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AN ITERATIVE METHOD FOR COMPUTING ZEROS OF OPERATORS SATISFYING AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. We use an iteration method to approximate zeros of operators satisfying autonomous differential equations. This iteration process has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, as the inverse of the operator involved is calculated once and for all. Our local and semilocal convergence results compare favorably with earlier ones under the same computational cost.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

We use the Newton-like method:

$$x_{n+1} = x_n - F'(y_n)^{-1} F(x_n) \quad (n \geq 0) \tag{2}$$

to generate a sequence approximating x^*.

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Here $F'(x) \in L(X,Y)$ denotes the Fréchet-derivative. We are interested in the case when:

$$y_n = \lambda_n x_n + (1-\lambda_n) z_n \qquad (n \geq 0) \tag{3}$$

where,

$$\lambda_n \in [0,1], \quad (n \geq 0) \tag{4}$$

$$z_n = x^* \tag{5}$$

or

$$z_n = x_n \qquad (n \geq 0), \tag{6}$$

or other suitable choice [1]-[4].

We provide a local and a semilocal convergence analysis for method (2) which compare favorably with earlier results [4], and under the same computational cost.

2. CONVERGENCE FOR METHOD (2) FOR z_n GIVEN BY (5) AND $\lambda_n = 0$ $(n \geq 0)$

We can show the following local result:

Theorem 1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*
there exists a solution x^* of equation

$$F(x) = 0 \text{ such that } F'(x^*)^{-1} \in L(Y,X)$$

and

$$\left\| F'(x^*)^{-1} \right\| \leq b; \tag{7}$$

$$\| F'(x) - F'(x^*) \| \leq L_0 \| x - x^* \| \quad \text{for all} \quad x \in D, \tag{8}$$

and

$$\overline{U}(x^*, r_0) = \left\{ x \in X \,\middle|\, \| x - x^* \| \leq r_0 = \tfrac{2}{bL_0} \right\} \subseteq D. \tag{9}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method (2) is well defined remains in $U(x^*, r_0)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r_0)$.

Moreover the following error bounds hold for all $n \geq 0$:

$$\| x_n - x^* \| \leq \theta_0^{2^n - 1} \| x_0 - x^* \| \quad (n \geq 1), \tag{10}$$

where

$$\theta_0 = \tfrac{1}{2} b L_0 \| x_0 - x^* \|. \tag{11}$$

Proof. By (2) and $F(x^*) = 0$ we get for all $n \geq 0$:

(12)

$$x_{n+1} - x^* = -F'(x^*)^{-1} \left[\int_0^1 \left(F'(x^* + t(x_n - x^*)) - F'(x^*) \right)(x_n - x^*) \right] dt$$

from which it follows

$$\|x_{n+1} - x^*\| \leq \tfrac{1}{2} b L_0 \|x_n - x^*\|^2 \tag{13}$$

from which (10) follows.

By (9) and (11) $\theta_0 \in [0,1)$. hence it follows from (10) that $x_n \in U(x^*, r_0)$ $(n \geq 0)$ and $\lim\limits_{n\to\infty} x_n = x^*$ (by using induction on the integer $n \geq 0$). ☐

Remark 1. *Method (2) has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, since the operator* $F'(x^*)^{-1}$ *is computed only once. It turns out that method (2) can be used for operators* F *which satisfy an autonomous differential equation*

$$F'(x) = G(F(x)), \tag{14}$$

where G *is a known continuous operator on* Y. *As* $F'(x^*) = G(0)$ *can be evaluated without knowing the value of* x^*.

Moreover in order for us to compare Theorem 1 with earlier results, consider the condition

$$\|F'(x) - F'(y)\| \leq L \|x - y\| \quad \textit{for all} \quad x \in D \tag{15}$$

used in [4] *instead of (8). The corresponding radius of convergence is given by*

$$r_R = \frac{2}{bL}. \tag{16}$$

since

$$L_0 \leq L \tag{17}$$

holds in general we obtain

$$r_R \leq r_0. \tag{18}$$

Furthermore in case strict inequality holds in (17), so does in (18). We showed in [1] *that the ration* $\frac{L}{L_0}$ *can be arbitrarily large. Hence we managed to enlarge the radius of convergence for method (2) under the same computational cost as in Theorem 1 in [4, p.113].*

This observation is very important in computational mathematics since a under choice of initial guesses x_0 *can be obtained.*

Below we give an example of a case where strict inequality holds in (17) and (18).

Example 1. *Let* $X = Y = R$, $D = U(0,1)$ *and define* F *on* D *by*

$$F(x) = e^x - 1. \tag{19}$$

Note that (19) satisfies (14) for $T(x) = x + 1$. *Using (7), (8), (9), (15) and (16) we obtain*

$$b = 1, L_0 = e - 1, L = e, \tag{20}$$

$$r_0 = 1.163953414 \tag{21}$$

and

$$r_R = .735758882. \tag{22}$$

In order to keep the iterates inside D *we can restrict* r_0 *and choose*

$$r_0 = 1. \tag{23}$$

In any case (17) and (18) holds as a strict inequalities.

We can show the following global result:

Theorem 2. *Let* $F : X \to Y$ *be Fréchet-differentiable operator, and* G *a continuous operator from* Y *into* Y. *Assume:*

condition (14) holds;

$G(0)^{-1} \in L(Y,X)$ *so that (7) holds;*

$$F(x) \le c \quad \text{for all } x \in X; \tag{24}$$

$$\|G(0) - G(z)\| \le a_0 \|z\| \quad \text{for all} \quad z \in Y \tag{25}$$

and

$$h_0 = \alpha_0 bc < 1. \tag{26}$$

Then, sequence $\{x_n\}$ $(n \ge 0)$ *generated by method (2) is well defined and converges to a unique solution* x^* *of equation* $F(x) = 0$.

Moreover the following error bounds hold for all $n \ge 0$:

$$\|x_n - x^*\| \le \frac{h_0^n}{1 - h_0} \|x_1 - x_0\| \quad (n \ge 0). \tag{27}$$

Proof. It follows from the contraction mapping principle [2] by using (25), (26) instead of

$$\|G(v) - G(z)\| \le a \|v - z\| \quad \text{for all} \quad v, z \in Y \tag{28}$$

and

$$h = abc < 1 \tag{29}$$

respectively in the proof of Theorem 2 in [4, p.113]. □

Remark 2. *If* F' *is* L_0 *Lipschitz continuous in a ball centered at* x^**, then the convergence of method (2) will be quadratic as soon as*

$$bL_0 \left\| x_0 - x^* \right\| < 2 \tag{30}$$

holds with x_0 *replaced by an iterate* x_n *sufficiently close to* x^**.*

Remark 3. *If (25) is replaced by the stronger (28), Theorem 2 reduces to Theorem 2 in* [4]*. Otherwise our Theorem is weaker than Theorem 2 in* [4] *since*

$$a_0 < a \tag{31}$$

holds in general.

We note that if (25) holds and

$$\left\| F(x) - F(x_0) \right\| \leq \gamma_0 \left\| x - x_0 \right\| \tag{32}$$

then

$$\left\| F(x) \right\| \leq \left\| F(x) - F(x_0) \right\| + \left\| F(x_0) \right\| \leq \gamma_0 \left\| x - x_0 \right\| + \left\| F(x_0) \right\|. \tag{33}$$

Let $r = \left\| x - x_0 \right\|$, and define

$$P(r) = a_0 b \left(\left\| F(x_0) \right\| + \gamma_0 r \right). \tag{34}$$

If $P(0) = a_0 b \left\| F(x_0) \right\| < 1$, then as in Theorem 3 in [4, p.114] inequality (26) and the contraction mapping principle we obtain the following semilocal result:

Theorem 3. *If*

$$q = \left(1 - a_0 b \left\| F(x_0) \right\| \right)^2 - 4 b a_0 \gamma_0 \left\| G(0)^{-1} F(x_0) \right\| \geq 0, \tag{35}$$

then a solution x^* *of equation*

$$F(x) \text{ exists in } U(x_0, r_1),$$

and is unique in $U(x_0, r_2)$*, where*

$$r_1 = \frac{1 - a_0 b \left\| F(x_0) \right\| - \sqrt{q}}{2 b a_0 \gamma_0} \tag{36}$$

and

$$r_2 = \frac{1 - a_0 b \left\| F(x_0) \right\|}{b a_0 \gamma_0}. \tag{37}$$

Remark 4. *Theorem 3 reduces to Theorem 3 in [4, p.114] if (25) and (32) are replaced by the stronger (28) and*

$$\left\| F(x) - F(y) \right\| \leq \gamma \left\| x - y \right\| \tag{38}$$

respectively. Otherwise our Theorem is weaker than Theorem 3 in [4]*.*

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