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# SEMI-ABELIAN MONADIC CATEGORIES

*Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday.*

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ABSTRACT. We characterize semi-abelian monadic categories and their localizations. These results are then used to obtain a characterization of pointed protomodular quasi-monadic categories, and in particular of protomodular quasivarieties.

## 1. Introduction

The notion of semi-abelian category can be considered as intermediate between the notion of Barr-exact category and the one of abelian category. Semi-abelian categories were introduced by Janelidze, Márki and Tholen [12] in a closed connection with the more general protomodular categories due to Bourn [4]. A finitely complete category with a zero object is protomodular if and only if it satisfies the split short five lemma. Semi-abelian categories are defined as exact protomodular categories with finite coproducts and a zero object. These categories are suitable to develop several basic aspects of homological algebra of groups and rings [6], as well as an abstract theory of commutators and of ideals [5]. Among the examples of semi-abelian categories there are the categories of groups, rings, commutative rings, Lie algebras, Heyting semilattices, crossed modules and compact Hausdorff groups.

Every variety of universal algebras is an exact category, and abelian varieties are precisely those whose theories contain abelian group operations $0$, $-$ and $+$ in such a way that these operations are homomorphisms. When the theory of a variety $\mathcal{V}$ only contains group operations $0$, $-$ and $+$, then $\mathcal{V}$ is semi-abelian. Bourn and Janelidze recently characterized semi-abelian varieties [7] as those whose theories contain a unique constant $0$, binary operations $\alpha_0, \alpha_1, ..., \alpha_{n-1}$ for $n \geq 1$ and a $(n+1)$-ary operation $\beta$ satisfying the equations $\alpha_i(x,x) = 0$ for $i = 0, 1, ..., n-1$ and $\beta(\alpha_0(x,y), \alpha_1(x,y), ..., \alpha_{n-1}(x,y), y) = x$. The case $n = 1$ shows that the above-mentioned existence of a group operation suffices to guarantee that $\mathcal{V}$ is semi-abelian, by setting $\alpha_0(x,y) = x - y$ and $\beta(x,y) = x + y$. Varieties of algebras satisfying these axioms have been also studied by Ursini [19], who called them classically ideal determined.

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In the present paper we show that the characterization of semi-abelian varieties can be extended to infinitary and many sorted ones. As a consequence we prove that $C^*$-algebras form a semi-abelian category, and we provide explicit operations witnessing this fact. Every variety of infinitary many-sorted algebras is exact and locally presentable. Having a general exact locally presentable category $\mathcal{C}$, we can consider the varietal hull $\mathcal{V}$ of $\mathcal{C}$ with respect to a chosen regular generator $G$ of $\mathcal{C}$. Following [21] the category $\mathcal{C}$ is a localization of $\mathcal{V}$ and, thus, $\mathcal{C}$ is semi-abelian whenever $\mathcal{V}$ is semi-abelian. Using our characterization of semi-abelian infinitary many-sorted varieties we get a characterization of their localizations. In particular, any exact locally presentable category $\mathcal{C}$ containing a regular generator $G$ which is a cogroup in $\mathcal{C}$ is semi-abelian.

In the last section we characterize protomodular quasimonadic categories, and then protomodular quasivarieties and their localizations.

## 2. Semi-abelian monadic categories

Let us recall that a functor $U\colon \mathcal{V} \to \mathrm{Sets}$ is monadic if it has a left adjoint $F$ and the comparison functor from $\mathcal{V}$ to the category of algebras $\mathrm{Alg}(T)$ of the induced monad $T = UF$ is an equivalence. A category $\mathcal{V}$ is monadic over sets if there exists a monadic functor $U\colon \mathcal{V} \to \mathrm{Sets}$. Monadic categories are precisely those given by a class of single-sorted infinitary operations and a class of equations such that free algebras exist [14]. Free algebras always exist if $\mathcal{V}$ is determined by a set of operations and a set of equations. In any case, the elements of $UF(n)$ (where $n$ is a cardinal and $F(n)$ is a free algebra over $n$) correspond to $n$-ary terms.

The first result we are going to prove is a straightforward generalization of the characterization of semi-abelian varieties given in [7]. We shall follow the presentation given in [3].

Let us recall that in any finitely complete pointed category the *split short five lemma* means the following statement: given a diagram (1)

$$\begin{tikzcd}
A' \arrow[r, "k'"'] \arrow[d, "f"] & B' \arrow[r, shift right, "p'"'] \arrow[d, "g"] & C' \arrow[l, shift right, "s'"'] \arrow[d, "h"'] \\
A \arrow[r, "k"'] & B \arrow[r, shift right, "p"'] & C \arrow[l, shift right, "s"']
\end{tikzcd}$$

where all squares are commutative, $p \circ s = 1_C$, $p' \circ s' = 1_{C'}$, $k = ker(p)$, $k' = ker(p')$ and $f$ and $h$ are isomorphisms, then $g$ is an isomorphism.

2.1. Theorem. *Let $U\colon \mathcal{V} \to \mathrm{Set}$ be a monadic functor. Then $\mathcal{V}$ is semi-abelian if and only if the corresponding theory has a unique constant $0$, binary terms $\alpha_i$ $i \in n$, where $n \geq 1$ is a cardinal, and a $(n+1)$-ary term $\beta$ satisfying the equations*

$$\alpha_i(x,x) = 0 \quad for \quad i \in n$$

*and*

$$\beta(\alpha_0(x,y),\alpha_1(x,y),...,\alpha_i(x,y),...,y) = x.$$

Proof. Let $F(x,y)$ and $F(y)$ be the free algebras on $\{x,y\}$ and $\{y\}$ respectively, and let $p\colon F(x,y) \to F(y)$ be the homomorphism determined by $p(x) = p(y) = y$. Then $p$ is split by the inclusion $s\colon F(y) \to F(x,y)$. Let $k\colon K \to F(x,y)$ be a kernel of $p$ and $A$ a subalgebra of $F(x,y)$ generated by $UK \cup UF(y)$. Since the codomain restriction of $k$ is a kernel of the domain restriction $p'\colon A \to F(y)$ of $p$, $A = F(x,y)$ by the split short five lemma. Hence there are elements $k_i \in K$, with $i \in n$, (where $n \geq 1$ is a cardinal) and a $(n+1)$-ary term $\beta$ such that $x = \beta(k_0,k_1,...,k_i,...,y)$. Since $k_i \in K$, there are binary terms $\alpha_i(x,y)$ such that $\alpha_i(x,y) = k_i$ for $i \in n$. Moreover, one obviously has that $\alpha_i(x,x) = 0$.

Conversely, assume that the terms satisfying the conditions in the theorem exist. Let us then consider the diagram (1), and we are going to prove that the arrow $g$ is an isomorphism.

First consider $a$ and $b$ in $B'$ such that $g(a) = g(b)$. Then

$$(h \circ p')(\alpha_i(a,b)) = (p \circ g)(\alpha_i(a,b)) = p(\alpha_i(g(a),g(b))) = p(0) = 0$$

and then $\alpha_i(a,b)$ is in $A'$ for $i \in n$. Since $(k \circ f)(\alpha_i(a,b)) = 0$, it follows that $\alpha_i(a,b) = 0$ for $i \in n$, which implies that $a = b$ because

$$b = \beta(\alpha_0(b,b),\alpha_1(b,b),...,b) = \beta(0,0,...,b) = \beta(\alpha_0(a,b),\alpha_1(a,b),...,b) = a.$$

Consequently, $g$ is injective.

In order to check that $g$ is surjective, let us consider any $b \in B$. We define $a = (s' \circ h^{-1} \circ p)(b)$. We have

$$p(\alpha_i(b,g(a)) = \alpha_i(p(b),(p \circ g)(a)) = \alpha_i(p(b),p(b)) = 0.$$

Hence $\alpha_i(b,g(a))$ is in $A$ for $i \in n$ and then

$$b = \beta(\alpha_0(b,g(a)),\alpha_1(b,g(a)),...,g(a)) = \beta(g(a_0),g(a_1),...,g(a)) = g(\beta(a_0,a_1...,a))$$

where $f(a_i) = \alpha_i(b,g(a))$ for $i \in n$. Thus $g$ is surjective. ∎

2.2. Remark. *a*) The same argument applies to varieties of $S$-sorted algebras, i.e. to monadic categories $U\colon \mathcal{V} \to \mathrm{Set}^S$. One just needs terms $\alpha_i$ for $i \in n$ and $\beta$ in each sort $s \in S$.
*b*) A similar argument allows one to characterize protomodular monadic categories: one simply has to replace the single constant 0 by constants $e_i$ for $i \in n$, with the properties $\alpha_i(x,x) = e_i$ and one keeps the axiom

$$\beta(\alpha_0(x,y),\alpha_1(x,y),...,\alpha_i(x,y),...,y) = x.$$

The following well-known simple lemma [6] immediately follows from the definitions:

2.3. Lemma. *Let $H: \mathcal{C} \to \mathcal{L}$ be a conservative pullback preserving functor, where $\mathcal{C}$ and $\mathcal{L}$ are pointed categories with pullbacks and $\mathcal{L}$ satisfies the split short five lemma. Then $\mathcal{C}$ satisfies the split short five lemma.*

2.4. Example. Let $\mathcal{C}$ be the category of non-unital $C^*$-algebras, where arrows are continuous homomorphisms of involutive algebras. The forgetful functor from $\mathcal{C}$ to the category of involutive algebras preserves finite limits and reflects isomorphisms (see 1.3.3 and 1.3.7 in [9]). Following Lemma 2.3, $\mathcal{C}$ satisfies the split short five lemma. Since $\mathcal{C}$ is monadic via the unit ball functor $U: \mathcal{C} \to \mathrm{Set}$ (see [15], [20]) it is exact and thus semi-abelian. We are now going to give explicitely the operations witnessing this fact.

Following [18], in the theory of non-unital $C^*$-algebras we have the operations

$$\alpha_0(x, y) = \frac{1}{2}x - \frac{1}{2}y$$

and

$$\beta(x, y) = \overline{2}\,\overline{2}(\frac{1}{2}x + \frac{1}{4}y)$$

where

$$\overline{2} = 2(1 \vee 2 \mid z \mid)^{-1}z$$

and

$$\mid z \mid = (z^* \cdot z)^{\frac{1}{2}}.$$

Then

$$\alpha_0(x, x) = \frac{1}{2}x - \frac{1}{2}x = 0$$

and

$$\beta(\alpha_0(x, y), y) = \beta(\frac{1}{2}x - \frac{1}{2}y, y) = \overline{2}\,\overline{2}(\frac{1}{4}x - \frac{1}{4}y + \frac{1}{4}y) = \overline{2}\,\overline{2}(\frac{1}{4}x) = \overline{2}(\frac{1}{2}x) = x.$$

2.5. Remark. The fact that commutative $C^*$-algebras form an exact Maltsev category was first observed in [8].

2.6. Example. The category *CompGrp* of compact groups is monadic via the usual forgetful functor $U: CompGrp \to \mathrm{Set}$. This immediately follows from the existence of free compact groups [11]. Since a group operation is present, *CompGrp* is semi-abelian.

## 3. Localizations of semi-abelian varieties

Let $\mathcal{C}$ be a cocomplete category with a regular generator $G$ and consider the functor $U: \mathcal{C}(G, -): \mathcal{C} \to \mathrm{Set}$. There is a left adjoint $F$ to $U$ sending a set $n$ to the $n$-th copower $n \cdot G$ of $G$. Let $T = UF$ be the induced monad and $H: \mathcal{C} \to \mathrm{Alg}(T)$ be the comparison functor. Since $G$ is a regular generator, $H$ is a full embedding. Theorem 2.1 tells us when $\mathrm{Alg}(T)$ is semi-abelian. In terms of a generator $G$, the conditions in Theorem 2.1 can be expressed as follows:

1. there is exactly one arrow $0\colon G \to 0$ (where 0 is the initial object in $\mathcal{C}$).

2. there exist arrows $\alpha_i\colon G \to 2 \cdot G$, $i \in n$ (where $n \geq 1$ is a cardinal) such that the square

$$\begin{tikzcd} G \arrow[r, "\alpha_i"] \arrow[d] & 2 \cdot G \arrow[d, "\nabla"] \\ 0 \arrow[r] & G \end{tikzcd}$$

commutes ($\nabla$ is the codiagonal).

3. there is an arrow $\beta\colon G \to n \cdot G$ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\beta"] \arrow[dr, "i_1"'] & & n \cdot G \arrow[dl, "{(\alpha_0,\alpha_1,\ldots,\alpha_i,\ldots)}"] \\ & 2 \cdot G & \end{tikzcd}$$

commutes ($i_1$ is the first injection into the coproduct).

We shall call a *semi-abelian generator* any regular generator $G$ satisfying these three conditions. Now, let us recall that a full reflective subcategory $\mathcal{C}$ of a category $\mathcal{L}$ is a localization if the reflector $L\colon \mathcal{L} \to \mathcal{C}$ preserves finite limits.

3.1. Proposition. *A category $\mathcal{C}$ is a localization of a semi-abelian monadic category over* Set *if and only if $\mathcal{C}$ is a cocomplete exact category with a semi-abelian generator.*

Proof. Due to the main result in [21] one already knows that localizations of monadic categories over Set are precisely cocomplete exact categories with a regular generator $G$. More precisely, having such a category $\mathcal{C}$ then the comparison functor $H\colon \mathcal{C} \to \mathrm{Alg}(T)$ is a localization. This immediately yields the sufficiency. For the necessity it suffices to observe that any localization of a semi-abelian category is semi-abelian by Lemma 2.3. ∎

3.2. Remark. There is an evident many-sorted version characterizing localizations of semi-abelian monadic categories over many-sorted sets as cocomplete exact categories having a semi-abelian generator. It is also possible to characterize localizations of semi-abelian varieties of universal algebras (see [22]).

3.3. Proposition. *A pointed category $\mathcal{C}$ having copowers, pullbacks and a semi-abelian generator satisfies the split short five lemma.*

Proof. Under our assumption there is still a left adjoint $F$ to $U = \mathcal{C}(G, -)$ and the comparison functor $H\colon \mathcal{C} \to \mathrm{Alg}(T)$ is a full embedding. Since $U$ preserves pullbacks, the forgetful functor $V\colon \mathrm{Alg}(T) \to \mathrm{Set}$ creates them and $V \circ H = U$, the functor $H$ preserves pullbacks. Since $\mathrm{Alg}(T)$ satisfies the split short five lemma, we conclude by Lemma 2.3 that $\mathcal{C}$ satisfies it as well. ∎

3.4. Remark. In order to give an example of a protomodular locally finitely presentable category $\mathcal{C}$ with a zero object which does not have a semi-abelian generator, we will present it as an essentially algebraic theory $\Gamma$ (see [1]). Let $\Gamma$ contain a unique constant 0, binary total operations $\alpha_0$, $\gamma$ and a binary partial operation $\beta(z,t)$ whose domain of definition $\mathrm{Def}(\beta)$ is given by the equation $\gamma(z,t) = 0$. Let $\Gamma$ contain the equations $\alpha_0(x,x) = 0$, $\gamma(\alpha_0(x,y),y) = 0$ and $\beta(\alpha_0(x,y),y) = x$. Then the argument used in Theorem 2.1 is still valid and $\mathrm{Alg}(T)$ then satisfies the split short five lemma. On the other hand, $\beta$ is not everywhere defined, which means that there is no reason for $\mathrm{Alg}(T)$ to have a semi-abelian generator.

## 4. Protomodular quasivarieties

A category is quasimonadic over Set if it is a full regular epireflective subcategory of a monadic category over Set (i.e. a full reflective subcategory with the property that the unit of the adjunction is a regular epimorphism). Quasimonadic categories over Set are precisely cocomplete regular categories $\mathcal{C}$ with a regular projective regular generator [10]. Again one uses $H\colon \mathcal{C} \to \mathrm{Alg}(T)$ to present $\mathcal{C}$ as a full regular epireflective subcategory of a monadic category.

4.1. Theorem. *Let $\mathcal{C}$ be a pointed category. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is a full regular epireflective subcategory of a semi-abelian monadic category*
2. *$\mathcal{C}$ is cocomplete, regular, and it has a regular projective semi-abelian generator.*

*Moreover, if $\mathcal{C}$ satisfies these equivalent conditions, then $\mathcal{C}$ is protomodular.*

Proof. Every cocomplete regular category $\mathcal{C}$ with a regular projective semi-abelian generator is a full regular epireflective subcategory of a semi-abelian monadic category $\mathrm{Alg}(T)$ by Theorem 2.1.

Conversely, let $\mathcal{C}$ be a regular epireflective subcategory of a semi-abelian monadic category $\mathrm{Alg}(T)$, and let $F(1)$ be a free $T$-algebra on 1. Then the reflection of $F(1)$ to $\mathcal{C}$ is a regular projective semi-abelian generator of $\mathcal{C}$. Moreover, $\mathcal{C}$ is clearly cocomplete and regular.

The last statement in the Theorem follows from Lemma 1.3. ∎

In order to give a characterization of semi-abelian quasivarieties, let us recall that an object $G$ is *abstractly finite* if for any small set $n$ there exists the $n$-th copower $S \cdot G$ of $G$ and, moreover, any arrow $G \to S \cdot G$ factors through $S' \cdot G$ for some finite subset $S'$ of $S$ [13]. Any finitely presentable object is abstractly finite. Then from Corollary 4.4, Corollary 4.6 in [17] and Theorem 4.1 above the following results easily follow:

4.2. Corollary. *A pointed category $\mathcal{C}$ is a regular epireflective subcategory of a protomodular finitary variety of universal algebras if and only if it is cocomplete, regular and has a regular projective abstractly finite semi-abelian generator.*

4.3. Corollary. *A pointed category $\mathcal{C}$ is a protomodular quasivariety if and only if it is cocomplete, regular and has a finitely presentable regular projective semi-abelian generator.*

4.4. Example. The category $\mathrm{Ab}_{tf}$ of torsion-free abelian groups is an example of a pointed protomodular quasivariety. Indeed, $\mathrm{Ab}_{tf}$ is reflective in the category Ab of abelian groups, and it is closed in it under subobjects.

We conclude with the following

4.5. Theorem. *A pointed category $\mathcal{C}$ is a localization of a protomodular quasimonadic category if and only if $\mathcal{C}$ is a cocomplete regular category with a semi-abelian generator.*

Proof. Necessity is clear. Let then $\mathcal{C}$ be a cocomplete regular category with a semi-abelian generator. Following the proof of Theorem 1.1 in [16], the category $\mathcal{C}$ is a localization of its regular epireflective hull in $\mathrm{Alg}(T)$. Then $\mathcal{C}$ is a localization of a protomodular quasimonadic category over Set. ∎

All the results in this section have obvious many-sorted versions.

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