Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 201 – 213

SEMI-INVARIANT SUBMANIFOLDS OF (g,F)-MANIFOLDS

Novac-Claudiu Chiriac

Abstract. We introduce (g, F)-manifolds and initiate a study of their semi-invariant submanifolds. These submanifolds are generalizations of CR-submanifolds of Kaehler manifolds. We obtain necessary and sufficient conditions for the integrability of distributions on a semi-invariant submanifold and study the geometry of foliations defined by these distributions. In particular, for a large class of (g, F)-manifolds we prove the existence of a natural foliation on their semi-invariant submanifolds.

1 Introduction

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published (cf. Yano - Kon [12]). An important class of such manifolds is formed by Kaehler manifolds. The geometry of submanifolds of a Kaehler manifold is rich and interesting, as well. CR - submanifolds introduced by Bejancu [2] have had a great impact on the developing of the theory of submanifolds in a Kaehler manifold.

In the present paper we first introduce the concept of (g, F)-manifold which contains as particular cases: almost Hermitian and almost parahermitian manifolds, almost contact and almost paracontact manifolds, almost sympletic manifolds, etc. Then we study semi-invariant submanifolds of a (g, F) - manifold, which are extensions of CR-submanifolds to this general class of manifolds. We find necessary and sufficient conditions for the integrability of both distributions on a semi-invariant submanifold (cf. Theorem 10 and Theorem 11). In particular, for (g, F) - manifolds wich are generalizations of Kaehler manifolds and parakaehler manifolds we prove that semi-invariant submanifolds carry a natural foliation (cf. Theorem 13). Finally, we obtain characterizations of totally geodesic foliations on semi-invariant submanifolds (cf. Theorem 16 and Theorem 17).

2010 Mathematics Subject Classification: 53C40, 53C15.
Keywords: semi-invariant submanifold; (g,F)-manifold; Distribution; Foliation.

2 Preliminaries

Let M be an m-dimensional Riemannian manifold with Riemannian metric g. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $F(M)$-module of smooth sections of the tangent bundle TM of M. We use the same notation for any other vector bundle over M. All manifolds and mappings are supposed to be differentiable of class C^∞.

Next, we consider an n-dimensional submanifold N of M. Then the main objects induced by the Levi-Civita connection $\widetilde{\nabla}$ of (M, g) on N are involved in the well known Gauss-Weingarten equations:

$$(a) \quad \widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and } (b) \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.1}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$. Here, ∇ is the Levi-Civita connection on N, h is the second fundamental form of N, A_V is the Weingarten operator with respect to the normal section V, and $\nabla^\perp$ is the normal connection in the normal bundle $TN^\perp$ of N. The two geometric objects h and A_V are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.2}$$

If h vanishes identically on N, then N is called totally geodesic.

Several studies have been developed on submanifolds of manifolds endowed with some geometrical structures. We recall here some of these structures. First, we consider an almost Hermitian manifold (M, g, J), where g is a Riemannian metric, J is an almost complex structure, that is, $J^2 = -I$, satisfying (cf. Yano - Kon [12], p. 124).

$$g(JX, JY) = g(X, Y), \quad \forall\ X, Y \in \Gamma(TM). \tag{2.3}$$

In 1978, Bejancu [2] has introduced the concept of CR-submanifold of an almost Hermitian manifold as follows. A real submanifold N of (M, g, J) is called a CR-submanifold (Cauchy-Riemann submanifold) if it is endowed with a distribution D satisfying the following conditions:

(i) D is invariant with respect to J, that is,

$$J(D_x) = D_x, \quad \forall\ x \in N.$$

(ii) The complementary orthogonal distribution $D^\perp$ to D in TM is anti-invariant with respect to J, that is,

$$J(D^\perp_x) \subset T_x N^\perp, \quad \forall\ x \in N.$$

When $D^\perp = \{0\}$ (resp. $D = \{0\}$), N is called invariant submanifold (resp. anti-invariant submanifold). Any real hypersurface of (M, g, J) is a CR-submanifold which is neither invariant nor anti-invariant submanifold. Many papers have been published on the geometry of CR-submanifolds, some of the important results being brought together in the books of Bejancu [3], Chen [7] and Yano - Kon [11].

Next, we consider a manifold M equipped with a semi-Riemannian metric g (cf. O'Neill [9], p.54) and an almost product structure P, that is $P^2 = I$, $(P \neq \pm I\ )$. Then (M, g, P) is called an almost parahermitian manifold if we have

$$g(PX, PY) = -g(X, Y), \quad \forall\ X, Y \in \Gamma(TM). \tag{2.4}$$

The concept of CR-submanifold has been considered by Bejan [1] in case the ambient manifold is an almost parahermitian manifold. Both, the almost Hermitian and almost parahermitian manifolds are necessarily of even dimension. The odd dimensional coanterparts of these manifolds can be introduced as follows.

Let M be a real $(2m+1)$-dimensional manifold endowed with a Riemannian metric g, a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the conditions

$$\begin{aligned} (a)\ \ \varphi^2 &= -I + \eta \otimes \xi, \quad (b)\ \ \eta(\xi) = 1, \\ (c)\ \ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{2.5}$$

Then $(M, g, \varphi, \xi, \eta)$ is called an almost contact metric manifold (cf. Blair [5, p.33]). The concept of semi-invariant submanifold of an almost contact metric manifold (cf. Bejancu-Papaghiuc [4]) represents an extension of the concept of CR-submanifold to the case of odd dimensional ambient manifold. Similarly, consider a $(2m+1)$-dimensional manifold M endowed with (g, φ, ξ, η) satisfying:

$$\begin{aligned} (a)\ \ \varphi^2 &= I - \eta \otimes \xi, \quad (b)\ \ \eta(\xi) = \varepsilon, \\ (c)\ \ g(\varphi X, \varphi Y) &= -g(X, Y) + \varepsilon\eta(X)\eta(Y), \end{aligned} \tag{2.6}$$

where $\varepsilon = +1$ or $\varepsilon = -1$, according as ξ is spacelike or timelike vector field with respect to the semi-Riemannian metric g. Then $(M, g, \varphi, \xi, \eta)$ is called an almost paracontact metric manifold (cf. Sato [10]). Semi-invariant submanifolds of almost paracontact manifolds (cf. Ianuş-Mihai [8]) are extensions of CR-submanifolds to this class of odd dimensional manifolds.

Finally, we recall that a real $2m$-dimensional manifold M is called an almost symplectic manifold if it is endowed with a nondegenerate 2-form Ω.

3 (g,F) - Manifolds and their submanifolds

Let M be a real m-dimensional manifold and g be a semi-Riemannian metric on M. Thus g might be a Riemannian metric or nondegenerate of constant index at any point of M. Suppose that there exists on M a non zero tensor field F of type $(1,1)$ satisfying

$$g(FX, Y) + g(X, FY) = 0, \quad \forall\ X, Y \in \Gamma(TM). \tag{3.1}$$

Then we say that M is a (g, F)-manifold. If in particular, F_x is nondegenerate at any point $x \in M$ then we say that M is a nondegenerate (g, F)-manifold. Otherwise, M is called degenerate (g, F)-manifold.

In literature there is an abundance of examples of (g, F)-manifolds. Some of these examples are presented here.

Example 1. *An almost Hermitian manifold (M, g, J) is a nondegenerate (g, F)-manifold. Indeed, take $F = J$ and from 2.3 we deduce 3.1.*

Example 2. *An almost parahermitian manifold (M, g, P) is a nondegenerate (g, F)-manifold. In this case we take $F = P$ and by using 2.4 and taking into account that $P^2 = I$ we obtain 3.1.*

Example 3. *An almost contact metric manifold $(M, g, \varphi, \xi, \eta)$ is a degenerate (g, F)-manifold. We put $F = \varphi$ and by using 2.5 we deduce 3.1. As $\varphi(\xi) = 0$, M is a degenerate (g, F)-manifold.*

Example 4. *An almost paracontact manifold $(M, g, \varphi, \xi, \eta)$ is a degenerate (g, F)-manifold. Here we take $F = \varphi$ and by 2.6 we obtain 3.1. As in the previous example we have $\varphi(\xi) = 0$, and therefore M is a degenerate (g, F)-manifold.*

Example 5. *Let (M, Ω) be an almost symplectic manifold endowed with a semi-Riemannian metric g. Then we define a tensor field F of type $(1, 1)$ by*

$$g(FX, Y) = \Omega(X, Y), \quad \forall\ X, Y \in \Gamma(TM). \tag{3.2}$$

As Ω is skew-symmetric, we deduce that F and g satisfy 3.1. Moreover, since Ω is nondegenerate we conclude that (M, Ω, g) is a nondegenerate (g, F)-manifold.

Remark 6. *Any $2m$-dimensional nondegenerate (g, F)-manifold is an almost symplectic manifold. Indeed, define Ω by 3.2 and by using 3.1 we deduce that Ω is a nondegenerate 2-form on M.*

Next, we consider a submanifold N of a (g, F)-manifold M. Suppose that g induces a semi-Riemannian metric on N which we denote by the same symbol g. Then, following the definition given by Bejancu [2] for CR-submanifolds we introduce a special class of submanifolds of M as follows.

Definition 7. *We say that N is a semi-invariant submanifold of the (g, F)-manifold M if there exists a distribution D on M satisfying the conditions:*

(i) D is a nondegenerate distribution with respect to g, and we have

$$F(D_x) \subset D_x\ , \quad \forall\ x \in N,$$

that is, D is F-invariant.

(ii) The complementary orthogonal distribution $D^\perp$ to D in TN is F-anti-invariant, that is,

$$F(D_x^\perp) \subset T_x N^\perp \ , \quad \forall\ x \in N.$$

(iii) $F^2(D^\perp)$ is a distribution on N.

If in particular, M is an almost Hermitian manifold, then we obtain the concept of CR-submanifold. In this case, the condition (iii) is a consequence of (i) and (ii). Moreover, the above concept of semi-invariant submanifold is a generalization of all the extensions of the concept of CR-submanifold to almost parahermitian manifolds, almost contact metric manifolds, almost paracontact metric manifolds, etc. (see Bejancu [3]).

Some particular classes of semi-invariant submanifolds are defined as follows. Let p and q be the ranks of the distributions D and $D^\perp$ respectively. If $q = 0$, that is $D^\perp = \{0\}$, we say that N is an F-invariant submanifold of M. If $p = 0$, that is $D = \{0\}$, we call N an F-anti-invariant submanifold of M. Thus, N is an F-invariant (resp. F-anti-invariant) submanifold if and only if

$$F(TN) \subset TN \quad (\text{resp. } F(TN) \subset TN^\perp).$$

If $pq \neq 0$ then N is called a proper semi-invariant submanifold. Now, we denote by $\widetilde{D}$ the complementary orthogonal vector bundle to $F(D^\perp)$ in $TN^\perp$. If $\widetilde{D} = \{0\}$, then we say that N is a normal F-anti-invariant submanifold. Thus N is normal F-anti-invariant if and only if

$$F(D^\perp) = TM^\perp.$$

Taking into account the Definition 7 we deduce that the tangent bundle and the normal bundle of a semi-invariant submanifold N have the orthogonal decompositions:

$$(a) \quad TN = D \oplus D^\perp \quad \text{and} \quad (b) \quad TN^\perp = F(D^\perp) \oplus \widetilde{D}. \tag{3.3}$$

Then we denote by P and Q the projection morphisms of TN on D and $D^\perp$ respectively, and obtain

$$(a) \quad X = PX + QX \ , \quad (b) \quad FX = \varphi X + \omega X \ , \quad \forall \ \ X \in \Gamma(TN), \tag{3.4}$$

where we put

$$(a) \quad \varphi X = FPX \quad \text{and} \quad (b) \quad \omega X = FQX \tag{3.5}$$

Thus φ is a tensor field of type $(1, 1)$ on N, while ω is a $F(D^\perp)$-valued vector 1-form on N.

Next, we prove the following.

Proposition 8. *Let N be a semi-invariant submanifold of a (g, F)-manifold M. Then we have the following assertions:*

(i) N is a (g, φ)-manifold.

(ii) $F^2(D^\perp)$ is a vector subbundle of $D^\perp$.

(iii) The vector bundle $\widetilde{D}$ is F-invariant, that is, we have

$$F(\widetilde{D}_x) \subset \widetilde{D}_x \ , \quad \forall\ x \in N.$$

Proof. (i) By definition, g is a semi-Riemannian metric on N and φ is a tensor field of type $(1,1)$ on N, we need only to show 3.1. By using 3.5(a), 3.4(a) and 3.1 for F we obtain

$$\begin{aligned} g(\varphi X, Y) &= g(FPX, Y) = g(FPX, PY) = -g(PX, FPY) \\ &= -g(X, FPY) = -g(X, \varphi Y), \quad \forall\ \ X, Y \in \Gamma(TN). \end{aligned}$$

Thus 3.1 is satisfied for g and φ, and therefore N is a (g, φ)-manifold.
Take $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Then by using 3.1 we obtain

$$g(X, F^2Y) = -g(FX, FY) = 0 \ ,$$

since $FX \in \Gamma(D)$ and $FY \in \Gamma(TM^\perp)$. Hence $F^2(D^\perp)$ is orthogonal to D and by condition (iii) of Definition 7. we deduce that $F^2(D^\perp)$ is a vector subbundle of $D^\perp$.

(iii) Take $X \in \Gamma(TN)$, $Y \in \Gamma(D^\perp)$ and $V \in \Gamma(\widetilde{D})$. Then by using 3.1 and 3.4(b) we obtain

$$g(FV, X) = -g(V, FX) = -g(V, \varphi X + \omega X) = 0,$$

and

$$g(FV, FY) = -g(V, F^2Y) = 0,$$

since $\varphi X \in \Gamma(D)$, $\omega X \in \Gamma(FD^\perp)$ and $F^2Y \in \Gamma(D^\perp)$. Thus $F\widetilde{D}$ is orthogonal to $TN \oplus FD^\perp$, that is, $F\widetilde{D}$ is a vector subbundle of $\widetilde{D}$. This completes the proof of the proposition.

□

Taking into account that F is an automorphism of TM provided M is a non-degenerate (g, F)-manifold, by condition (i) of Definition 7 and by assertions (ii) and (iii) of 8 we can state the following.

Corollary 9. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold M. Then we have:*

$$F(D) = D \ , \qquad F^2(D^\perp) = D^\perp \qquad \textit{and} \qquad F(\widetilde{D}) = \widetilde{D}.$$

4 Integrability of distributions on a semi-invariant submanifold

Let N be a semi-invariant submanifold of a (g, F)-manifold M. Then we recall that the Nijenhuis tensor field of F is defined as follows (cf. Blair [5], p. 63).

$$[F, F](X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY], \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$. In a similar way, the Nijenhuis tensor field of φ on N is given by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \tag{4.2}$$

for any $X, Y \in \Gamma(TN)$. We recall that a tensor field of type $(1, 1)$ defines an integrable structure on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of D and $D^\perp$ in terms of Nijenhuis tensor fields of F and φ.

Theorem 10. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold M. Then the following assertions are equivalent:*
(i) D is an integrable distribution.
(ii) The Nijenhuis tensor field of φ satisfies the equality

$$Q[\varphi, \varphi](X, Y) = 0 \ , \quad \forall\ X, Y \in \Gamma(D). \tag{4.3}$$

(iii) The Nijenhuis tensor fields of F and φ satisfy the equality

$$[F, F](X, Y) = [\varphi, \varphi](X, Y) \ , \quad \forall\ X, Y \in \Gamma(D). \tag{4.4}$$

Proof. First, we note that D is integrable if and only if

$$Q[X, Y] = 0 \ , \quad \forall\ X, Y \in \Gamma(D). \tag{4.5}$$

Then from 4.2 we deduce that

$$Q[\varphi, \varphi](X, Y) = Q[\varphi X, \varphi Y] \ , \quad \forall\ X, Y \in \Gamma(D), \tag{4.6}$$

since the last three terms in the right side of 4.2 lie in $\Gamma(D)$. As M is nondegenerate, we deduce that φ is an automorphism on $\Gamma(D)$. Thus the equivalence of (i) and (ii) follows from 4.5 and 4.6. Next, by using 4.1, 4.2, 3.4 and 3.5 we obtain

$$[F, F](X, Y) = [\varphi, \varphi](X, Y) + F^2Q[X, Y] - \omega([\varphi X, Y] + [X, \varphi Y]), \tag{4.7}$$

for any $X, Y \in \Gamma(D)$. Suppose that D is integrable. Then by using 4.5 in 4.7 and taking into account 3.5(b) we obtain 4.4. Conversely, suppose that 4.4 is satisfied. Then from 4.7 we deduce that

$$F^2 Q[X, Y] = \omega([\varphi X, Y] + [X, \varphi Y]). \tag{4.8}$$

By assertion (ii) of 8 we deduce that the left side of 4.8 is in $\Gamma(D^\perp)$. On the other hand, by 3.5(b) the right side of 4.8 is in $\Gamma(FD^\perp)$. As $D^\perp$ is a subbundle of TN while $FD^\perp$ is a subbundle of $TN^\perp$, we conclude that both sides in 4.8 must vanish. Finally, from

$$F^2 Q[X, Y] = 0 , \quad \forall\ X, Y \in \Gamma(D),$$

we deduce 4.5, since F is an automorphism of $\Gamma(TM)$. Hence (i) is equivalent to (iii), and this completes the proof of the theorem. □

Now, consider $X, Y \in \Gamma(D^\perp)$. Then taking into account that $\varphi X = \varphi Y = 0$, and by using 3.5(a) into 4.2 we obtain

$$[\varphi, \varphi](X, Y) = F^2 P[X, Y].$$

This enables us to state the following.

Theorem 11. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold. Then $D^\perp$ is integrable if and only if the Nijenhuis tensor field of φ vanishes identically on $D^\perp$.*

5 A natural foliation on a semi-invariant submanifold

Let M be a (g, F)-manifold and $\widetilde{\nabla}$ be the Levi-Civita connection on M with respect to the Riemannian metric g. Then we say that F is a parallel tensor field on M if

$$(\widetilde{\nabla}_X F)Y = 0 , \quad \forall\ X, Y \in \Gamma(TM). \tag{5.1}$$

Kaehler manifolds, para-Kaehler manifolds and cosymplectic manifolds are examples of (g, F)-manifolds with parallel tensor field F.

In the present section we study the geometry of semi-invariant submanifolds of (g, F)-manifolds with parallel tensor field F. First, we prove the following.

Proposition 12. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold with parallel tensor field F. Then we have*

$$A_{FX} Y - A_{FY} X = \varphi([X, Y]) , \quad \forall\ X, Y \in \Gamma(D^\perp). \tag{5.2}$$

Proof. By using 5.1, 2.1 and 3.4(b) we obtain

$$-A_{FY}X + \nabla^{\perp}_X FY = \varphi(\nabla_X Y) + \omega(\nabla_X Y) + Fh(X, Y),$$

for any $X, Y \in \Gamma(D^{\perp})$. Write a similar equation by interchanging X and Y, and then subtracting obtain

$$A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX = \varphi([X, Y]) + \omega([X, Y]),$$

since ∇ is a torsion-free linear connection and h is a symmetric $F(N)$-bilinear mapping on $\Gamma(TN)$. Thus 5.2 is obtained by equalizing the tangent parts to N in the above equation. □

Now, we can state the following important result.

Theorem 13. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold with parallel tensor field F. Then the F- anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. By using 2.1(b), 5.1 and 3.1, and taking into account that $\widetilde{\nabla}$ is a torsion-free and metric connection, we obtain

$$\begin{aligned} g(A_{FX}Y, Z) &= -g(\widetilde{\nabla}_Y FX, Z) = g(\widetilde{\nabla}_Y X, FZ) = -g(X, \widetilde{\nabla}_Y FZ) \\ &= g(FX, \widetilde{\nabla}_Y Z) = g(FX, [Y, Z] + \widetilde{\nabla}_Z Y) \\ &= g(FX, \widetilde{\nabla}_Z Y), \end{aligned} \tag{5.3}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$. Also, we have

$$g(A_{FY}X, Z) = g(FY, \widetilde{\nabla}_Z X) = -g(F\widetilde{\nabla}_Z Y, X) = g(\widetilde{\nabla}_Z Y, FX). \tag{5.4}$$

Comparing 5.3 and 5.4 we deduce that

$$g(A_{FX}Y - A_{FY}X, Z) = 0 \ , \quad \forall\ X, Y \in \Gamma(D^{\perp}) \ , \ Z \in \Gamma(D).$$

On the other hand, from 5.2 we conclude that

$$A_{FX}Y - A_{FY}X \in \Gamma(D).$$

Thus we proved that

$$A_{FX}Y - A_{FY}X = 0 \ , \quad \forall\ X, Y \in \Gamma(D^{\perp}). \tag{5.5}$$

Finally, by using 5.5 in 5.2 and taking into account that F is nondegenerate, we deduce that

$$P[X,Y]=0\ ,\quad \forall\ X,Y\in\Gamma(D^{\perp}),$$

that is, $D^{\perp}$ is integrable. □

Regarding the integrability of D we prove the following.

Theorem 14. *Let N be a semi-invariant submanifold of a nondegenerate (g,F)-manifold M with parallel tensor field F. Then the F-invariant distribution D is integrable if and only if the second fundamental form h of N satisfies*

$$g(h(X,\varphi Y)-h(Y,\varphi X),FZ)=0\ , \tag{5.6}$$

for any $X,Y\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$.

Proof. By using 5.1 and 2.1 we deduce that

$$\nabla_X\varphi Y+h(X,\varphi Y)=\varphi(\nabla_X Y)+\omega(\nabla_X Y)+Fh(X,Y),$$

for any $X,Y\in\Gamma(D)$. Write a similar equation by interchanging X and Y, and then subtracting obtain

$$\nabla_X\varphi Y-\nabla_Y\varphi X+h(X,\varphi Y)-h(Y,\varphi X)=\varphi([X,Y])+\omega([X,Y])\ ,$$

since h is symmetric and ∇ is a torsion-free linear connection. Equalize the normal parts in the above equation and obtain

$$h(X,\varphi Y)-h(Y,\varphi X)=\omega([X,Y]). \tag{5.7}$$

Now, suppose that D is integrable. Then by using 3.5(b) and 4.5 in 5.7 we deduce 5.6. Conversely, if 5.6 is satisfied, then from 5.7 we deduce that

$$g(Q[X,Y],F^2Z)=-g(FQ[X,Y],FZ)=-g(\omega[X,Y],FZ)=0\ .$$

Since M is nondegenerate, from Corollary 9 we infer that F^2 is an automorphism of $\Gamma(D^{\perp})$. Hence the above equality implies 4.5, that is D is integrable. □

Remark 15. *In particular, if F is an almost complex structure on M, from Theorem 13 and Theorem 14 we obtain the results of Blair-Chen [6] and Bejancu [2] respectively, for CR-submanifolds.*

Now, we denote by $F^{\perp}$ the natural foliation defined by the F-anti-invariant distribution $D^{\perp}$, and call it the F-anti-invariant foliation on N. We recall that $F^{\perp}$ is called a totally geodesic foliation if each leaf of $F^{\perp}$ is totally geodesic immersed in N. Thus $F^{\perp}$ is totally geodesic if and only if the Levi-Civita connection ∇ on N satisfies

$$\nabla_Y Z \in \Gamma(D^{\perp}) \ , \quad \forall\ Y, Z \in \Gamma(D^{\perp}). \tag{5.8}$$

Theorem 16. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold M with parallel tensor field F. Then the following assertions are equivalent:*

(i) The F-anti-invariant foliation is totally geodesic.
(ii) The second fundamental form h of N satisfies

$$h(X, Y) \in \Gamma(\widetilde{D}) \ , \quad \forall\ X \in \Gamma(D),\ Y \in \Gamma(D^{\perp}). \tag{5.9}$$

(iii) $D^{\perp}$ is A_V-invariant for any $V \in \Gamma(FD^{\perp})$, that is we have

$$A_V Y \in \Gamma(D^{\perp}) \ , \quad \forall\ Y \in \Gamma(D^{\perp}).$$

Proof. By using 2.1, 3.1, 5.1 and 2.2 we obtain

$$\begin{aligned} g(\nabla_Y Z, FX) &= g(\widetilde{\nabla}_Y Z, FX) = -g(\widetilde{\nabla}_Y FZ, X) \\ &= g(A_{FZ} Y, X) = g(h(X, Y), FZ)\ , \end{aligned} \tag{5.10}$$

for any $X \in \Gamma(D)$ and Y,$Z \in \Gamma(D^{\perp})$. Now, suppose that $F^{\perp}$ is totally geodesic. Then by 5.8, the first term of 5.10 vanishes since $FX \in \Gamma(D)$ for any $X \in \Gamma(D)$. Hence the last term is 5.10 vanishes, which implies 5.9. Conversely, suppose 5.9 is satisfied. Then from 5.10 we deduce 5.8 since F is an automorphism of $\Gamma(D)$. This proves the equivalence of (i) and (ii). Due to 2.2 we obtain the equivalence of (ii) and (iii). This completes the proof of the theorem. □

Finally, we can prove the following.

Theorem 17. *Let N be a semi-invariant submanifold of a nondegenerate (g, F)-manifold with parallel tensor field F. Then the F-invariant distribution D is integrable and the foliation F defined by D is totally geodesic if and only if the second fundamental form h of N satisfies*

$$h(X, Y) \in \Gamma(\widetilde{D}) \ , \quad \forall\ X, Y \in \Gamma(D). \tag{5.11}$$

Proof. D is integrable and F is totally geodesic if and only if

$$\nabla_X U \in \Gamma(D)\ , \quad \forall\ X, U \in \Gamma(D).$$

By 2.1(a), this is equivalent to

$$g(\widetilde{\nabla}_X U, Z) = 0\ , \quad \forall\ Z \in \Gamma(D^{\perp}).$$

As F is an automorphism of $\Gamma(D)$ we can write the above equality as follows

$$g(\widetilde{\nabla}_X FY, Z) = 0\ , \quad \forall\ X, Y \in \Gamma(D),\ Z \in \Gamma(D^{\perp}),$$

which via 5.1 and 3.1 is equivalent to

$$g(\widetilde{\nabla}_X Y, FZ) = 0.$$

Finally, by using 2.1 we conclude that D is integrable and F is totally geodesic if and only if

$$g(h(X,Y), FZ) = 0,$$

which completes the proof of the theorem. □

References

[1] C.L. Bejan, *CR-Submanifolds of hyperbolic almost Hermitian manifolds*, Demonstratio Mathematica, **23** (1990), 335-343. MR1101496 (92b:53036). Zbl 0754.53023.

[2] A. Bejancu, *CR-Submanifolds of a Kaehler manifold* I, Proc. Amer. Math. Soc., **69** (1978), 134-142. MR0467630 (57 #7486). Zbl 0368.53040.

[3] A. Bejancu, *Geometry of CR-Submanifolds*, D. Reidel Publish. Comp., Dordrecht, 1986. MR0861408 (87k:53126). Zbl 0605.53001.

[4] A. Bejancu and N. Papaghiuc, *Semi-invariant submanifolds of a Sasakian manifold*, An. St. Univ. "Al. I. Cuza" Iaşi, **27** (1981), 163-170. MR0618723(82i:53037).

[5] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002. MR1874240 (2002m:53120). Zbl 1011.53001.

[6] D.E. Blair and B.Y. Chen, *On CR-submanifolds of Hermitian manifolds*, Israel J. Math., **34** (1979), 353-363. MR0570892(81f:53049). Zbl 0453.53018.

[7] B.Y. Chen, *Geometry of Submanifolds and Its Applications*, Science Univ. Tokyo, 1981. MR0627323 (82m:53051). Zbl 0474.53050.

[8] S. Ianuş and I. Mihai, *Semi-invariant submanifolds of an almost paracontact manifolds*, Tensor N.S., **39** (1982), 195-200. MR0836935 (87c:53102). Zbl 0514.53015.

[9] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983. MR0719023(85f:53002). Zbl 0531.53051.

[10] I. Sato, *On a structure similar to the almost contact structure*, Tensor, **30** (1976), 219-224. MR0442845(56 #1221). Zbl 0344.53025

[11] K. Yano and M. Kon, *CR-Submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhäuser, Boston, 1983. MR0688816 (84k:53051). Zbl 0496.53037.

[12] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984. MR0794310(86g:53001). Zbl 0557.53001.

Novac-Claudiu Chiriac
University Constantin Brâncuşi of Târgu-Jiu
Str. Geneva, Nr. 3, 210136 Târgu-Jiu, Romania.
e-mail: novac@utgjiu.ro
http://www.utgjiu.ro/math/nchiriac/