ISSN 1842-6298
Volume **2** (2007), 1 – 9

FIXED TIME IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In the paper we study weak and strong invariance of differential inclusions with fixed time impulses and with state constraints.

We also investigate some properties of the solution set of impulsive system without state constraints. When the right-hand side is one sided Lipschitz we prove also the relaxation theorem and study the funnel equation of the reachable set.

1 Introduction

In the paper we will study the system

$$\dot{x}(t) \in F(t, x(t)),\ x(0) = x_0 \in D,\ \text{a.e. } t \in I = [0,1],\ t \neq t_i, \tag{1}$$

$$\Delta x|_{t=t_i} = S_i(x(t_i - 0)),\ i = 1, \ldots, p,\ x(t) \in D, \tag{2}$$

where $x \in D$ (closed subset of a Banach space E) and $F : I \times D \rightrightarrows E$ is a multifunction with nonempty values, $0 = t_0 < t_1 < \ldots < t_m < 1 = t_{m+1}$, $S_i : D \to D$ is continuous function and $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$. First we study the existence of solutions, i.e. absolutely continuous on every (t_i, t_{i+1}) (and left continuous at t_i) functions which satisfy (1) for a.a. t with (possible) jumps (impulses) at t_i.

When E is a Hilbert space we study weak and strong invariance when the right-hand side is almost upper semi-continuous (USC) or almost lower semi-continuous (LSC). If $D \equiv E$ and if $F(\cdot,\cdot)$ is almost continuous we study the so called funnel equation (with solution the reachable set). In this case we use the so called one sided Lipschitz (OSL) condition.

The multifunction $G : E \rightrightarrows E$ with nonempty closed bounded values is said to be upper semi-continuous (USC) at x_0, when for every $\varepsilon > 0$ there exists $\delta > 0$ with $G(x_0) + \varepsilon\mathbb{B} \supset G(x_0 + \delta\mathbb{B})$. Here $\mathbb{B}$ is the open unit ball. The multifunction $G(\cdot)$ is said to be lower semi-continuous (LSC) at x_0 when for every $f \in G(x_0)$ and every sequence $\{x_i\}_{i=1}^{\infty}$ converging to x_0 there exist $f_i \in G(x_i)$ such that $f_i \to f_0$. When $G(\cdot)$ is USC (LSC) at every $x \in D$ it is called USC (LSC). The multifunction $F(\cdot,\cdot)$

2000 Mathematics Subject Classification: 34A60, 34A37, 34B15, 49K24.
Keywords: Impulsive differential inclusions, state constraints, invariant solutions.

is said to be almost USC when for every $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with Lebesgue measure $\text{meas}(I_\varepsilon) > 1 - \varepsilon$ such that $F(\cdot,\cdot)$ is USC on $I_\varepsilon \times D$. The almost LSC maps are defined analogously. The multifunction is called continuous when it is USC and LSC.

The multifunction F is said to be OSL if there exists a Kamke function $w(\cdot,\cdot)$ such that:

for every $x, y \in E$ and every $f_x \in F(t,x)$ there exists $f_y \in F(t,y)$ such that

$$\langle x - y, f_x - f_y \rangle \le w(t, |x - y|).$$

Recall that the almost continuous function $w : I \times \mathbb{R}^+ \to \mathbb{R}$ is said to be Kamke function when $w(t,0) \equiv 0$ and the only solution of $s(0) = 0$, $\dot{s} = w(t,s)$ is $s(t) \equiv 0$.

Let F be defined on the whole E. The solution $x(\cdot)$ is said to be viable if $x(t) \in D$. The system (1)–(2) is said to be weakly invariant if there exists at least one viable solution. It is called (strongly) invariant if all the solutions are viable.

$T_D(x) := \left\{ v : \liminf_{h \downarrow 0} \dfrac{\text{dist}(x + hv; D)}{h} = 0 \right\}$ is the Bouligand contingent cone of D at x and

$N_D^P(x) := \{\zeta \in E : \exists\, \sigma > 0 \text{ and } \eta > 0 \text{ satisfying } \langle \zeta, y - x \rangle \leqslant \sigma |y - x|^2 \quad \forall y \in S \cap x + \eta B\}$

is the proximal normal cone of D at x.

2 Existence of solutions

In this section we prove the existence of solutions for (1)–(2). In case of Hilbert space and when F is defined on the whole E we study weak and strong invariance.

In this section we need the following hypotheses:

H1. There exists a Kamke function $\omega(\cdot,\cdot)$ such that $\chi(F(t,A)) \le \omega(t, \chi(A))$ for every bounded $A \subset D$ and a.e. $t \in I$. Here

$$\chi(A) = \inf\{r > 0 : \ A \text{ can be covered by finitely many balls of radius } \le r\}$$

is the Hausdorff measure of non-compactness.

H2. There exists a null set $\mathcal{N} \subset I$ such that $F(t,x) \bigcap T_D(x) \neq \emptyset$ for every $t \in I \setminus \mathcal{N}$ and every $x \in D$.

B. $|F(t,x)| \le p(t) w(|x|)$ for all $x \in D$ and a.e. $t \in I$, where $p(\cdot) \in L_1(I, \mathbb{R}^+)$ and $w : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, satisfying $\displaystyle\int_{I_i} p(s)\, ds < \int_{N_i}^{\infty} \frac{du}{w(u)}$, $i = 0, i, \dots, m$, where $N_0 = |x_0|$ and $N_i = \displaystyle\max_{x \in (M_{i-1}\mathbb{B}) \bigcap D} |S_i(x)|$. Here $M_{i-1} = \Gamma_I^{-1}\left(\displaystyle\int_{I_{i-1}} p(s)\, ds \right)$ and $\Gamma_L(z) = \displaystyle\int_{N_l}^{z} \frac{du}{w(u)}$.

The following theorem is proved in [3] (theorem 3.1).

Theorem 1. *For any I_i there exists a constant M_i such that $|x(t)| \le M_i$ on I_i for every (possible) solution $x(\cdot)$ of (1) – (2).*

Remark 2. *Given $\varepsilon > 0$. Using obvious modifications of the proof given in [3] one can show that for every $i = 0, 1, \dots, m$ there exist constants M_i and integrable functions $\lambda_i(\cdot)$ such that $|x(\cdot)| \le M_i$ and $|\overline{co}\ F(t,, x + \varepsilon\mathbb{B})| \le \lambda_i(t)$*

Lemma 3. *(Theorem 2 of [7]). Let X, Z be two Banach spaces, let $\Omega \subset I \times X$ be nonempty and let $M > 0$. Then any closed valued LSC multifunction from Ω into Z admits a Γ^M-continuous selection.*

Denote $I_i := [t_i, t_{i+1})$. The existence of solutions under hypotheses **H1**, **H2**, **B** is proved. We will follow (with essential modifications) the proofs given in [10] for differential inclusions without impulses.

Theorem 4. *If $F(\cdot,\cdot)$ is almost USC with closed convex values, then under* **H1**, **H2** *and* **B** *the differential inclusion (1) – (2) has a solution.*

Proof. Assume first that $F(\cdot,\cdot)$ is uniformly bounded with a constant $M > 0$. We will prove the existence of solution $x(\cdot)$ on $I_i = [t_i, t_{i+1}]$ assuming that the existence on $[0, t_i]$ is already proved. Fix $\varepsilon > 0$. There exists a set $I_\varepsilon \subset I_i$ wit Lebesgue measure $meas(I_\varepsilon) > t_{i+1} - t_i - \varepsilon$ such that $F(\cdot,\cdot)$ is USC on $I_\varepsilon \times E$. One can suppose also without loss of generality that $\omega(\cdot,\cdot)$ is (uniformly) continuous on $I_\varepsilon \times [0, 2M]$. Denote $x_i = x(t_i + 0)$. Then the problem

$$\dot{x}(t) \in F(t, x(t)),\ x(t_i) = x_i, \tag{3}$$

on the interval $[t_i, t_{i+1}]$ is a Cauchy problem without impulses. The proof of the existence of solutions follows [10] and will be given without full details.

Due to the *tangential condition* one can find $f_i \in F(t_i, x_i)$ and sequences $h_n \to 0^+$ $y_n \to 0$ such that $x_i + h_n(f_i - y_n) \in D$. Let $\delta > 0$ be such that $|\omega(t, s) - \omega(\tau, \xi)| < \varepsilon$ when $|t - \tau| < \delta$ $(t, \tau \in I_\varepsilon$ and $|s - \xi| < 2M\delta$. We take $h_n < \delta$, $t_i + h_n \le t_{i+1}$ and define $v(t) = x_i + (t - t_i)(f_i - y_n)$ on $[t_i, t_i + h_n]$. Obviously $v(t_i + h_n) \in D$. If $t_i \notin I_\varepsilon$ then we choose h_n such that $[t_i, t_i + h_n) \bigcap I_\varepsilon = \emptyset$. Using the same fashion and with trivial application of Zorn lemma one can construct an approximate solution $v(\cdot)$ on $[t_i, t_{i+1}]$.

Given a sequence $\{\varepsilon_i\}_{i=1}^\infty$ monotone decreasing to zero, we construct a sequence of approximate solutions $v_n(\cdot)$. Their derivatives $\dot{v}_n(\cdot)$ are strongly measurable and hence almost separable valued. Therefore there exists a full measure set on which $\dot{v}_n(t)$ are in separable subspace X_0. Therefore we can assume without loss of generality that $v_n(t)$ are in X_0 (it is true if $x_i \in X_0$). Define $B(t) = \chi\left(\bigcup_{n=1}^\infty \{v_n(t)\}\right)$.

From Proposition 9.3 of [10] we know that

$$\chi\left(\left\{\int_t^{t+h} \dot{v}_k(t): \ k \geq 1\right\}\right) dt \leq \int_t^{t+h} \chi\left(\{\dot{v}_k(t): \ k \geq 1\}\right) dt.$$

As we have constructed the sequence of the approximate solutions $\{v_n(\cdot)\}_{n=1}^{\infty}$ given $\varepsilon > 0$ one has $\dot{B}(t) \leq \omega(t, B(t)) + \varepsilon$ on a set I_ε with measure greater than $t_{i+1} - t_i - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, one has that $B(t_i) = 0$ and $\dot{B}(t) \leq \omega(t, B(t))$ for a.a. $t \in [t_i, t_{i+1})$.

Due to Arzela theorem the sequence $\{v_n(\cdot)\}_{n=1}^{\infty}$ is $C(I_i, E)$ precompact. Hence passing to subsequences $v_n(t) \to x(t)$ uniformly on $[t_i, t_{i+1}]$. It is straightforward to prove that $x(\cdot)$ is a solution of (1)–(2) also on $[0, t_{i+1}]$.

Now the assumption that $F(\cdot,\cdot)$ is uniformly bounded will be dispensed with. Due to Theorem 1 there exists a constant M_i such that every (possible) solution $x(\cdot)$ of (1)–(2) satisfies $|x(t)| \leq M_i$ on I_i. We redefine

$$G(t,x) = \begin{cases} F(t,x) & (t,x) \in I_i \times D \text{ and } |x| \leq M_i \\ F\left(t, \dfrac{M_i x}{|x|}\right) & (t,x) \in I_i \times D \text{ and } |x| > M_i. \end{cases}$$

Obviously (1)–(2) preserves the solution set and all other conditions when $F(\cdot,\cdot)$ is replaced by $G(\cdot,\cdot)$. So one can assume without loss of generality that F is bounded by Lebesgue integrable function $\lambda(\cdot)$. Next one can modify G on every $I_i \times D$ to obtain $|G(t,x)| \leq 1$ without destroying the other hypotheses. We will follow [6]. Namely define $\varphi(t) = \max\{1, \lambda(t)\} > 0$. The map $t \to \int_{t_i}^t \varphi(s)\, ds$ is continuous and strongly monotone increasing, i.e. invertible. Let $\Phi(\cdot)$ be its inverse, i.e. $\Phi\left(\int_0^t \varphi(s)\, ds\right) = t$, define $\tilde{F}(t,x) = \dfrac{1}{\varphi(\Phi(t))} F(\Phi(t), x)$ for $(t,x) \in I \times D$. Evidently $\tilde{F}$ satisfies all the conditions mentioned above with $\lambda(t) \equiv 1$. Moreover the set of trajectories, as curves in the phase space, is preserved. □

The following corollary is trivial consequence of Theorem 4 and Lemma 3.

Corollary 5. *Let $F(\cdot,\cdot)$ be almost LSC with closed values. If it satisfies* **H1**, **B** *and* $F(t,x) \subset T_D(x)$ for a.e. $t \in I$ and every $x \in D$ *then the differential inclusion (1) – (2) has a solution.*

Proof. Since $F(\cdot,\cdot)$ is almost LSC, one has that there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of pairwise disjoint compacts $J_n \subset I$ such that $F(\cdot,\cdot)$ is LSC on $J_n \times D$ for every n. Furthermore, its union is of full measure. Without loss of generality one can assume that $|F(t,x)| \leq M_k$ on $J_k \times D$. Consequently on $J_k \times D$ there exist Γ^{M_k+1}–continuous selection $f_k(t,x) \in F(t,x)$. We let $G_k(t,x) = \bigcap_{\varepsilon>0} \overline{co}\ f_k([t-\varepsilon, t+\varepsilon] \bigcap J_k, x + \varepsilon\mathbb{B} \bigcap D)$.

Define $G(t,x) = G_k(t,x)$ for $(t,x) \in J_k \times D$, $k = 1, 2, \ldots$. Obviously the so defined multifunction $G(\cdot,\cdot)$ is almost USC with nonempty convex compact values. Further G satisfies the *compactness* and the *tangential* conditions. Due to Theorem 4 the Cauchy problem (1)–(2) with F replaced by G has a solution $x(\cdot)$. As it is shown in [6] (theorem 4.1) and in [10] (lemma 6.1) $x(\cdot)$ is also a solution to

$$\dot{x}(t) = f_n(t, x(t)),\ x(0) = x_0,\ \text{for } (t,x) \in J_n \times D.$$

Indeed let $\tilde{I} \subset I_k$ be such that $\dot{x}(\cdot)$ is continuous on $\tilde{I}$ and $f_k(\cdot,\cdot)$ is Γ^{M_k+1}–continuous on $\tilde{I} \times D$. If t' is its point of density, then there exists $t_i \searrow t$, $\dot{x}(t_i) \to \dot{x}(t)$. Furthermore $(t_n - t, x_n - x) \in \Gamma^{M_k+1}$ and hence $f(t_n, x_n) \to f_k(t,x)$ as $n \to \infty$. Thus $\dot{x}(t) \in f_k(t, x(t))$. Since $f_n(t,x) \in F(t,x)$ one has that $x(\cdot)$ is also a solution of (1)–(2). □

3 Weak and strong invariance. Funnel equation

In this section we study existence of viable solutions when F is defined on $I \times E$. We assume that E is a Hilbert space.

We will said that the multi-map $G(t,x) \subset F(t,x)$ is a sub-multifunction (of F) when $G(\cdot,\cdot)$ is almost USC with nonempty convex compact values.

We assume that:

B1. $F : I \times E \rightrightarrows E$ is almost USC with nonempty convex compact values.

B2. The multifunction F is OSL.

The following lemma is then valid:

Lemma 6. *(Invariance principle) Suppose that* **B**, **B1**, **B2** *hold. The system (1)–(2) is invariant if and only if the system:*

$$\dot{x}(t) \in G(t, x(t)),\ x(0) = x_0 \in D,\ a.e.\ t \in I = [0,1],\ t \neq t_i, \tag{4}$$

$$\Delta x|_{t=t_i} = S_i(x(t_i - 0)),\ i = 1, \ldots, p,\ x(t) \in D, \tag{5}$$

is weakly invariant for every sub-multifunction G.

This Lemma is proved in [14] in case without impulses and in [13] in case of non-fixed time impulses. The proof here is a trivial extension of the these in [13, 14] and it is omitted.

Proposition 7. *Let $G(\cdot,\cdot)$ be a sub-multifunction of F. Under* **B, B1, B2** *the system (4)–(5) is weakly invariant (on $[0,T]$) iff there exists a null set $A_G \subset I$ such that $\sigma(-p, F(t,x)) \geq 0\ \forall\ p \in N_D^P(x)\ \forall\ t \in I \setminus A_G$.*

Proof. Let this proposition be proved in $[0, t_i)$. By virtue of proposition 3 of [14] this solution exists also on $[t_i, t_{i+1})$. The proof is complete. □

As a simple corollary of Lemma 6 and Proposition 7 we obtain the following characterization of invariance:

Theorem 8. *Let us assume that **B, B1, B2** hold. The system (1)–(2) is invariant if and only if for every sub-multifunction $G(t,x)$ of $F(t,x)$ there exists a null set $A_G \subset I$ such that $\sigma(-p, G(t,x)) \geq 0$, $\forall p \in N_D^P(x)$, $\forall x \in D$, $\forall t \in I \setminus A_G$.*

Further we assume that F is almost continuous. We show that the reachable set $\mathcal{S}_t$ of (1) is a unique solution of the so called funnel equation (cf [17]). The funnel equation for (1)–(2) has the form

$$\lim_{h \to 0^+} h^{-1} D_H \left(\Gamma(t+h), \bigcup_{x \in \Gamma(t)} \{x + hF(s,x)\} \right) = 0, \ t \neq t_i. \tag{6}$$

If $t = t_i$ then $\Gamma(t_i + 0) = \bigcup_{x \in \Gamma(t_i - 0)} \{x + S_i(x)\}$.

Using theorems 1 and 2 of [11], and the approach of [21] (with obvious modifications) one can prove:

Theorem 9. *Under* **B, B1, B2** *the reachable set $\mathcal{S}_t$ of (1) is a solution of the funnel equation (6). If moreover $w(t,\cdot)$ is monotone nondecreasing then $t \to \mathcal{S}_t$ is the unique closed solution of (6).*

Proof. The fact that S_t is a solution of the funnel equation (in case without impulses) is given in [21]. Here we will modify the proof of [12] to show the uniqueness.

Let $L(t)$ be other solution of (6). Following the proof of theorem 5.4.4 of [21] we let $r(t) = \mathrm{ex}\,(L(t), \mathcal{S}_t)$. If $x \in S_t$ then $|x-y| \leq r(t)$ for some $y \in L(t)$. If $f_x \in F(t,x)$ then there exists $f_y \in F(t,y)$ such that

$$\langle x - y, f_x - f_y \rangle \leq w(t, |x-y|),$$

because $F(t,\cdot)$ is OSL. Consequently

$$\lim_{h \to 0^+} \frac{r(t+h) - r(t)}{h} \leq \lim_{h \to 0^+} \frac{|x - y - h(f_x - f_y)| - |x-y|}{h} \leq w(t, |x-y|).$$

Since $r(0) = 0$ one has that $r(t) \equiv 0$. Analogously $\mathrm{ex}\Big(L(t), \mathcal{S}_t\Big) \equiv 0$. □

Remark 10. *The funnel equation is considered in [17, 21]. The solution $L(\cdot)$ of (6) is called R–solution of (1). In this light theorem 9 may be reformulated:*

Under **B, B1, B2** *the set valued map $t \to S_t$ is the unique R–solution of (1).*

Notice that the almost continuity of the right-hand side is essential.

Example 11. *(Example 1 of [12]) Consider the following differential inclusion*

$$\dot{x}(t) \in F(x(t)), \ x(0) = 0,$$

where $F(\cdot)$ is defined as follows:

$$F(x) := \begin{cases} -1 & x > 0 \\ [-1, 1] & x = 0 \\ 1 & x < 0 \end{cases}$$

It is easy to see that the right-hand side $F(\cdot)$ is OSL with a constant 0. Furthermore the unique solution of (1) is $x(t) \equiv 0$.

Thus the reachable set $S_t \equiv \{0\}$. However, $\lim_{h \to 0^+} h^{-1} D_H \left(\mathcal{S}_{t+h}, \bigcup_{x \in S_t} x + hF(x) \right) = 1 \ \forall t > 0$, i.e. $\mathcal{S}_t$ does not satisfy the funnel equation.

Recall that $\mathrm{ex}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b|$.

The following proposition, however, is true.

Proposition 12. *Suppose* **B, B2** *hold. If $F(\cdot, \cdot)$ is USC then:*

$$\lim_{h \to 0^+} h^{-1} \mathrm{ex} \left(\mathcal{S}_{t+h}, \bigcup_{x \in \mathcal{S}_t} \{x + hF(s, x)\} \right) = 0.$$

If $F(\cdot, \cdot)$ is LSC then:

$$\lim_{h \to 0^+} h^{-1} \mathrm{ex} \left(\{x + hF(s, x)\}, \Gamma(t + h) \right) = 0, \ \forall x \in \mathcal{S}_t.$$

The jump condition remains the same as in almost continuous case.

The proof in case without impulses is given in [12] and is valid (with obvious modifications) in our case.

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