Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **4** (2009) , 89 – 97

MULTIVALUED PEROV-TYPE THEOREMS IN GENERALIZED METRIC SPACES

Liliana Guran

Abstract. In this paper we present some fixed point results for multivalued operators, which extend the ones given by A.I. Perov and A.V. Kribenko, as well as some recent contributions due to A. Bucur, L. Guran and A. Petruşel.

1 Introduction

In 1996 the Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the w-distance (see [2]) and discussed some properties of this new generalized metric. In [4] A. Perov gave some fixed point results for contractions on spaces endowed with vector-valued metrics.

The purpose of this paper is to present some new Perov-Kibenko type fixed point results for multivalued operators. Our theorems extend some results given in [1] for operators which satisfy some contractivity conditions with respect to generalized w-distances.

2 Preliminaries

Let (X, d) be a metric space. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$\mathcal{P}(X) = P(X) \bigcup \emptyset$

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_b(X)$ - *the set of all nonempty bounded subsets of* X;

$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of* X;

For two subsets $A, B \in P_b(X)$ we recall the following functionals.

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A, B) := sup\{d(a, b) | x \in A, b \in B\}$ - *the diameter functional*;

2000 Mathematics Subject Classification: 47H10; 54H25.
Keywords: fixed point; multivalued operator; w-distance; Perov type generalized contraction.

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A,B) := max\{\sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b)\}$ - *the Pompeiu-Hausdorff functional*;

If $T : X \to P(X)$ is a multivalued operator, then we denote by $Fix(T)$ the fixed point set of T, i.e. $Fix(T) := \{x \in X | x \in T(x)\}$ and by $SFix(T)$ the strict fixed point set of T, i.e. $SFix(T) := \{x \in X | \{x\} = T(x)\}$. The symbol $Graph(T) := \{(x,y) \in X \times X : y \in T(x)\}$ denotes the graph of T.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [2]) as follows:

Let (X,d) be a metric space. A functional $w : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x,z) \leq w(x,y) + w(y,z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.

Some examples of w-distance are given in [2].

Example 1. *Let (X,d) be a metric space. Then the metric "d" is a w-distance on X.*

Example 2. *Let X be a normed linear space with norm $|| \cdot ||$. Then the function $w : X \times X \to [0, \infty)$ defined by $w(x,y) = ||x|| + ||y||$ for every $x, y \in X$ is a w-distance.*

Example 3. *Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0, \infty)$ defined by:*

$$w(x,y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ is a w-distance.

Let us recall a crucial lemma for w-distance (see [10] for more details).

Lemma 4. *Let (X,d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following statements hold:*

1. *If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*

4. If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

For the rest of the paper, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by $v \leq r$ means $v_i \leq r_i$, for each $i \in \{1, 2, \cdots, m\}$, while $v < r$ means $v_i < r_i$, for each $i \in \{1, 2, \cdots, m\}$.

Also, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$ and, if $c \in \mathbb{R}$ then $v \leq c$ means $v_i \leq c_i$, for each $i \in \{1, 2, \cdots, m\}$.

Definition 5. *A pair (X, d) is called a generalized metric space in Perov' sense if X is a nonempty set and $d : X \times X \to \mathbb{R}^m_+$ is a functional satisfying the usual axioms of a metric.*

Remark 6. *Notice that in a generalized metric space in Perov' sense the concepts of Cauchy sequence, convergent sequence, completeness, open and closed subsets are similar defined as those in a metric space.*

If $x_0 \in X$ and $r \in \mathbb{R}^m_+$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$ we will denote by $B(x_0; r) := \{x \in X |\ d(x_0, x) < r\}$ the open ball centered in x_0 with radius $r := (r_1, r_2, \cdots, r_m)$ and by $\widetilde{B}(x_0; r) := \{x \in X |\ d(x_0, x) \leq r\}$ the closed ball centered in x_0 with radius r.

In [3] we can find the notion of generalized w-distance as follows.

Definition 7. *Let (X, d) a generalized metric space. The mapping $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ defined by $\widetilde{w}(x, y) = (v_1(x, y), v_2(x, y), ..., v_m(x, y))$ is said to be a generalized w-distance if it satisfies the following conditions:*

(w_1) $\widetilde{w}(x, y) \leq \widetilde{w}(x, z) + \widetilde{w}(z, y)$, for every $x, y, z \in X$;

(w_2) $v_i : X \times X \to \mathbb{R}_+$ is lower semicontinuous, for $i \in \{1, 2, \ldots, m\}$;

(w_3) For any $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) > 0$, for $m \in \mathbb{N}$, there exists $\delta := (\delta_1, \delta_2, ..., \delta_m) > 0$ such that $\widetilde{w}(z, x) \leq \delta$ and $\widetilde{w}(z, y) \leq \delta$ implies $\widetilde{d}(x, y) \leq \varepsilon$.

Let us present now an important lemma for w-distances into the terms of generalized w-distances.

Lemma 8. *Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let (x_n) and (y_n) be two sequences in X, let $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)}, ..., \alpha_n^{(m)}) \in \mathbb{R}_+$ and $\beta_n = (\beta_n^{(1)}, \beta_n^{(2)}, ..., \beta_n^{(m)}) \in \mathbb{R}_+$ be two sequences such that $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ converge to zero for each $i \in \{1, 2, \ldots, m\}$. Let $x, y, z \in X$. Then the following assertions hold:*

1. *If $\widetilde{w}(x_n, y) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $\widetilde{w}(x_n, y_n) \leq \alpha_n$ and $\widetilde{w}(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $\widetilde{w}(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*

4. If $\widetilde{w}(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

Proof. Let us first prove (2). Let $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) > 0$ be given. From the definition of generalized w-distance, there exists $\delta := (\delta_1, \delta_2, ..., \delta_m) > 0$ such that $\widetilde{w}(u, v) \leq \delta$ and $\widetilde{w}(u, z) \leq \delta$ implies $\widetilde{d}(v, z) \leq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that the sequences α_n and β_n defined in the hypothesis satisfies the inequalities $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$. Then we have, for any $n \geq n_0$,
$\widetilde{w}(x_n, y_n) \leq \alpha_n \leq \delta$ and $\widetilde{w}(x_n, z) \leq \beta_n \leq \delta$ and hence $\widetilde{d}(y_n, z) \leq \varepsilon$. This implies that $\{y_n\}$ converges to z. It follows from (2) that (1) holds.

Let us prove (3). As previous let $\varepsilon > 0$ be given, choose $\delta > 0$ and $n_0 \in \mathbb{N}$. Then for any $m, n \geq n_0 + 1$ we have $\widetilde{w}(x_{n_0}, x_n) \leq \alpha_{n_0} \leq \delta$ and $\widetilde{w}(x_{n_0}, x_m) \leq \alpha_{n_0} \leq \delta$ and hence $\widetilde{d}(x_n, x_m) \leq \varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. As in the proof of (3), we can prove (4). □

Let us present some examples of generalized w-distances.

Example 9. *Let $(X, \widetilde{d})$ be a generalized metric space. Then $\widetilde{w} = \widetilde{d}$ is a generalized w-distance on X.*

Example 10. *Let $w_1, ..., w_m : X \times X \to \mathbb{R}_+$ be w-distances. Then $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ defined by $\widetilde{w}(x, y) = (w_1(x, y), w_2(x, y), ..., w_m(x, y))$ is a generalized w-distance.*

Throughout this paper we will denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ the zero $m \times m$ matrix and by I the identity $m \times m$ matrix. If $A \in M_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A.

Recall that a matrix A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

For the proof of the main result we need the following theorem, see [7].

Theorem 11. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

(i) A is a matrix convergent to zero;

(i) $A^n \to 0$ as $n \to \infty$;

(ii)The eigen-values of A are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;

(iii) The matrix $I - A$ is non-singular and

$$(I - A)^{-1} = I + A + ... + A^n + ...;$$

(iv) The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnenegative elements.

(v) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

3 Main results

Throughout this section (X, d) is a generalized metric space in Perov's sense and w is a generalized w-distance on the generalized metric space.

Let $x_0 \in X$ and $r > 0$. Let us define:

$B_w(x_0; r) := \{x \in X | w(x_0, x) < r\}$ the open ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}(x_0; r) := \{x \in X | w(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}^d(x_0; r)$- the closure in (X, d) of the set $B_w(x_0; r)$.

Definition 12. *Let $Y \subset X$ and $T : Y \to P(X)$ be a multivalued operator. Then, T is called a multivalued w-left A-contraction if $A = (a_{i,j})_{i,j \in \{1,2,\ldots,m\}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for each $x, y \in Y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $w(u, v) \leq Aw(x, y)$. This means, that for each $x, y \in Y$ and each $u \in T(x)$, there exists $v \in T(y)$ such that* $\begin{pmatrix} w_1(u,v) \\ \cdots \\ w_m(u,v) \end{pmatrix} \leq$

$$\begin{pmatrix} a_{11} & \cdots & a_{1,m} \\ \cdots & & \\ a_{m1} & \cdots & a_{m,m} \end{pmatrix} \cdot \begin{pmatrix} w_1(x,y) \\ \cdots \\ w_m(x,y) \end{pmatrix}$$

The notion of generalized w-distance with his properties was discussed in [3].

Theorem 13. *Let (X, d) be a complete generalized metric space, $w : X \times X \to \mathbb{R}^m_+$ a generalized w-distance, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$. Let $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X)$ be a multivalued w-left A-contraction. Suppose that:*

(i) if $v, r \in \mathbb{R}^m_+$ are such that $v^\tau \cdot (I - A)^{-1} \leq (I - A)^{-1} \cdot r$, then $v \leq r$;

(ii) there exists $x_1 \in T(x_0)$ such that $w(x_0, x_1)(I - A)^{-1} \leq r$.

Then, T has at least one fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$ such that $w(x_0, x_1)(I - A)^{-1} \leq r \leq (I - A)^{-1} \cdot r$. Then, by (i), $x_1 \in \widetilde{B_w}(x_0; r)$. Now, by the contraction condition, there exists $x_2 \in T(x_1)$ such that $w(x_1, x_2) \leq Aw(x_0, x_1)$.

Thus $w(x_1, x_2)(I - A)^{-1} \leq Aw(x_0, x_1)(I - A)^{-1} \leq Ar$. Notice that $x_2 \in \widetilde{B_w}(x_0; r)$. Indeed, since $w(x_0, x_2) \leq w(x_0, x_1) + w(x_1, x_2)$ we get that $w(x_0, x_2)(I - A)^{-1} \leq w(x_0, x_1)(I - A)^{-1} + w(x_1, x_2)(I - A)^{-1} \leq Ir + Ar \leq (I - A)^{-1}r$, which immediately implies (by (i)) that $w(x_0, x_2) \leq r$.

By induction, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{B_w}(x_0; r)$ having the properties:

(a) $x_{n+1} \in T(x_n), n \in \mathbb{N}$;

(b) $w(x_0,x_n)(I-A)^{-1} \leq (I-A)^{-1}r$, for each $n \in \mathbb{N}^*$, that means (by (i)) $w(x_0,x_n) \leq r$;

(c) $w(x_n,x_{n+1})(I-A)^{-1} \leq A^n r$, for each $n \in \mathbb{N}$.

By (c), for every $m,n \in \mathbb{N}$, with $m>n$, we get that

$$w(x_n,x_m)(I-A)^{-1} \leq A^n(I-A)^{-1}r.$$

By Lemma 8(3) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space. Denote by x^* its limit in $\widetilde{B_w}^d(x_0;r)$.

Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ is a sequence in $\widetilde{B_w}(x_0;s)$ which converge to x^* and $w(x_n,\cdot)$ is lower semicontinuous we have

$$w(x_n,x^*) \leq \lim_{m\to\infty} \inf w(x_n,x_m) \leq A^n(I-A)^{-1}r, \text{ for every } n \in \mathbb{N}.$$

For $x^* \in \widetilde{B_w}^d(x_0;s)$ and $x_n \in T(x_{n-1})$, $n \in \mathbb{N}^*$, there exists $u_n \in T(x^*)$ such that $w(x_n,u_n) \leq Aw(x_{n-1},x^*) \leq ... \leq A^n(I-A)^{-1}r$.

So, we have the following two relations:

$$w(x_n,x^*) \leq A^n(I-A)^{-1}r, \text{ for every } n \in \mathbb{N}.$$
$$w(x_n,u_n) \leq A^n(I-A)^{-1}r, \text{ for every } n \in \mathbb{N}.$$

Then, by Lemma 8(2) we obtain that $u_n \xrightarrow{d} x^*$. As $u_n \in T(x^*)$ for each $n \in \mathbb{N}$ and using that T has closed values in (X,d) it follows that $x^* \in T(x^*)$ Then T has a fixed point. □

Remark 14. *Some examples of matrix convergent to zero are:*

a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, *where* $a,b \in \mathbb{R}_+$ *and* $a+b<1$*;*

b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, *where* $a,b \in \mathbb{R}_+$ *and* $a+b<1$*;*

c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, *where* $a,b,c \in \mathbb{R}_+$ *and* $\max\{a,c\}<1$.

Notice that, for some $a,b,c \in \mathbb{R}_+$ condition (i) also takes place.

As a consequence of the previous theorem, if $T: X \to P_{cl}(X)$ is a multivalued w-left A-contraction on the complete generalized metric space (X,d), then we have the following result:

Corollary 15. *Let (X, d) be a complete generalized metric space, $w : X \times X \to \mathbb{R}^m_+$ a generalized w-distance and $T : X \to P_{cl}(X)$ be a multivalued w-left A-contraction. Then, T has at least one fixed point, moreover, for every $x^* \in T(x^*)$ we have that $w(x^*, x^*) = 0$.*

Remark 16. *This global result is proved in [3].*

A dual concept is given in the following definition.

Definition 17. *Let $Y \subset X$ and $T : Y \to P(X)$ be a multivalued operator. Then, T is called a multivalued w-right A-contraction if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and for each $x, y \in Y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $w(u, v) \leq w(x, y)A$.*

Remark 18. *Notice that, since $w(x, y)A = A^\tau w(x, y)$, the w-right A-contraction condition on the multivalued operator T is equivalent to the w-left A^τ-contraction condition given in Definition 12. It is also obvious that the matrix A converges to zero if and only if the matrix A^τ converges to zero (since A and A^τ have the same eigenvalues) and $[(I - A)^{-1}]^\tau = (I - A^\tau)^{-1}$.*

From Remark 18 and Theorem 13 we get the following dual result:

Theorem 19. *Let (X, d) be a complete generalized metric space, $w : X \times X \to \mathbb{R}^m_+$ a generalized w-distance, $x_0 \in X$ and $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ with $r_i > 0$ for each $i \in \{1, 2, \cdots, m\}$. Let $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X)$ be a multivalued w-right A-contraction. Suppose that:*

(i) if $v, r \in \mathbb{R}^m_+$ are such that $(I - A)^{-1} \cdot v^\tau \leq r \cdot (I - A)^{-1}$, then $v \leq r$;

(ii) there exists $x_1 \in T(x_0)$ such that $(I - A)^{-1} w(x_0, x_1) \leq r$.

Then, T has at least one fixed point.

Then we can give another global result for multivalued w-right A-contractions.

Corollary 20. *Let (X, d) be a complete generalized metric space, $w : X \times X \to \mathbb{R}^m_+$ a generalized w-distance and $T : X \to P_{cl}(X)$ be a multivalued w-right A-contraction. Then, T has at least one fixed point, moreover, for every $x^* \in T(x^*)$ we have that $w(x^*, x^*) = 0$.*

We are now interested for the problem that, if $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X)$ is a multivalued A-contraction, whether the closed ball $\widetilde{B_w}(x_0; r)$ is invariant with respect to T.

For example, for the case of a multivalued w-right A-contraction, we have:

Theorem 21. *Let* (X, d) *be a complete generalized metric space,* $w : X \times X \to \mathbb{R}^m_+$ *a generalized* w*-distance,* $x_0 \in X$ *and* $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ *with* $r_i > 0$ *for each* $i \in \{1, 2, \cdots, m\}$*. Let* $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X)$ *be a multivalued* w*-right* A*-contraction. Suppose also that* $w(x_0, u)(I - A)^{-1} \leq r$*, for each* $u \in T(x_0)$*.*

Then, the following assertions hold:

a) $\widetilde{B_w}(x_0; r)$ *is invariant with respect to* T*;*

b) T *has at least one fixed point.*

Proof. a) In order to prove that $\widetilde{B_w}(x_0; r)$ is invariant with respect to T, let us consider $x \in \widetilde{B_w}^d(x_0; r)$. Then, we have to show that $T(x) \subseteq \widetilde{B_w}(x_0; r)$. For this purpose, let $y \in T(x)$ be arbitrarily chosen. Then, by the contraction condition, there exists $u \in T(x_0)$ such that $w(y, u) \leq w(x_0, x)A$. Then, by the triangle inequality, we get that:

$w(x_0, y)(I - A)^{-1} \leq w(x_0, u)(I - A)^{-1} + w(u, y)(I - A)^{-1} \leq r + w(x_0, x)A(I - A)^{-1} \leq r + rA(I - A)^{-1} = r[I + A(I - A)^{-1}] = r[I + A(I + A + A^2 + \cdots)] = r(I - A)^{-1}$.

Thus, we get that $w(x_0, y) \leq r$. Hence, the proof of a) is complete.

b) Since $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(\widetilde{B_w}(x_0; r))$, Corollary 20 applies and the conclusion follows. □

A dual result is:

Theorem 22. *Let* (X, d) *be a complete generalized metric space,* $w : X \times X \to \mathbb{R}^m_+$ *a generalized* w*-distance,* $x_0 \in X$ *and* $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ *with* $r_i > 0$ *for each* $i \in \{1, 2, \cdots, m\}$*. Let* $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X)$ *be a multivalued* w*-left* A*-contraction. Suppose also that* $(I - A)^{-1}w(x_0, u) \leq r$*, for each* $u \in T(x_0)$*.*

Then, the following assertions hold:

a) $\widetilde{B_w}(x_0; r)$ *is invariant with respect to* T*;*

b) T *has at least one fixed point.*

References

[1] A. Bucur, L. Guran and A. Petruşel, *Fixed point theorems for multivalued operators on a set endowed with vector-valued metrics and applications*, Fixed Point Theory, **10** (2009), No.1, 19-34.

[2] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica **44** (1996) 381-391. MR1416281(97j:49011). Zbl 0897.54029.

[3] L. Guran, *A multivalued Perov-type theorems in generalized metric spaces*-accepted for publication in Creative Mathematics and Informatics, Baia-Mare, România.

[4] A.I. Perov, *On Cauchy problem for a system of ordinary differential equations*, (in Russian), Priblizhen. Metody Reshen. Differ. Uravn., **2** (1964), 115-134. MR0216057(35#6892). Zbl 0196.34703.

[5] A.I. Perov and A.V. Kibenko, *On a certain general method for investigation of boundary value problems* (Russian), Izv. Akad. Nauk SSSR Ser. Mat., **30** (1966), 249-264. MR0196534(33#4721). Zbl 0154.09201.

[6] R. Precup, *The role of the matrices that are convergent to zero in the study of semilinear operator systems*, Mathematical and Computer Modelling, **49** (2009), 703-708. MR2483647.

[7] I.A. Rus, *Principles and Applications of Fixed Point Theory*, (in Romanian), Editura Dacia, Cluj-Napoca, 1979.

[8] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), 541-559. MR2464135. Zbl pre05505966.

[9] I.A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008. MR2494238. Zbl pre05530321.

[10] T. Suzuki and W. Takahashi, *Fixed point theorems and characterizations of metric completeness*, Topological Methods in Nonlinear Analysis, **8** (1996), 371-382. MR1483635(99c:54064). Zbl 0902.47050.

Liliana Guran
Department of Applied Mathematics,
Babeş-Bolyai University of Cluj-Napoca,
Kogălniceanu Street no. 1, 400084 Cluj-Napoca, Romania.
e-mail: lilyguran@yahoo.com, gliliana.math@gmail.com