}_{c(U_{2n})}(\mathcal{E})$, $U_{2n+1} \Rightarrow R_1 = R_2$, while $U_{2n+1}\ R_X\ U_{2n+2}$ (i.e. relating two quotients if they agree unless $X$ is finite), then there is a finite numeral $k$ such that all the resulting quotients are $\widetilde{K}_k$-finite. Therefore, given any subobject $T'$ of $T$, either $T'$ is $\widetilde{K}_k$-finite, or for almost every equivalence class of $R$ there are two elements of $T'$ that agree on all elements of this class. Therefore, the image of the restriction of $g$ to $T' \times T'$ is a $\widetilde{K}$-finite collection of subobjects of $X$ whose union is almost the whole of $X$. Thus, every fibre of $g$ has all of its cliques $\widetilde{K}_k$-finite (a clique is a subobject of $T$ any two elements of which are either related or equal). Thus, by Ramsey's theorem, there is some finite numeral $l$ such that $T$ is $\widetilde{K}_l$-finite. ∎

4.24. Theorem. *Let $U$ be a subterminal object. If $\Omega \times U$ is $K$-finite and $X$ is bounded pseudo-amorphous with support $U$, then $X$ is $K^{**}$-finite.*

Proof. As explained in the sketch proof, we need only show that $K^*_{\text{fin}}X$ is $\widetilde{K}$-finite.

Let $U(\widetilde{num}_{\Omega^X})$ denote the object of inhabited upsets in $\widetilde{num}_{\Omega^X}$, with the reverse of the inclusion ordering. We will consider a morphism $K^*_{\text{fin}}X \xrightarrow{f} (U(\widetilde{num}_{\Omega^X})^X)^*$, where $(U(\widetilde{num}_{\Omega^X})^X)^*$ is the quotient of $U(\widetilde{num}_{\Omega^X})^X$ by the relation which relates two functions if the subobject on which they agree is finitely different from the whole of $X$. The morphism $f$ is given by $f(R)(x) = g(\Omega^{[x]_R})$ where $g$ is the morphism sending a $\widetilde{K}$-finite subobject $A$ of $\Omega^X$ to the object $\{i : \widetilde{num}_{\Omega^X} | \phi_{\Omega^X}(j, A)\}$ ($\Omega^{[x]_R}$ is $\widetilde{K}$-finite because $[x]_R$ is a subobject of $X$, and therefore has support $\leqslant U$, so $\Omega^{[x]_R} = (\Omega \times U)^{[x]_R}$ which is $\widetilde{K}$-finite).

Because almost all the $[x]_R$ are $\widetilde{K}_j$-finite, almost all the elements in the image of $f(R)$ are below some fixed $\overline{j} : U(\widetilde{num}_{\Omega^X})$. Let $L$ be the subobject of elements of $U(\widetilde{num}_{\Omega^X})$ that are less than or equal to $\overline{j}$. $L$ is $\widetilde{K}$-finite, so by considering the morphisms $\Omega^L \xrightarrow{(Ph)^*} P^*X$, we see that the quotient $(L^X)^*$ of $L^X$ under the finite difference relation is a $\widetilde{K}$-finite partial order with a top element. We need to show that the inverse image of each element of $(L^X)^*$ under $f$ is $\widetilde{K}$-finite. We do this using the generalised form of induction in Lemma 4.20, where $\Phi(x)$ is the assertion that $Pf(\{x\})$ is $\widetilde{K}$-finite.

We note that if $R'$ is a refinement of $R$ and $f(R') = f(R)$, then $R'$ and $R$ differ finitely, because $[x]'_R \subseteq [x]_R$ and if $f(R')(x) = f(R)(x)$ then $[x]'_R = [x]_R$ by Lemma 4.19. $\Phi$ therefore holds for the top element of $L$ because any $R$ and $R'$ are both refinements of $R \wedge R'$.

Now suppose that $\Phi(h')$ holds for every $h' \in \uparrow(h)$. Let

$$V = [(\exists R : K^*_{\text{fin}}X)(f(R) \in \uparrow(h))]$$

In $\mathcal{S}\mathbf{h}_{c(V)}(\mathcal{E})$, there are no relations $R$ with $f(R) \in \uparrow(h)$, so given $R_1$ and $R_2$ with $f(R_1) \geqslant h$ and $f(R_2) \geqslant h$, $f(R_1 \wedge R_2) \geqslant h$, so $f(R_1 \wedge R_2) = h$, and therefore, $R_1$ and $R_2$ differ finitely in $\mathcal{S}\mathbf{h}_{c(V)}(\mathcal{E})$. On the other hand, in $\mathcal{E}/V$, any $R'$ with $f(R') = h$ is a refinement of one of the relations $R$ with $f(R) \in \uparrow(h)$, so there are only $\widetilde{K}$-finitely many possible values for $R' \times V$ by Lemma 4.23 (it is $\widetilde{K}$-finite in $\mathcal{E}$, not just in $\mathcal{E}/V$, as the object $\uparrow(x)$ is $\widetilde{K}$-finite in $\mathcal{E}$). Therefore the preimage of $h$ under $f$ has $\widetilde{K}$-finite product with $V$, and $\widetilde{K}$-finite associated $c(V)$-sheaf, so some $\widetilde{K}$-finite quotient of it is $\widetilde{K}$-finite, and it is therefore $\widetilde{K}$-finite.

Therefore, by induction, all the preimages are $\widetilde{K}$-finite, and thus, $K^*_{\text{fin}}X$ is $\widetilde{K}$-finite, and hence $X$ is $K^{**}$-finite. ■

4.25. Lemma. *Any weakly amorphous subobject of a numeral is $\widetilde{K}$-finite.*

Proof. Let $(n, 0, s)$ be a numeral, and let $X$ be an $S$-weakly amorphous subobject of $n$, where $S$ is a numeral $(S, 0, t)$ with top element $\top$. Let $X'$ be the downset generated by $X$. As the downset generated by a single element is $\widetilde{K}$-finite, $X'$ is an $S$-weakly amorphous union of $\widetilde{K}$-finite objects, so it is $S$-weakly amorphous.

If $S$ satisfies $t^m(0) = \top$ for some standard natural number $m$, then we can partition $n$ into the numerals contained in the prenumerals $(n, s^i(0), s^m)$ for $i = 0, 1, \ldots, m-1$. This gives a morphism $m \xrightarrow{f} Pn$ such that for any $i, j : m$, either $i = j$ or $f(i) \cap f(j)$ is $\widetilde{K}$-finite. We now form a morphism $S \xrightarrow{g} Pn$ by $g(i) = \bigcup\{f(j) | t^j(0) = i\}$. For any $i_1, i_2 : S$, either $g(i_1) \cap g(i_2)$ is $\widetilde{K}$-finite, or there are $j_1, j_2 : m$ with $t^{j_1}(0) = i_1, t^{j_2}(0) = i_2$ and $j_1 = j_2$, in which case $i_1 = i_2$. Therefore, as $X'$ is $S$-weakly amorphous, there is an $i : S$ for which $g(i) \cap X'$ is $\widetilde{K} - finite$. This means that one of the numerals contained in the prenumerals $(n, s^i(0), s^m) \cap X'$ is $\widetilde{K}$-finite. However, if the numeral contained in $(n, s^i(0), s^m) \cap X'$ is $\widetilde{K}$-finite, then so are the numerals contained in $(n, s^{i+1}(0), s^m) \cap X'$ and $(n, s^{i-1}(0), s^m) \cap X'$. Therefore $X'$ is a $\widetilde{K}$-finite union of $\widetilde{K}$-finite objects, so it is $\widetilde{K}$-finite, and therefore, so is $X$. ■

Now we are ready to prove Theorem 4.17.

1. The fact that a $K^{**}$-finite object is bounded pseudo-amorphous and usually finite was shown in the sketch proof, Lemma 4.25 and Theorem 4.22.

4.26. Lemma. *For any object $X$ and any subterminal $U$, $K^*(X \times U) \times U \cong K^*X \times U$.*

Proof. The objects $Q(X \times U) \times U$ and $QX \times U$ are isomorphic via the morphisms $(\llcorner R(X \times U) \times U \lrcorner, \pi_2)$ and $(\llcorner RX \times U \lrcorner, \pi_2)$. These isomorphisms send the quotient $R_{ab}$ for $a$ and $b$ elements of $\widetilde{X \times U} \times U$ to the quotient $R_{a'b'}$ where $a'$ and $b'$ are the corresponding elements of $\widetilde{X} \times U$, and vice versa. Therefore, they preserve the relation on $QX \times U$ used in constructing $K^*X \times U$, and therefore, restrict to an isomorphism between $K^*(X \times U) \times U$ and $K^*X \times U$. ∎

2. A bounded pseudo-amorphous usually finite object is $K^{**}$-finite.

Proof. Let $X$ be a bounded pseudo-amorphous usually finite object. Consider the collection $F$ of subterminal objects $U$ such that $X \times U$ is $K^{**}$-finite. It is clearly downwards closed. It is also closed under $R_X$, since if $X \times U$ is $K^{**}$-finite, then $K^*X \times U$ is $\widetilde{K}$-finite, so if in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, $X \times V$ is $K^{**}$-finite, then $K^*X \times V$ is $\widetilde{K}$-finite in $\mathcal{E}$. Also, by Theorem 4.24, this collection contains all $U$ such that $\Omega \times U$ is $\widetilde{K}$-finite. Indeed, if it contains $U$ and in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, $\Omega \times V$ is $\widetilde{K}$-finite, then it will also contain $V$, as $X \times V$ will be $K^{**}$-finite in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$. Therefore, since $X$ is usually finite, $F$ will contain 1, and thus $X$ will be $K^{**}$-finite. ∎

## 5. Related notions

There are a number of obvious variations on the notion of a copower object – what about the object of all retracts of $X$, or the object of all subquotients of $X$? In this section, we briefly look at these ideas, and show that the former is a weaker notion, in that it does not allow us to construct power objects. However, this means that we may be able to use it in constructions in categories which are not topoi. The latter is as strong a notion as a power object, so perhaps an approach to topos theory based on per objects is possible. The theory of these objects still needs to be developped.

5.1. Definition. *In a cartesian category $\mathcal{C}$, a* potency object *for an object $A$ is an object $BA$ equipped with a retract $BA \times A \twoheadrightarrow CA \rightarrowtail BA \times A$ over $BA$, such that for any other retract $X \times A \twoheadrightarrow Y \rightarrowtail X \times A$ over $X$, there is an unique morphism $X \xrightarrow{\ulcorner Y \urcorner} BA$, such that the retract is a pullback of $BA \times A \twoheadrightarrow CA \rightarrowtail BA \times A$ along $X \times A \xrightarrow{\ulcorner Y \urcorner \times 1} BA \times A$.*

5.2. Proposition. *Any cartesian closed category with all finite limits has potency objects.*

Proof. Given $X$, $BX$ is the object of idempotent endomorphisms of $X$, i.e. the object $\{f : X^X | ff = f\}$. The retract $CX$ is the image of $(\pi_1, \mathrm{ev}|_{BX})$, where $\pi_1$ is the projection from $BX \times X$ to $BX$, and ev is the evaluation morphism $X^X \longrightarrow X$. This image exists because it is the equalizer of $(\pi_1, \mathrm{ev}|_{BX})$ and $1_{BX \times X}$. Given a retract $A \times X \twoheadrightarrow B \rightarrowtail A \times X$, the morphism $A \longrightarrow BX$ is the factorization of the exponential transpose of the composite with $\pi_2$. ∎

5.3. Proposition. *Any regular category with potency objects, such that subobjects of 1 form a Heyting semilattice, is cartesian closed.*

Proof. Let $k$ denote the generic endomorphism of $B(X \times Y) \times X \times Y$. The exponential $X^Y$ can be obtained from the object of graphs. A graph is an idempotent endomorphism of $X \times Y$ that factors through $\pi_2$, and whose composite with $\pi_2$ is $\pi_2$, so $X^Y$ is contained in

$$B'(X \times Y) = \{f : B(X \times Y)|(\exists g : (X \times Y)^Y)(f = g\pi_2)\}$$

The idea here is that $B'(X \times Y)$ is the object of retracts of $X \times Y$ contained in $Y$. It therefore admits a cover to $BY$ sending a retract of $X \times Y$ to its factorization through $\pi_2$. The exponential $X^Y$ will then be the pullback of $\ulcorner 1_Y \urcorner$ along this cover, i.e. the retracts whose image is isomorphic to $Y$.

To form $B'(X \times Y)$, we note that if $\sigma(X) \leqslant \sigma(Y)$ then $\pi_2$ is the coequalizer of

$$X \times Y \times Y \underset{\pi_{13}}{\overset{\pi_{12}}{\rightrightarrows}} X \times Y$$

Therefore, $f$ will factor through $\pi_2$ if $f$ has equal composites with $\pi_{12}$ and $\pi_{13}$. We can test this by composing these composites with the monomorphism $1_X \times \Delta_Y$, where $\Delta_Y$ is the diagonal map $Y \overset{\Delta_Y}{\rightarrowtail} Y \times Y$. This gives a retract of $X \times Y \times Y$. Therefore, we construct the morphisms $B(X \times Y) \xrightarrow{h_2} B(X \times Y \times Y)$ and $B(X \times Y) \xrightarrow{h_3} B(X \times Y \times Y)$ corresponding to the idempotent endomorphisms $\Delta_Y k\pi_{123}$ and $\Delta_Y k\pi_{124}$ respectively. Now $B'(X \times Y)$ will be the product $E \times (\sigma(X) \Rightarrow \sigma(Y))$, where $E$ is the equalizer of $h_2$ and $h_3$. We now construct $X^Y$ as the pullback $P$ of $\ulcorner 1_y \urcorner$ along $B'(X \times Y) \xrightarrow{\ulcorner g \urcorner} B(Y)$, where $g$ is the factorization of the endomorphism of $B'(X \times Y) \times X \times Y$ through $\pi_{13}$ composed with $\pi_{13}$ on the other side.

A morphism $Z \longrightarrow P$ corresponds to a morphism $Z \longrightarrow B'(X \times Y)$, corresponding to a retract of $Z \times X \times Y$, that factors through $Z \times Y$. composing this factorization with $\pi_1$ gives a morphism $Z \times Y \longrightarrow X$. Therefore, $P$ is indeed the exponential $X^Y$. ∎

5.4. Proposition. *Any cartesian category with disjoint coproducts and potency objects is cartesian closed.*

Proof. Let $k$ denote the generic endomorphism of $B(X \amalg Y) \times (X \amalg Y)$. The exponential $X^Y$ can be obtained from the object of cographs. A cograph is an idempotent endomorphism of $X \amalg Y$ that factors through $\nu_1$, and whose composite with $\nu_1$ is $\nu_1$, so $X^Y$ is contained in

$$B'(X \amalg Y) = \{f : B(X \amalg Y)|\mathrm{Im} f \subseteq \nu_1\}$$

Since $\nu_1$ is the equalizer of

$$X \amalg Y \underset{\nu_{13}}{\overset{\nu_{12}}{\rightrightarrows}} X \amalg Y \amalg Y$$

an endomorphism $f$ of $X \amalg Y$ will factor through $\nu_1$ if and only if it has equal composites with $\nu_{12}$ and $\nu_{13}$. If we form the composites $\nu_{12} f(1 \amalg \nabla_Y)$ and $\nu_{13} f(1 \amalg \nabla_Y)$, where $Y \amalg Y \xrightarrow{\nabla_Y} Y$ is the codiagonal map, we get two retracts of $X \amalg Y \amalg Y$ which will be equal if and only if $f$ factors through $\nu_1$. We can therefore construct $B'(X \amalg Y)$ as the equalizer of the two morphisms from $B(X \amalg Y)$ to $B(X \amalg Y \amalg Y)$ corresponding to the composites $\nu_{12}k(1 \times \nabla_Y)$ and $\nu_{13}k(1 \times \nabla_Y)$.

Factorization through $\nu_1$ gives a morphism from $B'(X \amalg Y)$ to $BX$, and the pullback along this of $\ulcorner 1_X^{\urcorner}$ is easily seen to be the exponential $X^Y$. ∎

5.5. Remark. Some condition besides cartesianess is required to get cartesian closure, since in any partial order with a top element, the top element is a potency object.

5.6. Definition. *Given an object $A$ in a cartesian category $\mathcal{C}$, a* per object *for $A$ is an object $\Psi A$ equipped with a cover $\Psi A \times A \xrightarrow{q} \ni$ through which the projection $\Psi A \times A \xrightarrow{\pi_1} \Psi A$ factors, and a subobject $\Phi A \rightarrowtail \ni$ such that for any other cover $X \times A \xrightarrow{x} Y$ through which $\pi_1$ factors, and any subobject $Z \rightarrowtail Y$, there is an unique morphism $X \xrightarrow{\ulcorner Z \lrcorner} \Psi A$ such that in the following diagram:*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, tail] & \Phi A \arrow[d, tail] \\ Y \arrow[r] \arrow[d] & \ni \arrow[d] \\ X \arrow[r] & \Psi A \end{tikzcd}$$

*both small squares are pullbacks, and the morphism $Y \longrightarrow \ni$ commutes with $x$ and $q(\ulcorner Z \lrcorner \times 1_A)$.*

5.7. Remark. The object $\Phi A$ is the object of ordered pairs $(\psi, x)$ where $\psi$ is a subquotient of $A$ and $x$ is an element of $\psi$, while $\ni$ is the object of ordered pairs $(\psi, a)$, where $\psi$ is still a subquotient of $A$, but $a$ is now any element of the quotient of $A$ involved in the construction of $\psi$.

5.8. Proposition. *A topos has per objects.*

Proof. The per object $\Psi X$ is just the object of symmetric transitive relations on $X$, which is a subobject of $P(X \times X)$. ∎

5.9. Proposition. *A cartesian category with per objects is a topos.*

Proof. Given an object $X$ in $\mathcal{E}$, let $\Psi X \xrightarrow{\xi} \Psi X$ correspond to the subobject $\in$ of $\Psi X \times X$, which is the pullback of $\Phi X \rightarrowtail \ni$ along $\Psi X \times X \longrightarrow \ni$, viewed as a subquotient of $\Psi X \times X$ whose quotient part is the identity. The image of $\xi$ is then the power object $PX$. Given any $R \rightarrowtail A \times X$, $\xi \ulcorner R \lrcorner = \ulcorner R \lrcorner$, so $\ulcorner R \lrcorner$ factors uniquely through $PX$, so $PX$ is the power object. ∎

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*Department of Mathematics and Statistics, Chase Building, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada*
Email: `tkenney@mathstat.dal.ca`

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