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SIMPLICIAL APPROXIMATION

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ABSTRACT. This paper displays an approach to the construction of the homotopy theory of simplicial sets and the corresponding equivalence with the homotopy theory of topological spaces which is based on simplicial approximation techniques. The required simplicial approximation results for simplicial sets and their proofs are given in full. Subdivision behaves like a covering in the context of the techniques displayed here.

Introduction

The purpose of this paper is to display a different approach to the construction of the homotopy theory of simplicial sets and the corresponding equivalence with the homotopy theory of topological spaces. This approach is an alternative to existing published proofs [4],[10], but is of a more classical flavour in that it depends heavily on simplicial approximation techniques.

The verification of the closed model axioms for simplicial sets has a reputation for being one of the most difficult proofs in abstract homotopy theory. In essence, that difficulty is a consequence of the traditional approach of deriving the model structure and the equivalence of the homotopy theories of simplicial sets and topological spaces simultaneously. The method displayed here starts with using an idea from localization theory (specifically, a bounded cofibration condition) to show that the cofibrations and weak equivalences of simplicial sets, as we've always known them, together generate a model structure for simplicial sets which is quite easy to derive (Theorem 1.6).

The fibrations for the theory are those maps which have the right lifting property with respect to all maps which are simultaneously cofibrations and weak equivalences. This is the correct model structure, but it is produced at the cost of initially forgetting about Kan fibrations. Putting the Kan fibrations back into the theory in the usual way, and deriving the equivalence of homotopy categories is the subject of the rest of the paper. The equivalence of the combinatorial and topological approaches to constructing homotopy theory is really the central issue of interest, and is the true source of the observed difficulty.

Recovering the Kan fibrations and their basic properties as part of the theory is done in a way which avoids the usual theory of minimal fibrations. Historically, the theory of minimal fibrations has been one of the two known general techniques for recovering

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information about the homotopy types of realizations of simplicial sets. The other is simplicial approximation.

Simplicial approximation theory is a part of the classical literature [1],[2], but it was never developed in a way that was systematic enough to lead to results about model structures. That gap is addressed here: the theory of the subdivision and dual subdivision is developed, both for simplicial complexes and simplicial sets, in Sections 2 and 3, and the fundamental result that the double subdivision of a simplicial set factors through a polyhedral complex in the same homotopy type (Lemma 4.4 and Proposition 4.5) appears in Section 4. The simplicial approximation theory for simplicial sets is most succinctly expressed here in Theorem 4.7 and Corollary 4.8.

The double subdivision result is the basis for everything that follows, including excision (Theorem 5.2), which leads directly to the equivalence of the homotopy categories of simplicial sets and topological spaces in Theorem 5.4 and Corollary 5.5. The Milnor Theorem which asserts that the combinatorial homotopy groups of a fibrant simplicial set coincide with the ordinary homotopy groups of its topological realization (Theorem 6.7) is proved in Section 6, in the presence of a combinatorial proof of the assertion that the subdivision functors preserve anodyne extensions (Lemma 6.4).

One of the more interesting outcomes of the present development is that, with appropriately sharp simplicial approximation tools in hand, the subdivisions of a finite simplicial set behave like coverings. In particular, from this point of view, every simplicial set is locally a Kan complex (Lemma 7.1), and the methods for manipulating homotopy types then follow almost by exact analogy with the theory of locally fibrant simplicial sheaves or presheaves [5], [6]. In that same language, we can show that every fibration which is a weak equivalence has the "local right lifting property" with respect to all inclusions of finite simplicial sets (Lemma 7.3), and then this becomes the main idea leading to the coincidence of fibrations as defined here and Kan fibrations (Corollary 7.6). The same collection of techniques almost immediately implies the Quillen result (Theorem 7.7) that the realization of a Kan fibration is a Serre fibration[1]. The development of Kan's Ex^{∞} functor (Lemma 7.9, Theorem 7.10) is also accomplished from this point of view in a simple and conceptual way.

This paper is not a complete exposition, even of the basic homotopy theory of simplicial sets. I have chosen to rely on existing published references for the development of the simplicial (or combinatorial) homotopy groups of Kan complexes [4], [8], and of other basic constructions such as long exact sequences in simplicial homotopy groups for fibre sequences of Kan complexes, as well as the standard theory of anodyne extensions. Other required combinatorial tools which are not easily recovered from the literature are developed here.

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[1]There is an erratum to this proof on p.72.

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1. Closed model structure

Say that a map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ of topological realizations is a weak equivalence. A *cofibration* of simplicial sets is a monomorphism, and a *fibration* is a map which has the right lifting property with respect to all trivial cofibrations. All fibrations are Kan fibrations in the usual sense; it comes out later (Corollary 7.6) that all Kan fibrations are fibrations. As usual, we say that a fibration (respectively cofibration) is *trivial* if it is also a weak equivalence.

1.1. Lemma. *Suppose that X is a simplicial set with at most countably many non-degenerate simplices. Then the set of path components $\pi_0|X|$ and all homotopy groups $\pi_i(|X|, x)$ of the realization of X are countable.*

Proof. The statement about path components is trivial. We can assume that X is connected to prove the statement about the homotopy groups, with respect to a fixed base point $x \in X_0$.

The fundamental group $\pi_1(|X|, x)$ is countable, by the Van Kampen theorem. The space $|X|$ plainly has countable homology groups

$$H_*(|X|, \mathbb{Z}) \cong H_*(X, \mathbb{Z})$$

in all degrees.

Suppose that the continuous map $p : Y \to Z$ is a Serre fibration with connected base Z such that Z and the fibre F have countable integral homology groups in all degrees, and such that $\pi_1 Z$ is countable. Then a Serre spectral sequence argument (with twisted coefficients) shows that the homology groups $H_*(Y, \mathbb{Z})$ are countable in all degrees.

This last statement applies in particular to the universal cover $p : Y_1 \to |X|$ of the realization $|X|$. Then the Hurewicz theorem (in its classical form — see [14], for example) implies that

$$\pi_2|X| \cong \pi_2 Y_1 \cong H_2(Y_1, \mathbb{Z})$$

is countable.

Inductively, one shows that the n-connected covers $Y_n \to |X|$ have countable homology, and in particular the groups

$$\pi_{n+1}|X| \cong \pi_{n+1} Y_n \cong H_{n+1}(Y_n, \mathbb{Z})$$

are countable. ■

The class of trivial cofibrations of simplicial sets satisfies a bounded cofibration condition:

1.2. LEMMA. *Suppose that A is a countable simplicial set, and that there is a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of simplicial set maps in which i is a trivial cofibration. Then there is a countable subcomplex $D \subset Y$ such that $A \to Y$ factors through D, and such that the map $D \cap Y \to D$ is a trivial cofibration.

PROOF. We can assume that A is a connected subcomplex of Y. The homotopy groups $\pi_i(|A|)$ are countable by Lemma 1.1.

Suppose that x is a vertex of $A = B_0$. Then there is a finite connected subcomplex $L_x \subset Y$ which contains a homotopy $x \to i(y)$ where y is a vertex of X. Write $C_1 = A \cup (\bigcup_x L_x)$. Suppose that w, z are vertices of $C_1 \cap X$ which are homotopic in C_1. Then there is a finite connected subcomplex $K_{w,z} \subset X$ such that $w \simeq z$ in $K_{w,z}$. Let $B_1 = C_1 \cup (\bigcup_{w,z} K_{w,z})$. Then every vertex of A is homotopic to a vertex of $C_1 \cap X$ inside C_1, and any two vertices $z, w \in C_1 \cap X$ which are homotopic in C_1 are also homotopic in $B_1 \cap X$. Observe also that the maps $B_0 \subset C_1 \subset B_1$ are π_0 isomorphisms.

Repeat this process countably many times to find a sequence

$$A = B_0 \subset C_1 \subset B_1 \subset C_2 \subset B_2 \subset \dots$$

of countable subcomplexes of Y. Set $B = \bigcup B_i$. Then B is a countable subcomplex of Y such that $\pi_0(B \cap X) \cong \pi_0(B) \cong \pi_0(A) = *$.

Pick $x \in B \cap X$. The same argument (which does not disturb the connectivity) can now be repeated for the countable list of elements in all higher homotopy groups $\pi_q(B, x)$, to produce the desired countable subcomplex $D \subset Y$. ■

1.3. LEMMA. *Suppose that $p : X \to Y$ is a map of simplicial sets which has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. Then p is a weak equivalence.*

PROOF. The map p is a homotopy equivalence, by a standard argument. In effect, there is a commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "1_Y"'] \arrow[ur, dotted, "i"] & Y \end{tikzcd}$$

and then a commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(1_X, ip)}"] \arrow[d] & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[r, "p\sigma"'] \arrow[ur, dotted, "H"] & Y \end{tikzcd}$$

so that $pi = 1_Y$ and then H is a homotopy $1_X \simeq ip$. Here, $\sigma : X \times \Delta^1 \to X$ is the projection onto X. ■

1.4. Lemma. *Every map $f : X \to Y$ of simplicial sets has factorizations*

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where i is a trivial cofibration and p is a fibration, and j is a cofibration and q is a trivial fibration.

Proof. A standard transfinite small object argument based on Lemma 1.2 produces the factorization $f = p \cdot i$. Also, f has a factorization $f = q \cdot j$, where j is a cofibration and q has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. But then q is a trivial fibration on account of Lemma 1.3. ■

1.5. Lemma. *Every trivial fibration $p : X \to Y$ has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$.*

Proof. Find a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "p"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where j is a cofibration and the fibration q has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$. Then q is a trivial fibration by Lemma 1.3, so that j is a trivial cofibration. The lifting r exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] \arrow[ur, "r"] & Y \end{tikzcd}$$

It follows that p is a retract of q, and so p has the desired lifting property. ■

1.6. Theorem. *With these definitions, the category* $\mathbf{S}$ *of simplicial sets satisfies the axioms for a closed simplicial model category.*

Proof. The axioms **CM1**, **CM2** and **CM3** have trivial verifications. The factorization axiom **CM5** is a consequence of Lemma 1.4, while the axiom **CM4** is a consequence of Lemma 1.5.

The function spaces $\mathbf{hom}(X, Y)$ are exactly as we know them: an n-simplex of this simplicial set is a map $X \times \Delta^n \to Y$.

If $i : A \to B$ and $j : C \to D$ are cofibrations, then the induced map

$$(B \times C) \cup_{A \times C} (A \times D) \to B \times D$$

is a cofibration, which is trivial if either i or j is trivial. The first part of the statement is obvious set theory, while the second part follows from the fact that the realization functor preserves products. ■

1.7. Lemma. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[d] \\ B \arrow[r, "g_*"'] & D \end{tikzcd}$$

where i is a cofibration and g is a weak equivalence. Then g_* is a weak equivalence.

Proof. All simplicial sets are cofibrant, and this result follows from the standard formalism for categories of cofibrant objects [4, II.8.5]. ■

The other axiom for properness, which says that weak equivalences are stable under pullback along fibrations, is proved in Corollary 7.8.

2. Subdivision operators

Write NX for the poset of non-degenerate simplices of a simplicial set X, ordered by the face relationship. Here "x is a face of y" means that the subcomplex $\langle x\rangle$ of X which is generated by x is a subcomplex of $\langle y\rangle$. Let $BX = BNX$ denote its classifying space. Any simplex $x \in X$ can be written uniquely as $x = s(y)$ where s is an iterated degeneracy and y is non-degenerate. It follows that any simplicial set map $f : X \to Y$ determines a functor $f_* : NX \to NY$ where $f_*(x)$ is uniquely determined by $f(x) = t \cdot f_*(x)$ with t an iterated degeneracy and $f_*(x)$ non-degenerate.

Say that a simplicial set K is a *polyhedral complex* if K is a subcomplex of BP for some poset P. The simplices of a polyhedral complex K are completely determined by their vertices; in this case the non-degenerate simplices of K are precisely those simplices x for which the list $(v_i x)$ of vertices of x consists of distinct elements.

If P is a poset there is a map $\gamma : BBP \to BP$ which is best described categorically as the functor $\gamma : NBP \to P$ which sends a non-degenerate simplex $x : \mathbf{n} \to P$ to the element $x(n) \in P$. This is the so-called "last vertex map", and is natural in poset morphisms $P \to Q$. In particular all ordinal number maps $\theta : \mathbf{m} \to \mathbf{n}$ induce commutative diagrams of simplicial set maps

$$\begin{tikzcd} B\Delta^m \arrow[r, "\theta_*"] \arrow[d, "\gamma"'] & B\Delta^n \arrow[d, "\gamma"] \\ \Delta^m \arrow[r, "\theta"'] & \Delta^n \end{tikzcd}$$

Similarly, if $K \subset BP$ is a polyhedral complex then $\gamma|_K$ takes values in K by the commutativity of all diagrams

$$\begin{tikzcd} B\Delta^n \arrow[r, "x_*"] \arrow[d, "\gamma"'] & BBP \arrow[d, "\gamma"] \\ \Delta^n \arrow[r, "x"'] & BP \end{tikzcd}$$

arising from simplices x of K.

For a general simplicial set X, we write

$$\operatorname{sd} X = \varinjlim_{\Delta^n \to X} B\Delta^n,$$

where the colimit is indexed over the simplex category of X. The object $\operatorname{sd} X$ is called the *subdivision* of X. The maps $\gamma : B\Delta^n \to \Delta^n$ together determine a natural map $\gamma : \operatorname{sd} X \to X$. Note that there is an isomorphism $\operatorname{sd} \Delta^n \cong B\Delta^n$.

Suppose that x is a non-degenerate simplex of X. Then the inclusion $\langle x \rangle \subset X$ induces an isomorphism $N\langle x \rangle = \langle x \rangle \cap NX$. Every simplicial set X is a colimit of the subcomplexes $\langle x \rangle$ generated by non-degenerate simplices x. Also the canonical maps $\operatorname{sd} \Delta^n \cong B\Delta^n \to BX$ which are induced by all simplices of X together induce a natural map

$$\pi : \operatorname{sd} X \to BX.$$

The map π is surjective, since every non-degenerate simplex x (and any string of its faces) is in the image of some simplex $\sigma : \Delta^n \to X$.

It follows that there is a commutative diagram

$$\begin{tikzcd} \varinjlim_{x \in NX} \operatorname{sd}\langle x \rangle \arrow[r, "\cong"] \arrow[d, "\pi_*"'] & \operatorname{sd} X \arrow[d, "\pi"] \\ \varinjlim_{x \in NX} B\langle x \rangle \arrow[r] & BX \end{tikzcd} \tag{1}$$

The bottom horizontal map $\varinjlim_x B\langle x \rangle \to BX$ is surjective, because any string $x_0 \leq \dots \leq x_n$ of non-degenerate simplices of X is in the image of the corresponding string of non-degenerate simplices of the subcomplex $\langle x_n \rangle$. If $\alpha \in B\langle x_n \rangle$ and $\beta \in B\langle y_n \rangle$ map to the same element of BX_n they are both images of a string $\gamma \in B(\langle x\rangle \cap \langle y\rangle)_n$. This element γ is in the image of some map $B\langle z\rangle_n \to B(\langle x\rangle \cap \langle y\rangle)_n$. Thus there is a $\zeta \in B\langle z\rangle_n$ which maps to both α and β. It follows that α and β represent the same element in $\varinjlim_x B\langle x\rangle$, and so the map $\varinjlim_x B\langle x\rangle \to BX$ is an isomorphism.

2.1. Lemma. *The map $\pi : \operatorname{sd} X \to BX$ is surjective in all degrees, and is a bijection on vertices. Consequently, two simplices $u, v \in \operatorname{sd} X_n$ have the same image in BX if and only if they have the same vertices.*

Proof. We have already seen that π is surjective.

For every vertex $v \in \operatorname{sd} X$ there is a unique non-degenerate n-simplex $x \in X$ of minimal dimension (the carrier of v) such that v lifts to a vertex of $\operatorname{sd}\Delta^n$ under the map $x_* : \operatorname{sd}\Delta^n \to \operatorname{sd} X$. Observe that

$$v = x_*([0, 1, \dots, n])$$

by the minimality of dimension of x. We see from the diagram

$$\begin{tikzcd} \operatorname{sd}\Delta^n \arrow[d, "x_*"'] \arrow[dr, "x_*"] & \\ \operatorname{sd} X \arrow[r, "\pi"'] & BX \end{tikzcd}$$

that $\pi(v) = \langle x\rangle$. It follows that the function $v \mapsto \pi(v) = \langle x\rangle$ is injective. ∎

Let K be a polyhedral complex with imbedding $K \subset BP$ for some poset P. Every non-degenerate simplex x of K can be represented by a monomorphism of posets $x : \mathbf{n} \to P$ and hence determines a simplicial set monomorphism $x : \Delta^n \to K$. In particular, the map x induces an isomorphism $\Delta^n \cong \langle x\rangle \subset K$. It follows from the comparison in the diagram (1) that the map $\pi : \operatorname{sd} K \to BK$ is an isomorphism for all polyhedral complexes K.

Suppose that L is obtained from K by attaching a non-degenerate n-simplex. The induced diagram

$$\begin{tikzcd} \operatorname{sd}\partial\Delta^n \arrow[r] \arrow[d, "i"'] & \operatorname{sd} K \arrow[d, "i_*"] \\ \operatorname{sd}\Delta^n \arrow[r] & \operatorname{sd} L \end{tikzcd}$$

is a pushout, in which the maps i and i_* are monomorphisms of simplicial sets. It follows in particular that the subdivision functor sd preserves monomorphisms as well as pushouts (sd has a right adjoint).

Let C and D be subcomplexes of a simplicial set X such that $X = C \cup D$. Then the diagram of monomorphisms

$$\begin{tikzcd} N(C \cap D) \arrow[r] \arrow[d] & ND \arrow[d] \\ NC \arrow[r] & NX \end{tikzcd}$$

is a pullback and a pushout of partially ordered sets, and the diagram

$$\begin{tikzcd} B(C \cap D) \arrow[r] \arrow[d] & BD \arrow[d] \\ BC \arrow[r] & BX \end{tikzcd} \tag{2}$$

is a pullback and a pushout of simplicial sets.

There is a homeomorphism $h : |\operatorname{sd}\Delta^n| \to |\Delta^n|$, which is the affine map that takes a vertex $\sigma = \{v_0, \dots, v_k\}$ to the barycentre $b_\sigma = \frac{1}{k+1}\sum v_i$. There is a convex homotopy $H : h \simeq |\gamma|$ which is defined by $H(\alpha, t) = th(\alpha) + (1-t)|\gamma|(\alpha)$. The homeomorphism h and the homotopy H respect inclusions of simplices. Instances of the map h and homotopy H can therefore be patched together to give a homeomorphism

$$h : |\operatorname{sd} K| \xrightarrow{\cong} |K|$$

and a homotopy

$$H : h \simeq |\gamma|$$

for each polyhedral complex K. The homeomorphism h and the homotopy H both commute with inclusions of polyhedral complexes.

3. Classical simplicial approximation

In this section, "simplicial complex" has the classical meaning: a simplicial complex K is a set of non-empty subsets of some vertex set V which is closed under taking subsets. In the presence of a total order $(V, \leq)$ on V, a simplicial complex K determines a unique polyhedral subcomplex $K \subset BV$ in which an n-simplex $\sigma \in BV$ is in K if and only if its set of vertices forms a simplex of the simplicial complex K.

Any map of simplicial complexes $f : K \to L$ in the traditional sense determines a simplicial set map $f : K \to L$ by first imposing an orientation on the vertices of L, and then by choosing a compatible orientation on the vertices of K. It is usually, however, better to observe that a simplicial complex map f induces a map $f_* : NK \to NL$ on the corresponding posets of simplices, and hence induces a map $f_* : BNK \to BNL$ of the associated subdivisions.

Suppose given maps of simplicial complexes

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & X \\ L & \end{tikzcd}$$

where i is a cofibration (or monomorphism) and L is finite. Suppose further that there is a continuous map $f : |L| \to |X|$ such that the diagram

$$\begin{tikzcd} |K| \arrow[r, "\alpha_*"] \arrow[d, "i_*"'] & |X| \\ |L| \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. There is a subdivision $\operatorname{sd}^n L$ of L such that in the composite

$$|\operatorname{sd}^n L| \xrightarrow{h^n} |L| \xrightarrow{f} |X|,$$

every simplex $|\sigma| \subset |\operatorname{sd}^n L|$ maps into the star $st(v)$ of some vertex $v \in X$.

Recall that $\operatorname{st}(v)$ for a vertex v can be characterized as an open subset of $|X|$ by

$$\operatorname{st}(v) = |X| - |X_v|,$$

where X_v is the subcomplex of X consisting of those simplices which do not have v as a vertex. One can also characterize $\operatorname{st}(v)$ as the set of those linear combinations $\sum \alpha_v v \in |X|$ such that $\alpha_v \neq 0$. Note that the star $\operatorname{st}(v)$ of a vertex v is convex.

The homeomorphism $h : |\operatorname{sd} K| \to |K|$ is defined on vertices by sending σ to the barycentre $b_\sigma \in |\sigma|$. Observe that if $\sigma_0 \leq \cdots \leq \sigma_n$ is a simplex of $\operatorname{sd} K$ and v is a vertex of some σ_i then the image of any affine linear combination $\sum \alpha_i \sigma_i$ is the affine sum $\sum \alpha_i b_{\sigma_i}$ of the barycentres. Then since v appears non-trivially in b_{σ_i} it must appear non-trivially in the sum of the barycentres. This means that $h(\operatorname{st}(\sigma)) \subset \operatorname{st}(\gamma(\sigma))$, where $\gamma : \operatorname{sd} K \to K$ is the last vertex map. In other words γ is a simplicial approximation of the homeomorphism h, as defined by Spanier [13].

It follows that γ^n is a simplicial approximation of h^n; in effect,

$$h^n(\operatorname{st}(v)) \subset h^{n-1}(\operatorname{st}(\gamma(v)) \subset h^{n-2}(\operatorname{st}(\gamma^2(v)) \subset \dots$$

There is a corresponding convex homotopy $H : |\gamma^n| \to h^n$ defined by

$$H(x,t) = (1-t)\gamma^n(x) + th^n(x)$$

which exists precisely because γ^n is a simplicial approximation of h^n.

The point is now that the composite

$$|\operatorname{sd}^n L| \xrightarrow{h^n} |L| \xrightarrow{f} |X|,$$

admits a simplicial approximation for n sufficiently large since $fh^n(\operatorname{st}(v)) \subset \operatorname{st}(\phi(w))$ for some vertex $\phi(w)$ of X, and the assignment $w \mapsto \phi(w)$ defines a simplicial complex map $\phi : \operatorname{sd}^n L \to \operatorname{sd} X \to X$ whose realization ϕ_* is homotopic to fh^n by a convex homotopy no matter how the individual vertices $\phi(w)$ are chosen subject to the condition on stars above. In particular, the function $w \mapsto \phi(w)$ can be chosen to extend the vertex map underlying the simplicial complex map $\alpha\gamma^n$. It follows that there is a simplicial complex map $\phi : \operatorname{sd}^n L \to X$ such that the diagram of simplicial complex maps

$$\begin{tikzcd}
\operatorname{sd}^n K \arrow[r, "\gamma^n"] \arrow[d, "i_*"'] & K \arrow[r, "\alpha"] & X \\
\operatorname{sd}^n L \arrow[urr, "\phi"'] & &
\end{tikzcd}$$

commutes, and such that $|\phi| \simeq fh^n$ via a homotopy H' that extends the homotopy $|\alpha|H : |\alpha||\gamma^n| \to |\alpha|h^n$.

The homotopy $fH : f|\gamma^n| \to fh^n$ also extends the homotopy αH. It follows that there is a commutative diagram

$$\begin{tikzcd}
(|\operatorname{sd}^n K| \times \Delta^2) \cup (|\operatorname{sd}^n L| \times \Lambda^2_2) \arrow[r, "{(s_0\alpha H,(fH,H'))}"] \arrow[d] & {|X|} \\
{|\operatorname{sd}^n L| \times \Delta^2} \arrow[ur, "K"'] &
\end{tikzcd}$$

Then the composite

$$|\operatorname{sd}^n L| \times \Delta^1 \xrightarrow{1\times d^2} |\operatorname{sd}^n L| \times \Delta^2 \xrightarrow{K} |X|$$

is a homotopy from $|\phi|$ to the composite $f|\gamma^n|$ rel $|\operatorname{sd}^n K|$, and we have proved

3.1. Theorem. *Suppose given simplicial complex maps*

$$\begin{tikzcd}
K \arrow[r, "\alpha"] \arrow[d, "i"] & X \\
L &
\end{tikzcd}$$

where i is an inclusion and L is finite. Suppose that $f : |L| \to |X|$ is a continuous map such that $f|i| = |\alpha|$. Then there is a commutative diagram of simplicial complex maps

$$\begin{tikzcd}
\operatorname{sd}^n K \arrow[r, "\gamma^n"] \arrow[d, "i"'] & K \arrow[r, "\alpha"] & X \\
\operatorname{sd}^n L \arrow[urr, "\phi"'] & &
\end{tikzcd}$$

such that $|\phi| \simeq f|\gamma^n|$ rel $|\operatorname{sd}^n K|$.

One final wrinkle: the maps in the statement of Theorem 3.1 are simplicial complex maps which may not reflect the orientations of the underlying simplicial set maps. One gets around this by subdividing one more time: the corresponding diagram

$$\begin{tikzcd}
N\operatorname{sd}^n K \arrow[r, "N\gamma^n"] \arrow[d, "Ni"'] & NK \arrow[r, "N\alpha"] & NX \\
N\operatorname{sd}^n L \arrow[urr, "N\phi"'] & &
\end{tikzcd}$$

of poset morphisms of non-degenerate simplices certainly commutes, and hence induces a commutative diagram of simplicial set maps

$$\begin{tikzcd} BN\,\mathrm{sd}^n K \arrow[r, "BN\gamma^n"] \arrow[d, "BNi"'] & BNK \arrow[r, "BN\alpha"] & BNX \\ BN\,\mathrm{sd}^n L \arrow[urr, "BN\phi"'] & & \end{tikzcd}$$

It follows that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \mathrm{sd}^{n+1} K \arrow[r, "\gamma^{n+1}"] \arrow[d, "i"'] & K \arrow[r, "\alpha"] & X \\ \mathrm{sd}^{n+1} L \arrow[urr, "\phi\gamma"'] & & \end{tikzcd}$$

provided that the original maps α and i are themselves morphisms of simplicial sets. Finally, there is a homotopy $|\phi| \simeq f|\gamma^n|$ rel $|\mathrm{sd}^n K|$, so that $|\phi\gamma| \simeq f|\gamma^{n+1}|$ rel $|\mathrm{sd}^{n+1} K|$. We have proved the following:

3.2. Corollary. *Suppose given simplicial set maps*

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"] & X \\ L & \end{tikzcd}$$

between polyhedral complexes, where i is a cofibration and L is finite. Suppose that $f : |L| \to |X|$ is a continuous map such that $f|i| = |\alpha|$. Then there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \mathrm{sd}^n K \arrow[r, "\gamma^n"] \arrow[d, "i"'] & K \arrow[r, "\alpha"] & X \\ \mathrm{sd}^n L \arrow[urr, "\phi"'] & & \end{tikzcd}$$

such that $|\phi| \simeq f|\gamma^n|$ rel $|\mathrm{sd}^n K|$.

4. Approximation results for simplicial sets

Note that $\mathrm{sd}(\Delta^n) = C\,\mathrm{sd}(\partial\Delta^n)$, where in general CK denotes the cone on a simplicial set K. This is a consequence of the following

4.1. Lemma. *Suppose that P is a poset, and that CP is the poset cone, which is constructed from P by formally adjoining a terminal object. Then there is an isomorphism $BCP \cong CBP$.*

Proof. Any functor $\gamma : \mathbf{n} \to CP$ determines a pullback diagram

$$\begin{tikzcd} \mathbf{k} \arrow[r] \arrow[d] & P \arrow[d] \\ \mathbf{n} \arrow[r] & CP \end{tikzcd}$$

where k is the maximum vertex in $\mathbf{n}$ which maps into P. It follows that

$$BCP_n = BP_n \sqcup BP_{n-1} \sqcup \cdots \sqcup BP_0 \sqcup \{*\},$$

where the indicated vertex $*$ corresponds to functors $\mathbf{n} \to CP$ which take all vertices into the cone point. The simplicial structure maps do the obvious thing under this set of identification, and so BCP is isomorphic to CBP (see [4], p.193). ∎

Following [2], say that a simplicial set X is *regular* if for every non-degenerate simplex α of X the diagram

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d_0\alpha"] \arrow[d, "d^0"'] & \langle d_0\alpha \rangle \arrow[d] \\ \Delta^n \arrow[r, "\alpha"'] & \langle \alpha \rangle \end{tikzcd} \qquad (3)$$

is a pushout.

It is an immediate consequence of the definition (and the fact that trivial cofibrations are closed under pushout) that all subcomplexes $\langle \alpha \rangle$ of a regular simplicial set X are weakly equivalent to a point. We also have the following:

4.2. Lemma. *Suppose that X is a simplicial set such that all subcomplexes $\langle \alpha \rangle$ which are generated by non-degenerate simplices α are contractible. Then the canonical map $\pi : \mathrm{sd}\, X \to BX$ is a weak equivalence.*

Proof. We argue along the sequence of pushout diagrams

$$\begin{tikzcd} \bigsqcup_{\alpha \in N_n X} \partial\langle \alpha \rangle \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{\alpha \in N_n X} \langle \alpha \rangle \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

The property that all non-degenerate simplices of X generate contractible subcomplexes is shared by all subcomplexes of X, so inductively we can assume that the natural maps $\pi : \mathrm{sd}\, \partial\langle \alpha \rangle \to B\partial\langle \alpha \rangle$ and $\pi : \mathrm{sd}\, \mathrm{sk}_{n-1} X \to B\, \mathrm{sk}_{n-1} X$ are weak equivalences.

But the comparison map $\gamma : \mathrm{sd}\langle \alpha \rangle \to \langle \alpha \rangle$ is a weak equivalence, and $\langle \alpha \rangle$ is contractible by assumption. At the same time $B\langle \alpha \rangle$ is a cone on $B\partial\langle \alpha \rangle$ by Lemma 4.1, so the comparison $\pi : \mathrm{sd}\langle \alpha \rangle \to B\langle \alpha \rangle$ is a weak equivalence for all non-degenerate simplices α. The gluing lemma (see also (2)) therefore implies that the map $\pi : \mathrm{sd}\, \mathrm{sk}_n X \to B\, \mathrm{sk}_n X$ is a weak equivalence. ∎

4.3. Corollary. *The canonical map $\pi : \mathrm{sd}\, X \to BX$ is a weak equivalence for all regular simplicial sets X.*

Write N_*K for the poset of non-degenerate simplices of K, with the opposite order, and write $B_*K = BN_*K$ for the corresponding polyhedral complex. The cosimplicial space $\mathbf{n} \mapsto B_*\Delta^n$ determines a functorial simplicial set

$$\mathrm{sd}_*\, X = \varinjlim_{\Delta^n \to X} B_*\Delta^n,$$

and the "first vertex maps" $\gamma_* : B_*\Delta^n \to \Delta^n$ together determine a functorial map $\gamma_* : \mathrm{sd}_*\, X \to X$. Similarly, the maps $B_*\Delta^n \to B_*X$ induced by the simplices $\Delta^n \to K$ of K together determine a natural simplicial set map $\pi_* : \mathrm{sd}_*\, X \to B_*X$. Observe that the map $\pi_* : \mathrm{sd}_*\, \Delta^n \to B_*\Delta^n$ is an isomorphism. We shall say that $\mathrm{sd}_*\, X$ is the *dual subdivision* of the simplicial set X.

4.4. Lemma. *The simplicial set $\mathrm{sd}_*\, X$ is regular, for all simplicial sets X.*

Proof. Suppose that α is a non-degenerate n-simplex of $\mathrm{sd}_*\, X$. Then there is a unique non-degenerate r-simplex y of X of minimal dimension (the carrier of α) and a unique non-degenerate n-simplex $\sigma \in \mathrm{sd}_*\, \Delta^r$ such that the classifying map $\alpha : \Delta^n \to \mathrm{sd}_*\, X$ factors as the composite

$$\Delta^n \xrightarrow{\sigma} \mathrm{sd}_*\, \Delta^r \xrightarrow{y_*} \mathrm{sd}_*\, X.$$

This follows from the fact that the functor sd_* preserves pushouts and monomorphisms. Observe that $\sigma(0) = [0, 1, \dots r]$, for otherwise $\sigma \in \mathrm{sd}\, \partial\Delta^r$ and r is not minimal.

The composite diagram

$$\begin{tikzcd} \Delta^{n-1} \arrow[r] \arrow[d, "d^0"'] & \mathrm{sd}_*\, \partial\Delta^r \arrow[r] \arrow[d] & \mathrm{sd}_*\, \partial\langle y\rangle \arrow[d] \\ \Delta^n \arrow[r, "\sigma"'] & \mathrm{sd}_*\, \Delta^r \arrow[r] & \mathrm{sd}_*\langle y\rangle \end{tikzcd} \tag{4}$$

is a pullback (note that all vertical maps are monomorphisms), and the diagram (3) factors through (4) via the diagram of monomorphisms

$$\begin{tikzcd} \langle d_0\alpha\rangle \arrow[r] \arrow[d] & \mathrm{sd}_*\, \partial\langle y\rangle \arrow[d] \\ \langle\alpha\rangle \arrow[r] & \mathrm{sd}_*\langle y\rangle \end{tikzcd}$$

It follows that the diagram (3) is a pullback.

If two simplices v, w of Δ^n map to the same simplex in $\langle\alpha\rangle$, then $\sigma(v)$ and $\sigma(w)$ map to the same simplex of $\mathrm{sd}_*\langle y\rangle$. But then $\sigma(v) = \sigma(w)$ or both simplices lift to $\mathrm{sd}_*\, \partial\Delta^r$, since sd_* preserves pushouts and monomorphisms. If $\sigma(v) = \sigma(w)$ then $v = w$ since σ is a non-degenerate simplex of the polyhedral complex $\mathrm{sd}_*\, \Delta^r$. Otherwise, $\sigma(v)$ and $\sigma(w)$ both lift to $\mathrm{sd}_*\, \partial\Delta^r$, and so v and w are in the image of d^0. Thus all identifications arising from the epimorphism $\Delta^n \to \langle\alpha\rangle$ take place inside the image of $d^0 : \Delta^{n-1} \to \Delta^n$, and the square (4) is a pushout. ∎

4.5. Proposition. *Suppose that X is a regular simplicial set. Then the dotted arrow exists in the diagram*

$$\begin{tikzcd} \operatorname{sd} X \arrow[r, "\pi"] \arrow[d, "\gamma"'] & BX \arrow[dl, dotted] \\ X & \end{tikzcd}$$

making it commute.

Proof. All subcomplexes of a regular simplicial set are regular, so it's enough to show (see the comparison (1)) that the dotted arrow exists in the diagram

$$\begin{tikzcd} \operatorname{sd}\langle\alpha\rangle \arrow[r, "\pi"] \arrow[d, "\gamma"'] & B\langle\alpha\rangle \arrow[dl, dotted] \\ \langle\alpha\rangle & \end{tikzcd}$$

for a non-degenerate simplex α, subject to the obvious inductive assumption on the dimension of α: we assume that there is a commutative diagram

$$\begin{tikzcd} \operatorname{sd}\langle d_0\alpha\rangle \arrow[r, "\pi"] \arrow[d, "\gamma"'] & B\langle d_0\alpha\rangle \arrow[dl, "\gamma_*"] \\ \langle d_0\alpha\rangle & \end{tikzcd}$$

Consider the pushout diagram

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d_0\alpha"] \arrow[d, "d^0"'] & \langle d_0\alpha\rangle \arrow[d] \\ \Delta^n \arrow[r, "\alpha"'] & \langle\alpha\rangle \end{tikzcd}$$

Then given non-degenerate simplices u, v of Δ^{n-1}_r, $\langle\alpha(u)\rangle = \langle\alpha(v)\rangle$ in $\langle\alpha\rangle$ if and only if either $u = v$ or $u, v \in d^0\Delta^{n-1}$ and $\langle d_0\alpha(u)\rangle = \langle d_0\alpha(v)\rangle$ in $\langle d_0\alpha\rangle$.

Suppose given two strings $u_1 \leq \dots \leq u_k$ and $v_1 \leq \dots \leq v_k$ of non-degenerate simplices of Δ^n such that $\langle\alpha(u_i)\rangle = \langle\alpha(v_i)\rangle$ in $\langle\alpha\rangle$ for $1 \leq i \leq k$. We want to show that these elements of $(\operatorname{sd}\Delta^n)_k$ map to the same element of $\langle\alpha\rangle$ under the composite map

$$\operatorname{sd}\Delta^n \xrightarrow{\gamma} \Delta^n \xrightarrow{\alpha} \langle\alpha\rangle.$$

If this is true for all such pairs of strings, then there is an induced commutative diagram of simplicial set maps

$$\begin{tikzcd} \operatorname{sd}\Delta^n \arrow[r, "\alpha_*"] \arrow[d, "\gamma"'] & B\langle\alpha\rangle \arrow[d, "\gamma_*"] \\ \Delta^n \arrow[r, "\alpha"'] & \langle\alpha\rangle \end{tikzcd}$$

and the Proposition is proved.

We assume inductively that the corresponding diagram

$$\begin{tikzcd} \mathrm{sd}\,\Delta^{n-1} \arrow[r, "d_0\alpha_*"] \arrow[d, "\gamma"'] & B\langle d_0\alpha\rangle \arrow[d, "\gamma_*"] \\ \Delta^{n-1} \arrow[r, "d_0\alpha"'] & \langle\alpha\rangle \end{tikzcd}$$

exists for $d_0\alpha$.

Set $i = k+1$ if all u_i and v_i are in $d^0\Delta^{n-1}$. Otherwise, let i be the minimum index such that u_i and v_i are not in $d^0\Delta^{n-1}$. Observe that a non-degenerate simplex w of Δ^n is outside $d^0\Delta^{n-1}$ if and only if 0 is a vertex of w.

If $i = k+1$ the strings $u_1 \leq \dots \leq u_k$ and $v_1 \leq \dots \leq v_k$ are both in the image of the map $d^0_* : \mathrm{sd}\,\Delta^{n-1} \to \mathrm{sd}\,\Delta^n$, and can therefore be interpreted as elements of $\mathrm{sd}\,\Delta^{n-1}$ which map to the same element of $B\langle d_0\alpha\rangle$. These strings therefore map to the same element in $\langle d_0\alpha\rangle$, and hence to the same element of $\langle\alpha\rangle$.

If $i = 0$ the strings are equal, and hence map to the same element of $\langle\alpha\rangle$.

Suppose that $0 < i < k+1$. Then the simplices $u_j = v_j$ have more than one vertex (including 0), and so the last vertices of u_j and d_0u_j coincide for $j \geq i$. It follows that the strings

$$u_1 \leq \dots \leq u_{i-1} \leq d_0u_i \leq \dots \leq d_0u_k$$

and

$$v_1 \leq \dots \leq v_{i-1} \leq d_0v_i \leq \dots \leq d_0v_k$$

determine elements of $\mathrm{sd}\,\Delta^{n-1}$ having the same images under the map $\gamma : \mathrm{sd}\,\Delta^n \to \Delta^n$ as the respective original strings. These strings also map to the same element of $B\langle d_0\alpha\rangle$ since $d_0u_j = d_0v_j$ for $j \geq i$. The strings $u_1 \leq \dots \leq u_k$ and $v_1 \leq \dots \leq v_k$ therefore map to the same element of $\langle\alpha\rangle$. ∎

4.6. Lemma. *Suppose given a diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

in which i is a cofibration and f is a weak equivalence between objects which are fibrant and cofibrant. Then there is a map $\theta : B \to X$ such that $\theta \cdot i = \alpha$ and $f \cdot \theta$ is homotopic to β rel A.

Proof. The weak equivalence f has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where q is a trivial fibration and j is a trivial cofibration. The object Z is both cofibrant and fibrant, so there is a map $\pi : Z \to X$ such that $\pi \cdot j = 1_X$ and $j \cdot \pi \simeq 1_Z$ rel X. Form the diagram

$$\begin{tikzcd} A \arrow[r, "j\alpha"] \arrow[d, "i"'] & Z \arrow[d, "q"] \\ B \arrow[r, "\beta"'] \arrow[ur, "\omega"] & Y \end{tikzcd}$$

Then the required lift $B \to X$ is $\pi \cdot \omega$. ■

4.7. Theorem. *Suppose given maps of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \\ B & \end{tikzcd}$$

where i is a cofibration of polyhedral complexes and B is finite, and suppose that there is a commutative diagram of continuous maps

$$\begin{tikzcd} {|A|} \arrow[r, "|\alpha|"] \arrow[d, "|i|"'] & {|X|} \\ {|B|} \arrow[ur, "f"'] & \end{tikzcd}$$

Then there is a diagram of simplicial set maps

$$\begin{tikzcd} \mathrm{sd}^m \mathrm{sd}_* A \arrow[r, "\gamma_*\gamma^m"] \arrow[d, "i_*"'] & A \arrow[r, "\alpha"] & X \\ \mathrm{sd}^m \mathrm{sd}_* B \arrow[urr, "\phi"'] & & \end{tikzcd}$$

such that

$$|\phi| \simeq f|\gamma_*\gamma^m| : |\,\mathrm{sd}^m \mathrm{sd}_* B| \to |X|$$

rel $|\,\mathrm{sd}^m \mathrm{sd}_* A|$

Proof. The simplicial set $\mathrm{sd}_* X$ is regular (Lemma 4.4), and there is a (natural) commutative diagram

$$\begin{tikzcd} \mathrm{sd}\,\mathrm{sd}_* X \arrow[r, "c"] \arrow[d, "\gamma"'] & B\,\mathrm{sd}_* X \arrow[dl, "\tilde{\gamma}"] \\ \mathrm{sd}_* X & \end{tikzcd}$$

by Proposition 4.5. On account of Lemma 4.6, there is a continuous map $\tilde{f} : |\operatorname{sd}\operatorname{sd}_* B| \to |\operatorname{sd}\operatorname{sd}_* X|$ such that the diagram

$$\begin{tikzcd} |\operatorname{sd}\operatorname{sd}_* A| \arrow[r, "|\alpha|"] \arrow[d, "|i|"'] & |\operatorname{sd}\operatorname{sd}_* X| \\ |\operatorname{sd}\operatorname{sd}_* B| \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutes and such that $|\gamma_*\gamma|\tilde{f} \simeq f|\gamma_*\gamma|$ rel $|\operatorname{sd}\operatorname{sd}_* A|$. Now consider the diagram

$$\begin{tikzcd} |\operatorname{sd}\operatorname{sd}_* A| \arrow[r, "|c\alpha_*|"] \arrow[d, "|i_*|"'] & |B\operatorname{sd}_* X| \\ |\operatorname{sd}\operatorname{sd}_* B| \arrow[ur, "|c|\tilde{f}"'] & \end{tikzcd}$$

Then by applying Corollary 3.2 to the continuous map $|c|\tilde{f}$ the polyhedral complex map $c\alpha_*$ and the cofibration of polyhedral complexes i_*, we see that there is a diagram of simplicial set maps

$$\begin{tikzcd} \operatorname{sd}^n\operatorname{sd}\operatorname{sd}_* A \arrow[r, "\gamma^n"] \arrow[d, "i_*"'] & \operatorname{sd}\operatorname{sd}_* A \arrow[r, "c\alpha_*"] & B\operatorname{sd}_* X \\ \operatorname{sd}^n\operatorname{sd}\operatorname{sd}_* B \arrow[urr, "\psi"'] & & \end{tikzcd}$$

such that $|\psi| \simeq |c|\tilde{f}|\gamma^n|$ rel $|\operatorname{sd}^n\operatorname{sd}\operatorname{sd}_* A|$. It follows that

$$|\gamma_*\tilde{\gamma}\psi| \simeq |\gamma_*\tilde{\gamma}c|\tilde{f}|\gamma^n| = |\gamma_*\gamma|\tilde{f}|\gamma^n| \simeq f|\gamma_*\gamma||\gamma^n|.$$

Thus $\phi = \gamma_*\tilde{\gamma}\psi$ is the required map of simplicial sets, where $m = n+1$. ∎

4.8. Corollary. *Suppose given maps of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \\ B & \end{tikzcd}$$

where i is a cofibration and B is finite, and suppose that there is a commutative diagram of continuous maps

$$\begin{tikzcd} |A| \arrow[r, "|\alpha|"] \arrow[d, "|i|"'] & |X| \\ |B| \arrow[ur, "f"'] & \end{tikzcd}$$

Then there is a diagram of simplicial set maps

$$\begin{tikzcd}
\mathrm{sd}^m \mathrm{sd}_* \mathrm{sd}\,\mathrm{sd}_* A \arrow[r, "\gamma_*\gamma\gamma_*\gamma^m"] \arrow[d, "i_*"'] & A \arrow[r, "\alpha"] & X \\
\mathrm{sd}^m \mathrm{sd}_* \mathrm{sd}\,\mathrm{sd}_* B \arrow[urr, "\phi"'] & &
\end{tikzcd}$$

such that

$$|\phi| \simeq f|\gamma_*\gamma\gamma_*\gamma^m| : |\,\mathrm{sd}^m \mathrm{sd}_* \mathrm{sd}\,\mathrm{sd}_* B| \to |X|$$

rel $|\,\mathrm{sd}^m \mathrm{sd}_* \mathrm{sd}\,\mathrm{sd}_* A|$

PROOF. The cofibration i induces a cofibration of polyhedral complexes

$$i_* : B\,\mathrm{sd}_* A \to B\,\mathrm{sd}_* B.$$

The simplicial set maps

$$\begin{tikzcd}
B\,\mathrm{sd}_* A \arrow[r, "\tilde{\gamma}"] \arrow[d, "i_*"'] & \mathrm{sd}_* A \arrow[r, "\gamma_*"] & A \arrow[r, "\alpha"] & X \\
B\,\mathrm{sd}_* B & & &
\end{tikzcd}$$

and the composite continuous map

$$|B\,\mathrm{sd}_* B| \xrightarrow{|\tilde{\gamma}|} |\,\mathrm{sd}_* B| \xrightarrow{|\gamma_*|} |B| \xrightarrow{f} |X|$$

satisfy the conditions of Theorem 4.7. ■

Suppose that K is a polyhedral complex, and recall that NK denotes the poset of non-degenerate simplices of K with face relations, with nerve $BK = BNK \cong \mathrm{sd}\,K$. Recall also that $N_*K = (NK)^{op}$ is the dual poset; it has the same objects as NK, namely the non-degenerate simplices of K, but with the reverse ordering. The nerve BN_*K coincides with the dual subdivision $\mathrm{sd}_* K$ of K.

The poset NBK of non-degenerate simplices of BK has as objects all strings

$$\sigma : \quad \sigma_0 < \sigma_1 < \cdots < \sigma_q \tag{5}$$

of strings of non-degenerate simplices of K with no repeats. The face relation in NBK corresponds to inclusion of strings. The poset NB_*K has as objects all strings

$$\tau_0 > \tau_1 > \cdots > \tau_p$$

of non-degenerate simplices of K with no repeats, with the face relation again given by inclusion of substrings. Reversing the order of strings defines a poset isomorphism

$$\phi_K : NBK \xrightarrow{\cong} NB_*K$$

which is natural in polyhedral complexes K. The poset isomorphism ϕ_K induces a natural isomorphism

$$\Phi_K : \operatorname{sd}\operatorname{sd} K \xrightarrow{\cong} \operatorname{sd}\operatorname{sd}_* K$$

of associated nerves.

The composite

$$\operatorname{sd}\operatorname{sd}\Delta^n \xrightarrow{\gamma} \operatorname{sd}\Delta^n \xrightarrow{\gamma} \Delta^n$$

is induced by the poset morphisms

$$NBN\Delta^n \xrightarrow{\gamma} N\Delta^n \xrightarrow{\gamma} \mathbf{n}$$

which are defined by successive application of the last vertex map. Thus, this composite sends the object σ (as in (5) to $\sigma_q(m) \in \mathbf{n}$, where the poset inclusion $\sigma_q : \mathbf{m} \to \mathbf{n}$ defines the m-simplex $\sigma_q \in \Delta^n$. The composite of poset morphisms

$$NBN\Delta^n \xrightarrow{\phi} NBN_*\Delta^n \xrightarrow{\gamma} N_*\Delta^n \xrightarrow{\gamma_*} \mathbf{n}$$

(where γ_* is the first vertex map) sends the object σ to the element $\sigma_0(0) \in \mathbf{n}$. There is a relation $\sigma_0(0) \leq \sigma_q(m)$ in the poset $\mathbf{n}$ which is associated to all such objects σ. These relations define a homotopy $NBN\Delta^n \times \mathbf{1} \to \mathbf{n}$ from $\gamma_*\gamma\phi$ to $\gamma\gamma$. The maps and the homotopy respect all ordinal number morphisms $\theta : \mathbf{m} \to \mathbf{n}$.

It follows, by applying the nerve construction that there is an explicit simplicial homotopy $H : \operatorname{sd}\operatorname{sd}\Delta^n \times \Delta^1 \to \Delta^n$ from $\gamma_*\gamma\Phi_*$ to $\gamma\gamma$, and that this homotopy is natural in ordinal number maps. Glueing together instances of the isomorphisms $\Phi_* : \operatorname{sd}\operatorname{sd}(\Delta^n) \to \operatorname{sd}\operatorname{sd}_*(\Delta^n)$ along the simplex for a simplicial set X therefore determines an isomorphism

$$\Phi_X : \operatorname{sd}\operatorname{sd} X \xrightarrow{\cong} \operatorname{sd}\operatorname{sd}_* X \tag{6}$$

and a natural homotopy

$$H : \operatorname{sd}\operatorname{sd} X \times \Delta^1 \to X \tag{7}$$

from the composite

$$\operatorname{sd}\operatorname{sd} X \xrightarrow[\cong]{\Phi_X} \operatorname{sd}\operatorname{sd}_* X \xrightarrow{\gamma} \operatorname{sd}_* X \xrightarrow{\gamma_*} X$$

to the composite

$$\operatorname{sd}\operatorname{sd} X \xrightarrow{\gamma} \operatorname{sd} X \xrightarrow{\gamma} X.$$

5. Excision

5.1. Lemma. *Suppose that U_1 and U_2 are open subsets of a topological space Y such that $Y = U_1 \cup U_2$. Suppose given a commutative diagram of pointed simplicial set maps*

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & S(U_1) \cup S(U_2) \arrow[d] \\ L \arrow[r, "\beta"'] & S(Y) \end{tikzcd}$$

where i is an inclusion of finite polyhedral complexes. Then for some n the composite diagram

$$\begin{tikzcd} \operatorname{sd}^n K \arrow[r, "\gamma^n"] \arrow[d, "i_*"'] & K \arrow[r, "\alpha"] & S(U_1) \cup S(U_2) \arrow[d] \\ \operatorname{sd}^n L \arrow[r, "\gamma^n"'] & L \arrow[r, "\beta"'] & S(Y) \end{tikzcd}$$

is pointed homotopic to a diagram

$$\begin{tikzcd} \operatorname{sd}^n K \arrow[r] \arrow[d, "i_*"'] & S(U_1) \cup S(U_2) \arrow[d] \\ \operatorname{sd}^n L \arrow[r] \arrow[ur, dotted] & S(Y) \end{tikzcd}$$

admitting the indicated lifting.

Proof. There is an n such that the composite

$$\operatorname{sd}^n L \xrightarrow{\eta} S|\operatorname{sd}^n L| \xrightarrow{Sh^n} S|L| \xrightarrow{S\beta_*} SY$$

factors uniquely through a map $\tilde{\beta} : \operatorname{sd}^n L \to S(U_1) \cup S(U_2)$, where $\beta_* : |L| \to Y$ is the adjoint of β.

Suppose that $\Delta^r \subset K$ is a non-degenerate simplex of K. The diagram

$$\begin{tikzcd} |\operatorname{sd}^n \Delta^r| \arrow[r, "h^n"] \arrow[d, "i_*"'] & |\Delta^r| \arrow[d, "i_*"] \\ |\operatorname{sd}^n L| \arrow[r, "h^n"'] & |L| \end{tikzcd}$$

is homotopic to the diagram

$$\begin{tikzcd} |\operatorname{sd}^n \Delta^r| \arrow[r, "|\gamma^n|"] \arrow[d, "i_*"'] & |\Delta^r| \arrow[d, "i_*"] \\ |\operatorname{sd}^n L| \arrow[r, "|\gamma^n|"'] & |L| \end{tikzcd}$$

and the homotopies of such diagrams respect inclusions between non-degenerate simplices of K. Thus, each composite diagram

$$\begin{tikzcd} \operatorname{sd}^n \Delta^r \arrow[r, "\gamma^n"] \arrow[d, "i_*"'] & \Delta^r \arrow[r, "\alpha"] & S(U_1) \cup S(U_2) \arrow[d] \\ \operatorname{sd}^n L \arrow[r, "\gamma^n"'] & L \arrow[r, "\beta"'] & S(Y) \end{tikzcd}$$

is homotopic to a diagram

$$\begin{tikzcd}
\mathrm{sd}^n \Delta^r \arrow[r, "\eta"] \arrow[d, "i_*"'] & S|\mathrm{sd}^n \Delta^r| \arrow[r, "Sh^n"] & S|\Delta^r| \arrow[r, "S\alpha_*"] & S(U_1) \cup S(U_2) \arrow[d] \\
\mathrm{sd}^n L \arrow[r, "\eta"'] \arrow[urrr, "\tilde{\beta}"] & S|\mathrm{sd}^n L| \arrow[r, "Sh^n"'] & S|L| \arrow[r, "S\beta_*"'] & S(Y)
\end{tikzcd}$$

and the homotopies respect inclusions between non-degenerate simplices of K. Note that the map $\alpha : \Delta^r \to S(U_1) \cup S(U_2)$ factors through some $S(U_i)$ so that the "adjoint" α_* is induced by a map $|\Delta^r| \to U_i$. Observe also that the maps h and $|\gamma^n|$ coincide, and the homotopy between them is constant on the vertices of K.

It follows that the composite diagram

$$\begin{tikzcd}
\mathrm{sd}^n K \arrow[r, "\gamma^n"] \arrow[d, "i_*"'] & K \arrow[r, "\alpha"] & S(U_1) \cup S(U_2) \arrow[d] \\
\mathrm{sd}^n L \arrow[r, "\gamma^n"'] & L \arrow[r, "\beta"'] & S(Y)
\end{tikzcd}$$

is pointed homotopic to a diagram

$$\begin{tikzcd}
\mathrm{sd}^n K \arrow[rrr, "(\tilde{\beta} i)"] \arrow[d, "i_*"'] & & & S(U_1) \cup S(U_2) \arrow[d] \\
\mathrm{sd}^n L \arrow[r, "\eta"'] \arrow[urrr, "\tilde{\beta}"] & S|\mathrm{sd}^n L| \arrow[r, "Sh^n"'] & S|L| \arrow[r, "S\beta_*"'] & S(Y)
\end{tikzcd}$$

■

5.2. Theorem. *Suppose that U_1 and U_2 are open subsets of topological space Y, and suppose that $Y = U_1 \cup U_2$. Then the induced inclusion of simplicial sets $S(U_1) \cup S(U_2) \subset S(Y)$ is a weak equivalence.*

Proof. First of all observe that the induced function

$$\pi_0 |S(U_1 \cup U_2)| \to \pi_0 |S(Y)|$$

is a bijection, by subdivision of paths.

Pick a base point $x \in Y$, and let $\mathcal{F}_x Y$ denote the category of all finite pointed subcomplexes of $S(Y)$ containing x, ordered by inclusion. This category is plainly filtered, and there is an isomorphism

$$\pi_n |S(Y)| \cong \varinjlim_{K \in \mathcal{F}_x Y} \pi_n |K|.$$

The natural weak equivalences $\gamma' = \gamma_* \tilde{\gamma} : B(\mathrm{sd}_* K) \to K$ resulting from Lemma 4.4 and Proposition 4.5 may be used to replace a finite simplicial set K by a finite polyhedral complex $B(\mathrm{sd}_* K)$.

Suppose that $[\alpha] \in \pi_q(|S(Y)|, x)$ is carried on a finite subcomplex $\omega : K \subset S(Y)$ in the sense that $[\alpha] = \omega_*[\alpha']$ for some $[\alpha'] \in \pi_q|K|$. Then it follows from Lemma 5.1 that there is an $r \geq 0$ such that the diagram

$$\begin{tikzcd}
\operatorname{sd}^r B(\operatorname{sd}_* \Delta^0) \arrow[r, "\gamma'\gamma^r", "\cong"'] \arrow[d] & \Delta^0 \arrow[r, "x"] \arrow[d, "x"] & S(U_1) \cup S(U_2) \arrow[d, "i"] \\
\operatorname{sd}^r B(\operatorname{sd}_* K) \arrow[r, "\gamma'\gamma^r"'] & K \arrow[r, "\omega"'] & S(Y)
\end{tikzcd}$$

is pointed homotopic to a diagram

$$\begin{tikzcd}
\operatorname{sd}^r B(\operatorname{sd}_* \Delta^0) \arrow[r, "x"] \arrow[d] & S(U_1) \cup S(U_2) \arrow[d, "i"] \\
\operatorname{sd}^r B(\operatorname{sd}_* K) \arrow[r] \arrow[ur, dotted, "\sigma"'] & S(Y)
\end{tikzcd}$$

in which the indicated lift σ exists. The composite $\gamma'\gamma^r$ is a weak equivalence, so $[\alpha'] = (\gamma'\gamma^r)_*[\alpha'']$ for some α''. But then $[\alpha] = \omega_*(\gamma'\gamma^r)_*[\alpha''] = i_*\sigma_*[\alpha'']$ so that i_* is surjective on homotopy groups.

Suppose that $[\beta] \in \pi_q|S(U_1) \cup S(U_2)|$ is carried on the subcomplex $K \subset S(U_1) \cup S(U_2)$ and suppose that $i_*[\beta] = 0$. Then there is a commutative diagram of simplicial set inclusions

$$\begin{tikzcd}
K \arrow[r, "i_1"] \arrow[d, "j"'] & S(U_1) \cup S(U_2) \arrow[d] \\
L \arrow[r, "i_2"'] & S(Y)
\end{tikzcd}$$

such that $[\beta] \mapsto 0$ in $\pi_q|L|$. There is an $s \geq 0$ such that the composite diagram

$$\begin{tikzcd}
\operatorname{sd}^s B(\operatorname{sd}_* K) \arrow[r, "\gamma'\gamma^s"] \arrow[d, "j_*"'] & K \arrow[r, "i_1"] & S(U_1) \cup S(U_2) \arrow[d, "i"] \\
\operatorname{sd}^s B(\operatorname{sd}_* L) \arrow[r, "\gamma'\gamma^s"'] & L \arrow[r, "i_2"'] & S(Y)
\end{tikzcd}$$

is pointed homotopic to a diagram

$$\begin{tikzcd}
\operatorname{sd}^s B(\operatorname{sd}_* K) \arrow[r, "i_1'"] \arrow[d, "j_*"'] & S(U_1) \cup S(U_2) \arrow[d, "i"] \\
\operatorname{sd}^s B(\operatorname{sd}_* L) \arrow[r, "i_2'"'] \arrow[ur, dotted, "\tau"'] & S(Y)
\end{tikzcd}$$

in which the indicated lifting exists. Again, the maps $\gamma'\gamma^s$ are weak equivalences, so that $[\beta] = (\gamma'\gamma^s)_*[\beta']$ for some $[\beta'] \in \pi_q|\operatorname{sd}^s B(\operatorname{sd}_* K)|$ and

$$i_{1*}[\beta] = i_{1*}(\gamma'\gamma^s)_*[\beta'] = i_{1*}'[\beta'] = \tau_* j_*[\beta'].$$

Finally, $(\gamma'\gamma^s)_* j_*[\beta'] = j_*[\beta] = 0$ so that $j_*[\beta'] = 0$ in $\pi_q|\operatorname{sd}^s B(\operatorname{sd}_* L)|$ and so $i_{1*}[\beta] = 0$ in $\pi_q|S(U_1) \cup S(U_2)|$. ∎

The category **S** of simplicial sets is a category of cofibrant objects for a homotopy theory, for which the cofibrations are inclusions of simplicial sets and the weak equivalences are those maps $f : X \to Y$ which induce weak equivalences $f_* : |X| \to |Y|$ of CW-complexes. As such, it has most of the usual formal calculus of homotopy cocartesian diagrams (specifically II.8.5 and II.8.8 of [4]).

5.3. Lemma. *Suppose that the diagram*

$$\begin{tikzcd} \bigsqcup_i S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \bigsqcup_i e^n \arrow[r] & Y \end{tikzcd}$$

is a pushout in the category of CW-complexes. Then the diagram

$$\begin{tikzcd} \bigsqcup_i S(S^{n-1}) \arrow[r] \arrow[d] & S(X) \arrow[d] \\ \bigsqcup_i S(e^n) \arrow[r] & S(Y) \end{tikzcd}$$

is a homotopy cocartesian diagram of simplicial sets.

Proof. The usual classical arguments say that one can find an open subset $U \subset Y$ such that $X \subset U$ and this inclusion is a homotopy equivalence. The set U is constructed by fattening up each sphere S^{n-1} to an open subset U_i of the n-cell e^n (by radial projection) such that $S^{n-1} \subset U_i$ is a homotopy equivalence. We can therefore assume that the inclusion

$$\bigsqcup_i S^{n-1} \subset (\bigsqcup_i e^n) \cap U$$

is a homotopy equivalence. We can also assume that there is an open subset $V_i \subset e^n$ such that the inclusion is a homotopy equivalence, such that $V_i \cap U_i \subset U_i$ is a homotopy equivalence, and such that $e^n = V_i \cup U_i$. The net result is a commutative diagram

$$\begin{tikzcd} & \bigsqcup_i S(S^{n-1}) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathbf{III}"] & S(X) \arrow[d, "\simeq"] \\ S(V \cap U) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\mathbf{I}"] & S(U \cap (\bigsqcup_i e^n)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathbf{II}"] & S(U) \arrow[d] \\ S(V) \arrow[r, "\simeq"'] & \bigsqcup_i S(e^n) \arrow[r] & S(Y) \end{tikzcd}$$

of simplicial set homomorphisms in which all vertical maps are cofibrations and the labelled maps are weak equivalences. The the composite diagram $\mathbf{I} + \mathbf{II}$ is homotopy cocartesian by excision (Lemma 5.2), so that the diagram $\mathbf{II}$ is homotopy cocartesian by the usual argument. It follows that the composite diagram $\mathbf{III} + \mathbf{II}$ is homotopy cocartesian, again by a standard argument. ∎

5.4. Theorem. *The adjunction map $\epsilon : |S(T)| \to T$ is a weak equivalence for all spaces T.*

Proof. The functor $T \mapsto S(T)$ preserves fibrations and trivial fibrations, and thus preserves weak equivalences since all spaces are fibrant. In particular, the functor $T \mapsto |S(T)|$ preserves weak equivalences. We can therefore presume that T is a CW-complex.

All cells e^n are contractible spaces, so that the natural maps $\epsilon : |S(e^n| \to e^n$ are weak equivalences. If the diagram

$$\begin{tikzcd} \bigsqcup_i S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \bigsqcup_i e^n \arrow[r] & Y \end{tikzcd} \tag{8}$$

is a pushout in the category of CW-complexes, then it follows from Lemma 5.3 that the induced diagram

$$\begin{tikzcd} \bigsqcup_i |S(S^{n-1})| \arrow[r] \arrow[d] & |S(X)| \arrow[d] \\ \bigsqcup_i |S(e^n)| \arrow[r] & |S(Y)| \end{tikzcd} \tag{9}$$

is homotopy cocartesian. The maps $\epsilon : |S(S^{n-1})| \to S^{n-1}$ are therefore weak equivalences, by induction on dimension. The general case follows by comparison of the homotopy cartesian diagrams (8) and (9), and the usual sort of transfinite induction. ∎

The following is now a consequence of Theorem 5.4 and a standard adjointness trick:

5.5. Corollary. *The canonical map $\eta : X \to S|X|$ is a weak equivalence for all simplicial sets X.*

6. The Milnor Theorem

Write $\mathbf{S}_f$ for the full subcategory of the simplicial set category whose objects are the fibrant simplicial sets. All fibrant simplicial sets X are Kan complexes, and therefore have combinatorially defined homotopy groups $\pi_n(X, x)$, $n \geq 1$, $x \in X_0$, as well as sets of path components $\pi_0 X$. Say that a map $f : X \to Y$ of fibrant objects is a *combinatorial weak equivalence* if it induces isomorphisms $\pi_0 X \cong \pi_0 Y$ and $\pi_n(X, x) \cong \pi_n(Y, f(x))$ for all n and x. Recall that any fibre sequence

$$\begin{tikzcd} F_y \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

(ie. pullback, with p a fibration) induces a long exact sequence in homotopy groups

$$\cdots \to \pi_2(Y,y) \xrightarrow{\partial} \pi_1(F_y,x) \xrightarrow{i_*} \pi_1(X,x) \xrightarrow{p_*} \pi_1(Y,y) \xrightarrow{\partial} \pi_0 F_y \xrightarrow{i_*} \pi_0 X \xrightarrow{p_*} \pi_0 Y$$

for any choice of vertex $x \in F_y$.

6.1. Lemma. *A map $p : X \to Y$ between fibrant simplicial sets is a fibration and a combinatorial weak equivalence if and only if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$.*

Proof. If p has the right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ then it has the right lifting property with respect to all cofibrations, and therefore has the right lifting property with respect to all trivial cofibrations. It follows that p is a fibration. The map p is also a homotopy equivalence since X and Y are fibrant, so it is a combinatorial weak equivalence.

The reverse implication is the standard argument: see [4, I.7.10], and also the proof of Lemma 7.3 below. ■

6.2. Lemma. *The category $\mathbf{S}_f$ of all fibrant simplicial sets, together with the classes of all fibrations and combinatorial weak equivalences in the category, satisfies the axioms for a category of fibrant objects for a homotopy theory.*

Proof. With Lemma 6.1 and the closed simplicial model structure of Theorem 1.6 in place, the only axiom that requires proof is the weak equivalence axiom. In other words we have only to prove that, given a commutative triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

of morphisms between fibrant simplicial sets, if any two of the maps are combinatorial weak equivalences then so is the third. This is again a standard argument [4, I.8.2], which uses a combinatorial construction of the fundamental groupoid. ■

A *finite anodyne extension* is an inclusion $K \subset L$ of simplicial sets, such that there are subcomplexes

$$K = K_0 \subset K_1 \subset \cdots \subset K_N = L$$

with pushout diagrams

$$\begin{tikzcd} \Lambda^m_s \arrow[r] \arrow[d] & K_i \arrow[d] \\ \Delta^m \arrow[r] & K_{i+1} \end{tikzcd}$$

The notation means that K_{i+1} is constructed from K_i by explicitly attaching a simplex to a horn in K_i.

Recall [4] that a cofibration is said to be an *anodyne extension* if it is a member of the saturation of the set of all inclusions $\Lambda^n_k \subset \Delta^n$. In other words, the class of anodyne extensions is generated by all inclusions of horns in simplices under processes involving disjoint union, pushout and filtered colimit, and is closed under retraction. All anodyne extensions are weak equivalences.

We want to show that the subdivision functors sd and sd_* preserve finite anodyne extensions. This will be accomplished in two stages.

6.3. Lemma. *Suppose that $v : \Delta^0 \to K$ is a finite anodyne extension for some choice of vertex v in a finite complex K. Then the canonical inclusion $K \to CK$ is a finite anodyne extension.*

Proof. Suppose given a pushout diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d] & K \arrow[d, "i"] \\ \Delta^n \arrow[r] & L \end{tikzcd}$$

where there is some vertex $v \in K$ such that the corresponding map $v : \Delta^0 \to K$ is finite anodyne. Assume inductively that the map $N \to CN$ is anodyne for all finite complexes constructed in fewer stages than L, and for all N constructed by adjoining simplices of dimension smaller than n. Then the inclusions $K \to CK$ and $\Lambda^n_k \to C\Lambda^n_k$ are both anodyne, and there are pushout diagrams

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] \\ CK \arrow[r] & CK \cup_K L \end{tikzcd}$$

and

$$\begin{tikzcd} C\Lambda^n_k \cup_{\Lambda^n_k} \Delta^n \arrow[r] \arrow[d] & CK \cup_K L \arrow[d] \\ C\Delta^n \arrow[r] & CL \end{tikzcd}$$

The cofibration

$$C\Lambda^n_k \cup_{\Lambda^n_k} \Delta^n \to C\Delta^n$$

is isomorphic to the anodyne extension $\Lambda^{n+1}_k \subset \Delta^{n+1}$. ∎

6.4. Lemma. *The functors* sd *and* sd_* *preserve finite anodyne extensions.*

Proof. We will prove that the subdivision functor sd preserves finite anodyne extensions. The corresponding statement for sd_* has a similar proof.

It suffices to show that all induced maps $\mathrm{sd}\,\Lambda^n_k \to \mathrm{sd}\,\Delta^n$ are finite anodyne extensions. This will be done by induction on n; the case $n = 1$ is obvious.

It is a consequence of Lemma 4.1 that $\operatorname{sd}\Delta^n$ coincides up to isomorphism with the cone $C\operatorname{sd}\partial\Delta^n$ on $\operatorname{sd}\Delta^n$. The cone functor C takes the inclusion $\partial\Delta^r \to \Delta^r$ to the anodyne extension $\Lambda^{r+1}_{r+1} \subset \Delta^{r+1}$, and hence takes all inclusions $K \subset L$ of finite simplicial sets to finite anodyne extensions $CK \to CL$. There is a commutative diagram

$$\begin{tikzcd} \operatorname{sd}\Lambda^n_k \arrow[r] \arrow[d] & \operatorname{sd}\Delta^n \\ C\operatorname{sd}\Lambda^n_k \arrow[r] & C\operatorname{sd}\partial\Delta^n \arrow[u, "\cong"'] \end{tikzcd}$$

It therefore suffices to show that the canonical map $\operatorname{sd}\Lambda^n_k \to C\operatorname{sd}\Lambda^n_k$ is a finite anodyne extension.

Note that Λ^n_k has a filtration by subcomplexes F_r, where F_r is generated by the non-degenerate r-simplices which have k as a vertex. Then $F_0 = \{k\}$, $F_{n-1} = \Lambda^n_k$, and there are pushout diagrams

$$\begin{tikzcd} \bigsqcup_{x\in F_r^{(r)}} \Lambda^r_j \arrow[r] \arrow[d] & F_{r-1} \arrow[d] \\ \bigsqcup_{x\in F_r^{(r)}} \Delta^r \arrow[r] & F_r \end{tikzcd}$$

where $F_r^{(r)}$ denotes the set of non-degenerate r-simplices in F_r. In particular, the map $\Delta^0 \subset \Lambda^n_k$ arising from the inclusion of the vertex k is a finite anodyne extension. It also follows, by induction, that the map

$$\Delta^0 = \operatorname{sd}\Delta^0 \to \operatorname{sd}\Lambda^n_k$$

which is induced by applying sd to the inclusion $\{k\} \subset \Lambda^n_k$ is a finite anodyne extension.

The proof may therefore be completed by applying Lemma 6.3. ∎

For a simplicial set X, the simplicial set $\operatorname{Ex} X$ has n-simplices $\operatorname{Ex} X_n = \hom(\operatorname{sd}\Delta^n, X)$. The functor $X \mapsto \operatorname{Ex} X$ is right adjoint to the subdivision functor $A \mapsto \operatorname{sd} A$. It follows from Lemma 6.4 that $\operatorname{Ex} X$ is a Kan complex if X is a Kan complex; it is easier to see that $\operatorname{Ex} X$ is fibrant if X is fibrant. Write $\gamma : X \to \operatorname{Ex} X$ for the natural simplicial set map which is adjoint to the map $\gamma : \operatorname{sd} X \to X$.

6.5. Lemma. *Suppose that X is a Kan complex. Then the map $\gamma : X \to \operatorname{Ex} X$ is a combinatorial weak equivalence.*

Proof. The functor Ex preserves Kan fibrations on account of Lemma 6.4, and the map γ plainly induces a bijection

$$\pi_0 X \cong \pi_0 \operatorname{Ex} X.$$

The functor Ex also preserves those fibrations which have the right lifting property with respect to all $\partial\Delta^n \to \Delta^n$, since the subdivision functor sd preserves inclusions of polyhedral complexes.

Pick a base point $x \in X$, and construct the corresponding comparison of fibre sequences

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d, "\gamma"'] & PX \arrow[r] \arrow[d, "\gamma"] & X \arrow[d, "\gamma"] \\ \operatorname{Ex}\Omega X \arrow[r] & \operatorname{Ex} PX \arrow[r] & \operatorname{Ex} X \end{tikzcd}$$

Then $\operatorname{Ex} PX$ is simplicially contractible, and so there is an induced diagram

$$\begin{tikzcd} \pi_1 X \arrow[r, "\cong"] \arrow[d] & \pi_0 \Omega X \arrow[d, "\cong"] \\ \pi_1 \operatorname{Ex} X \arrow[r, "\cong"'] & \pi_0 \operatorname{Ex}\Omega X \end{tikzcd}$$

It follows that the induced map $\pi_1 X \to \pi_1 \operatorname{Ex} X$ is an isomorphism for all choices of base points in all Kan complexes X.

This construction may be iterated to show that the induced map $\pi_n X \to \pi_n \operatorname{Ex} X$ is an isomorphism for all choices of base points in all Kan complexes X, and for all $n \geq 0$. ■

There is a similar description of a functorially constructed simplicial set $\operatorname{Ex}_* X$ has n-simplices $\operatorname{Ex}_* X_n = \hom(\operatorname{sd}_* \Delta^n, X)$. The functor $X \mapsto \operatorname{Ex}_* X$ is right adjoint to the (dual) subdivision functor $A \mapsto \operatorname{sd}_* A$. The dual subdivision functor also preserves weak equivalences, cofibrations and finite anodyne extensions, and the natural map $\gamma_* : \operatorname{sd}_* A \to A$ is a weak equivalence. It follows that $\operatorname{Ex}_* X$ is a Kan complex if X is a Kan complex, and that $\operatorname{Ex}_* X$ is fibrant if X is fibrant. Write $\gamma_* : Y \to \operatorname{Ex}_* Y$ for the adjoint of the natural map $\gamma_* : \operatorname{sd}_* Y \to Y$. The proof of the following result is formally the same as that for Lemma 6.5:

6.6. LEMMA. *Suppose that X is a Kan complex. Then the map $\gamma_* : X \to \operatorname{Ex}_* X$ is a combinatorial weak equivalence.*

6.7. THEOREM. [Milnor Theorem] *Suppose that X is a Kan complex. Then the canonical map $\eta : X \to S(|X|)$ induces an isomorphism*

$$\pi_i(X, x) \cong \pi_i(|X|, x)$$

for all vertices $x \in X$ and for all $i \geq 0$.

In other words, Theorem 6.7 asserts the existence of an isomorphism between the combinatorial homotopy groups of a Kan complex X and the ordinary homotopy groups of its topological realization $|X|$.

PROOF OF THEOREM 6.7. The vertical arrows in the comparison diagram

$$\begin{tikzcd} \pi_i(X, x) \arrow[r] \arrow[d] & \pi_i(S|X|, x) \arrow[d] \\ \pi_i(\operatorname{Ex}^m \operatorname{Ex}_* X, x) \arrow[r] & \pi_i(\operatorname{Ex}^m \operatorname{Ex}_* S|X|, x) \end{tikzcd}$$

are isomorphisms for all m by Lemma 6.5 and 6.6. The simplicial approximation result Theorem 4.7 says that any element $\pi_i(S|X|, x)$ lifts to some element of $\pi_i(\mathrm{Ex}^r \mathrm{Ex}_* X, x)$ for sufficiently large r, and that any element of $\pi_i(X, x)$ which maps to $0 \in \pi_i(S|X|, x)$ must also map to 0 in $\pi_i(\mathrm{Ex}^s \mathrm{Ex}_* X, x)$ for some s. ∎

7. Kan fibrations

Write $\mathrm{SD}(X)$ for either the subdivision $\mathrm{sd}\, X$ of a simplicial set X or for the dual subdivision $\mathrm{sd}_* X$, and let $\Gamma : \mathrm{SD}(X) \to X$ denote the corresponding canonical map. Similarly, write $\mathrm{EX}(X)$ for either $\mathrm{Ex}\, X$ or $\mathrm{Ex}_* X$, and also let $\Gamma : X \to \mathrm{EX}(X)$ denote the adjoint map.

Here is one of the more striking consequences of simplicial approximation (Theorem 4.7 or Corollary 4.8): every simplicial set X is a Kan complex up to subdivision. More explicitly, we have the following:

7.1. Lemma. *Suppose that $\alpha : \Lambda^n_k \to X$ is a map of simplicial sets. Then there is an $r \geq 0$ such that α extends to Δ^n up to subdivision in the sense that there is a commutative diagram*

$$\begin{tikzcd} \mathrm{SD}^r(\Lambda^n_k) \arrow[r, "\Gamma^r"] \arrow[d] & \Lambda^n_k \arrow[r, "\alpha"] & X \\ \mathrm{SD}^r(\Delta^n) \arrow[urr] & & \end{tikzcd}$$

of simplicial set maps.

Proof. All spaces are fibrant, so there is a diagram of continuous maps

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "|\alpha|"] \arrow[d] & {|X|} \\ {|\Delta^n|} \arrow[ur, "f"'] & \end{tikzcd}$$

Now apply Theorem 4.7. ∎

7.2. Remark. In fact, although it's convenient to do so for the moment we do not have to mix instances of sd and sd_* in the proof of Lemma 7.1 — see the proof of Lemma 7.9 below.

7.3. Lemma. *Suppose that $p : X \to Y$ is a Kan fibration and a weak equivalence. Suppose that there is a commutative diagram*

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y \end{tikzcd} \tag{10}$$

Then there is an $r \geq 0$ and a commutative diagram

$$\begin{tikzcd} \mathrm{SD}^r(\partial\Delta^n) \arrow[r, "\Gamma^r"] \arrow[d] & \partial\Delta^n \arrow[r, "\alpha"] & X \arrow[d, "p"] \\ \mathrm{SD}^r(\Delta^n) \arrow[r, "\Gamma^r"'] \arrow[urr] & \Delta^n \arrow[r, "\beta"'] & Y \end{tikzcd}$$

In other words all maps which are both Kan fibrations and weak equivalences have the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, up to subdivision. We will do better than that, in Theorem 7.4.

Proof of Lemma 7.3. Suppose that $i : K \subset L$ is an inclusion of finite polyhedral complexes. If the diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\ L \arrow[r, "\beta"'] & Y \end{tikzcd} \tag{11}$$

is homotopic up to subdivision to a diagram for which the lifting exists, then the lifting exists for the original diagram up to subdivision.

In effect, a homotopy up to subdivision is a diagram

$$\begin{tikzcd} \mathrm{SD}^k(K \times \Delta^1) \arrow[r, "h_1"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{SD}^k(L \times \Delta^1) \arrow[r, "h_2"'] & Y \end{tikzcd}$$

The homotopy starts (up to subdivision) at the original diagram

$$\begin{tikzcd} \mathrm{SD}^k(K) \arrow[r, "\Gamma^k"] \arrow[d] & K \arrow[r, "\alpha"] & X \arrow[d, "p"] \\ \mathrm{SD}^k(L) \arrow[r, "\Gamma^k"'] & L \arrow[r, "\beta"'] & Y \end{tikzcd} \tag{12}$$

If the lifting exists at the other end of the homotopy in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathrm{SD}^k(K) \arrow[r, "d^0_*"] \arrow[d] & \mathrm{SD}^k(K \times \Delta^1) \arrow[r, "h_1"] & X \arrow[d, "p"] \\ \mathrm{SD}^k(L) \arrow[r, "d^0_*"'] \arrow[urr, "\sigma"] & \mathrm{SD}^k(L \times \Delta^1) \arrow[r, "h_2"'] & Y \end{tikzcd}$$

then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{SD}^k(K) \arrow[r, "d^1_*"] \arrow[d] & \mathrm{SD}^k(K \times \Delta^1) \cup \mathrm{SD}^k(L) \arrow[r, "(h_1,\sigma)"] \arrow[d, "j"'] & X \arrow[d, "p"] \\
\mathrm{SD}^k(L) \arrow[r, "d^1_*"'] & \mathrm{SD}^k(L \times \Delta^1) \arrow[r, "h_2"'] \arrow[ur, "\sigma'"] & Y
\end{tikzcd}$$

The map labelled j is a finite anodyne extension by Lemma 6.4, so the lifting σ' exists. The outer square diagram is the diagram (12) and the composite $\sigma' d^1_*$ is the required lift.

Lemma 7.1 implies that the contracting homotopy $h_1 : \Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ onto the vertex 0 extends to a homotopy of diagrams up to subdivision from the diagram (10) to a diagram

$$\begin{tikzcd}
\mathrm{SD}^k(\partial\Delta^n) \arrow[r, "\alpha_1"] \arrow[d] & X \arrow[d, "p"] \\
\mathrm{SD}^k(\Delta^n) \arrow[r, "\beta_1"'] & Y
\end{tikzcd} \tag{13}$$

where the composite

$$\mathrm{SD}^k(\Delta^{n-1}) \xrightarrow{d^i_*} \mathrm{SD}^k(\partial\Delta^n) \xrightarrow{\alpha_1} X$$

factors through a fixed base point $* = \alpha(0)$ for $i \neq 0$.

The composite

$$\mathrm{SD}^k(\Delta^{n-1}) \xrightarrow{d^0_*} \mathrm{SD}^k(\partial\Delta^n) \xrightarrow{\alpha_1} X$$

represents an element $[|\alpha_1 d^0_*|] \in \pi_{n-1}|X|$, and this element maps to $0 \in \pi_{n-1}|X|$ since the diagram (13) commutes. The homotopy $|\,\mathrm{SD}^k\,\Delta^{n-1} \times \Delta^1| \to |X|$ from $|\alpha_1 d^0_*|$ to the base point is homotopic rel boundary and after subdivision to the realization of a simplicial map $\mathrm{SD}^r(\mathrm{SD}^k(\Delta^{n-1}) \times \Delta^1) \to X$, which extends after subdivision to a homotopy of diagrams

$$\begin{tikzcd}
\mathrm{SD}^s(\mathrm{SD}^k(\partial\Delta^n) \times \Delta^1) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{SD}^s(\mathrm{SD}^k(\Delta^n) \times \Delta^1) \arrow[r] & Y
\end{tikzcd}$$

from a subdivision of the diagram (13) to a diagram

$$\begin{tikzcd}
\mathrm{SD}^{s+k}\,\partial\Delta^n \arrow[r, "\alpha_2"] \arrow[d] & X \arrow[d, "p"] \\
\mathrm{SD}^{s+k}\,\Delta^n \arrow[r, "\beta_2"'] & Y
\end{tikzcd}$$

such that α_2 maps all of $\mathrm{SD}^{s+k}\,\partial\Delta^n$ to the base point of X.

The element $[|\beta_2|] \in \pi_n|Y|$ lifts to an element $[\gamma] \in \pi_n|X|$ since $p_* : \pi_n|X| \to \pi_n|Y|$ is an isomorphism. The map $\gamma : |\operatorname{SD}^{s+k}\Delta^n| \to |X|$ is homotopic rel boundary and after subdivision to the realization of a simplicial set map $f : \operatorname{SD}^{s+k+l}\Delta^n \to X$ which maps $\operatorname{SD}^{s+k+l}\partial\Delta^n$ into the base point. It follows that, after subdivision, $|\beta_2|$ is homotopic rel boundary to the map $|pf|$. The homotopy $|\operatorname{SD}^{s+k+l}\Delta^n \times \Delta^1| \to |Y|$ rel boundary is itself homotopic to the realization of a simplicial homotopy $\operatorname{SD}^m(\operatorname{SD}^{s+k+l}\Delta^n \times \Delta^1) \to Y$ rel boundary after further subdivision. It follows that β_2 lifts to X rel boundary after subdivision. ∎

7.4. THEOREM. *Suppose that $p : X \to Y$ is a Kan fibration and a weak equivalence. Then p has the right lifting property with respect to all inclusions $\partial\Delta^n \to \Delta^n$.*

PROOF. Suppose given a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

and let $x = \sigma(0) \in Y$. The fibre $F_{\sigma(0)}$ over $\sigma(0)$ is defined by the pullback diagram

$$\begin{tikzcd} F_{\sigma(0)} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "\sigma(0)"'] & Y \end{tikzcd}$$

and the Kan complex $F_{\sigma(0)}$ has the property that all maps $\partial\Delta^n \to F_{\sigma(0)}$ can be extended to a map $\operatorname{SD}^r\Delta^n \to F_{\sigma(0)}$ after a suitable subdivision, by Lemma 7.3.

All maps $\Gamma^r : F_{\sigma(0)} \to \operatorname{EX}^r F_{\sigma(0)}$ are weak equivalences of Kan complexes, while the extension up to subdivision property for $F_{\sigma(0)}$ implies that all elements of the combinatorial homotopy group $\pi_j F_{\sigma(0)}$ vanish in $\pi_j \operatorname{EX}^r F_{\sigma(0)}$ for some r. The Kan complex $F_{\sigma(0)}$ therefore has trivial combinatorial homotopy groups, and is contractible.

A standard (combinatorial) result about Kan fibrations [4, I.10.6] asserts that there is a fibrewise homotopy equivalence

$$\begin{tikzcd} F_\sigma \arrow[rr, "\theta", "\simeq"'] \arrow[dr] & & F_{\sigma(0)} \times \Delta^n \arrow[dl, "pr"] \\ & \Delta^n & \end{tikzcd}$$

where F_σ denotes the pullback of p over Δ^n. It follows that the induced lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & F_\sigma \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "1"'] \arrow[ur, dotted] & \Delta^n \end{tikzcd}$$

can be solved up to homotopy of diagrams, and can therefore be solved. ∎

7.5. Corollary. *Suppose that $i : A \to B$ is a cofibration and a weak equivalence. Then i has the left lifting property with respect to all Kan fibrations.*

Proof. The map i has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & X \arrow[d, "p"] \\ & B \end{tikzcd}$$

where j is anodyne and p is a Kan fibration. Then p is a weak equivalence as well as a Kan fibration, and therefore has the right lifting property with respect to all cofibrations by Theorem 7.4. The lifting θ therefore exists in the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "1"'] \arrow[ur, "\theta"] & B \end{tikzcd}$$

It follows that i is a retract of j, and so i has the left lifting property with respect to all Kan fibrations. ■

7.6. Corollary. *Every Kan fibration is a fibration of simplicial sets, and conversely.*

7.7. Theorem. [Quillen] *Suppose that $p : X \to Y$ is a fibration. Then the realization $|p| : |X| \to |Y|$ of p is a Serre fibration*[2].

Proof. We want to show that all lifting problems in continuous maps

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "\alpha"] \arrow[d] & {|X|} \arrow[d, "|p|"] \\ {|\Delta^n|} \arrow[r, "\beta"'] \arrow[ur, dotted] & {|Y|} \end{tikzcd} \tag{14}$$

can be solved. The idea is to show that all such problems can be solved up to homotopy of diagrams.

We can assume, first of all, that $\alpha(k)$ is a vertex of X. If it is not, there will be path in $|X|$ from $\alpha(k)$ to some vertex $x \in X$, and that path extends to a homotopy of diagrams in the usual way.

There is a simplicial set map $\alpha' : \mathrm{SD}^r \Lambda^n_k \to X$ such that the realization $\alpha'_* : |\mathrm{SD}^r \Lambda^n_k| \to |X|$ is homotopic to $\alpha|\Gamma^r|$ relative to the image of the cone point k in $|X|$. This homotopy extends to a homotopy from $\beta|\Gamma^r|$ to a map $\beta_1 : |\mathrm{SD}^r \Delta^n| \to |Y|$ which restricts to $|p\alpha'|$ on $|\mathrm{SD}^r \Lambda^n_k|$.

[2]There is an erratum to the following proof on p.72.

There is a further subdivision $\mathrm{SD}^{s+r}\,\Delta^n$ such that the composite map $\beta_1|\Gamma^s|$ is homotopic rel $|\,\mathrm{SD}^{s+r}\,\Lambda^n_k|$ to the realization of a simplicial map

$$\beta' : \mathrm{SD}^{s+r}\,\Delta^n \to Y.$$

It follows that there is a homotopy of diagrams from the diagram

$$\begin{tikzcd} |\,\mathrm{SD}^{s+r}\,\Lambda^n_k| \arrow[r, "|\Gamma^{s+r}|"] \arrow[d] & |\Lambda^n_k| \arrow[r, "\alpha"] & |X| \arrow[d, "|p|"] \\ |\,\mathrm{SD}^{s+r}\,\Delta^n| \arrow[r, "|\Gamma^{s+r}|"'] & |\Delta^n| \arrow[r, "\beta"'] & |Y| \end{tikzcd} \tag{15}$$

to the realization of the diagram of simplicial set morphisms

$$\begin{tikzcd} \mathrm{SD}^{s+r}\,\Lambda^n_k \arrow[r, "\alpha'\Gamma^s"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{SD}^{s+r} \arrow[r, "\beta'"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

The indicated lift exists in the diagram of simplicial set morphisms, since p is a fibration and the induced map $\mathrm{SD}^{s+r}\,\Lambda^n_k \to \mathrm{SD}^{s+r}\,\Delta^n$ is anodyne, by Lemma 6.4.

The lifting problem can therefore be solved for the diagram (15). The map $|\Gamma^{s+r}|$ is homotopic to a homeomorphism, and the homotopy and the homeomorphism are natural in simplicial complexes. It follows that there is a diagram homotopy from the diagram (15) to a diagram which is isomorphic to the original diagram (14), so the lifting problem can be solved for that diagram. ∎

The following result is an easy consequence of Theorem 7.7 and the formalism of categories of fibrant objects [4, II.8.6]. Its proof completes the proof of the assertion that the model structure on the category of simplicial sets is proper.

7.8. Corollary. *Suppose given a pullback diagram*

$$\begin{tikzcd} A \times_Y X \arrow[r, "f_*"] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "f"'] & Y \end{tikzcd}$$

where p is a fibration and f is a weak equivalence. Then the induced map $f_* : A \times_Y X \to X$ is a weak equivalence.

Write $\mathrm{Ex}^\infty X$ for the colimit of the system

$$X \xrightarrow{\gamma} \mathrm{Ex}\,X \xrightarrow{\gamma} \mathrm{Ex}^2\,X \to \dots$$

Write $\tilde{\gamma} : X \to \mathrm{Ex}^\infty X$ for the natural map. This is Kan's Ex^∞ construction, applied to the simplicial set X. The following result is well known [4], but has a remarkably easy proof in the present context.

7.9. Lemma. *The simplicial set* $\mathrm{Ex}^\infty X$ *is a Kan complex.*

Proof. The space $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$. By Corollary 3.2, there is a commutative diagram of simplicial set homomorphisms

$$\begin{tikzcd} \mathrm{sd}^r \Lambda^n_k \arrow[r, "\gamma^r"] \arrow[d] & \Lambda^n_k \\ \mathrm{sd}^r \Delta^n \arrow[ur] & \end{tikzcd}$$

This means that any map $\alpha : \Lambda^n_k \to Y$ sits inside a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d] & Y \arrow[r, "\gamma^r"] & \mathrm{Ex}^r Y \\ \Delta^n \arrow[urr] & & \end{tikzcd}$$

for some r. This is true for all simplicial sets Y, and hence for all $\mathrm{Ex}^r X$. ∎

7.10. Theorem. *The natural map* $\tilde{\gamma} : X \to \mathrm{Ex}^\infty X$ *is a weak equivalence, for all simplicial sets* X.

Proof. The Ex^∞ functor preserves fibrations on account of Lemma 6.4, and the map $\gamma : X \to \mathrm{Ex}\, X$ induces a bijection $\pi_0 X \cong \pi_0(\mathrm{Ex}\, X)$ for all simplicial sets X.

Suppose that $j : X \to \tilde{X}$ is a fibrant model for X, and let $x \in X$ be a choice of base point. The space of paths $P\tilde{X}$ starting at $x \in \tilde{X}$ and the fibration $\pi : P\tilde{X} \to \tilde{X}$ determines a pullback diagram

$$\begin{tikzcd} X \times_{\tilde{X}} P\tilde{X} \arrow[r, "j_*"] \arrow[d, "\pi_*"'] & P\tilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "j"'] & \tilde{X} \end{tikzcd}$$

in which the map π_* is a fibration and j_* is a weak equivalence by Corollary 7.8. The fibre $\Omega\tilde{X}$ for both π and π_* is a Kan complex, so that the map $\tilde{\gamma} : \Omega\tilde{X} \to Ex^\infty\Omega\tilde{X}$ is a weak equivalence by Lemma 6.5 and Theorem 6.7. It follows from Theorem 7.7 and the method of proof of Lemma 6.5 that the map $\tilde{\gamma} : X \to \mathrm{Ex}^\infty X$ is a weak equivalence if we can show that the simplicial set $\mathrm{Ex}^\infty(X \times_{\tilde{X}} P\tilde{X})$ is weakly equivalent to a point.

It is therefore sufficient to show that $\mathrm{Ex}^\infty Y$ is weakly equivalent to a point if the map $Y \to *$ is a weak equivalence. The object $\mathrm{Ex}^\infty Y$ is a Kan complex by Lemma 7.9, so it suffices to show that all lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & \mathrm{Ex}^\infty Y \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

can be solved if Y is weakly equivalent to a point. By an adjointness argument, this amounts to showing that the map $\alpha_* : \mathrm{sd}^r \partial\Delta^n \to Y$ can be extended over Δ^n after subdivision in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathrm{sd}^{s+r} \partial\Delta^n \arrow[r, "\gamma^s"] \arrow[d] & \mathrm{sd}^r \partial\Delta^n \arrow[r, "\alpha_*"] & Y \\ \mathrm{sd}^{s+r} \Delta^n \arrow[urr] & & \end{tikzcd}$$

There is a commutative diagram

$$\begin{tikzcd} \mathrm{sd}\,\mathrm{sd}_* \mathrm{sd}^r \partial\Delta^n \arrow[r, "\mathrm{sd}\,\mathrm{sd}_* \alpha_*"] \arrow[d, "\gamma_*\gamma"'] & \mathrm{sd}\,\mathrm{sd}_* Y \arrow[r, "\pi"] \arrow[d, "\gamma_*\gamma"'] & B\,\mathrm{sd}_* Y \arrow[dl] \\ \mathrm{sd}^r \partial\Delta^n \arrow[r, "\alpha_*"'] & Y & \end{tikzcd}$$

on account of Lemma 4.4 and Proposition 4.5. The map π is a weak equivalence by Corollary 4.3 and Lemma 4.4. The map $\gamma_*\gamma$ is a weak equivalence since its realization is homotopic to a homeomorphism. It follows that the polyhedral complex $B\,\mathrm{sd}_* Y$ is weakly equivalent to a point.

Corollary 3.2 and the contractibility of the space $|B\,\mathrm{sd}_* Y|$ together imply that there is a commutative diagram

$$\begin{tikzcd} \mathrm{sd}^t \mathrm{sd}^2 \mathrm{sd}^r \partial\Delta^n \arrow[r, "\gamma^t"] \arrow[d] & \mathrm{sd}^2 \mathrm{sd}^r \partial\Delta^n \arrow[r, "\Phi_*"] & \mathrm{sd}\,\mathrm{sd}_* \mathrm{sd}^r \partial\Delta^n \arrow[r, "\pi\,\mathrm{sd}\,\mathrm{sd}_*\,\alpha_*"] & B\,\mathrm{sd}_* Y \\ \mathrm{sd}^t \mathrm{sd}^2 \mathrm{sd}^r \Delta^n \arrow[urrr] & & & \end{tikzcd}$$

The natural homotopy (7) at the end of Section 4 induces a homotopy

$$h : \mathrm{sd}^2 \mathrm{sd}^r(\partial\Delta^n) \times \Delta^1 \to Y$$

from the composite $\alpha_*\gamma_*\gamma\Phi_*$ to $\alpha_*\gamma^2$. There is an obvious map

$$\mathrm{sd}^2 \mathrm{sd}^r(\partial\Delta^n \times \Delta^1) \to \mathrm{sd}^2 \mathrm{sd}^r(\partial\Delta^n) \times \Delta^1$$

which, when composed with h, and by taking adjoints gives a homotopy from $\alpha : \partial\Delta^n \to Y$ to a map $(\alpha_*\gamma_*\gamma\Phi_*)_* : \partial\Delta^n \to \mathrm{Ex}^\infty Y$ which extends to a map $\Delta^n \to \mathrm{Ex}^\infty Y$. The object $\mathrm{Ex}^\infty Y$ is a Kan complex, so the map α extends over Δ^n as well, by a standard argument. ∎

7.11. Corollary. *The map $\gamma : X \to \mathrm{Ex}\,X$ is a weak equivalence for all simplicial sets X.*

Proof. The map $\tilde{\gamma} : X \to \operatorname{Ex}^{\infty} X$ is a weak equivalence, as is the map $\tilde{\gamma} : \operatorname{Ex} X \to \operatorname{Ex}^{\infty} X$, and there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{\gamma}"] \arrow[d, "\gamma"'] & \operatorname{Ex}^{\infty} X \\ \operatorname{Ex} X \arrow[ur, "\tilde{\gamma}"'] & \end{tikzcd}$$

■

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ERRATUM

Theorem 7.7 above is the Quillen Theorem that the realization of a Kan fibration is a Serre fibration. The proof for this result which is given above has a fatal error in the last line of the argument, while Quillen's original argument is of course correct.

Quillen's argument depends on the Gabriel-Zisman Theorem that the realization of a minimal fibration is a Kan fibration. One can simplify Quillen's argument a little by showing that if a Kan fibration $p : X \to Y$ is fibrewise weakly equivalent to a Kan fibration $q : Z \to Y$ whose realization is a Serre fibration, then the realization of p is a Serre fibration. It appears, however, that the dependence of Quillen's Theorem on the theory of minimal fibrations cannot be removed with the techniques developed in this article.

I would like to thank Andre Henriques for bringing the error to my attention.

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