GROUPOID C^*-ALGEBRAS

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Abstract. The purpose of this paper is to recall the main ingredients of the construction of the C^*-algebra of a groupoid (introduced by Renault in [19]) and to collect some results on the independence of the C^*-algebra on the choice of Haar system.

1   Introduction

The construction of the (full) C^*-algebra of a locally compact Hausdorff groupoid (due to Renault [19]) extends the case of a group. The space of continuous functions with compact support on groupoid is made into a ∗-algebra and endowed with the smallest C^*-norm making its representations continuous. For this ∗-algebra the multiplication is convolution. For defining the convolution on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". Unlike the case of locally compact group, Haar system on groupoid need not exists, and if it does, it will not usually be unique. The purpose of this paper is to recall the main ingredients of the construction of the C^*-algebra of a groupoid (due to Renault [19]) and to collect the results on the independence of the C^*-algebra on the choice of Haar system.

2   Definitions and notation

2.1   The notion of groupoid

We include some definitions that can be found in several places (e.g. [19], [14]). A *groupoid* is a set G endowed with a *product map*

$$(x, y) \mapsto xy \ \left[: G^{(2)} \to G\right]$$

where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map*

$$x \mapsto x^{-1} \ [: G \to G]$$

such that the following conditions hold:

(1) If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy) z = x (yz)$.

(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.

(3) For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx) x^{-1} = z$.

(4) For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1} (xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range* and the *source (domain) maps*. It follows easily from the definition that they have a common image called the *unit space* of G, which is denoted $G^{(0)}$. Its elements are *units* in the sense that $xd(x) = r(x) x = x$.

From a more abstract point of view, a groupoid is simply a category in which every morphism is an isomorphism (that is, invertible).

It is useful to note that a pair (x, y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$).

The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. Also for $u, v \in G^{(0)}$, $G_v^u = G^u \cap G_v$ More generally, given the subsets $A, B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. G_A^A becomes a groupoid (called the *reduction* of G to A) with the unit space A, if we define $\left(G_A^A\right)^{(2)} = G^{(2)} \cap \left(G_A^A \times G_A^A\right)$.

For each unit u, $G_u^u = \{x : r(x) = d(x) = u\}$ is a group, called *isotropy group* at u. The group bundle

$$\{x \in G : \ r(x) = d(x)\}$$

is denoted G', and is called the *isotropy group bundle* of G.

If A and B are subsets of G, one may form the following subsets of G:

$$\begin{aligned} A^{-1} &= \left\{x \in G : \ x^{-1} \in A\right\} \\ AB &= \left\{xy : (x, y) \in G^{(2)} \cap (A \times B)\right\} \end{aligned}$$

The relation $u \sim v$ iff $G_v^u \neq \emptyset$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called *orbits* and the orbit of a unit u is denoted $[u]$. A subset A of $G^{(0)}$ is said *saturated* if it contains the orbits of its elements. For any subset A of $\Gamma^{(0)}$, we denote by $[A]$ the union of the orbits $[u]$ for all $u \in A$. The quotient space for the equivalence relation induced on $G^{(0)}$ ($u \sim v$ iff $G_v^u \neq \emptyset$) is called the *orbit space* of G and denoted $G^{(0)}/G$. The graph of this equivalence relation will be denoted in this paper by

$$R = \{(r(x), d(x)), \ x \in G\}$$

A groupoid is said *transitive* if and only if it has a single orbit, or equivalently if the map $\theta : G \to G^{(0)} \times G^{(0)}$, defined by

$$\theta(x) = (r(x), d(x)) \text{ for all } x \in G,$$

is surjective. Generally, for each orbit $[u]$ of a groupoid G, the reduction of G to $[u]$, $G|_{[u]}$, is a transitive groupoid called *transitivity component* of G. It is easy to see that (algebraically) G is the disjoint union of its transitivity components. A groupoid is said *principal* if the map θ (defined above) is injective.

By a *homomorphism of groupoids* we mean a map $\varphi : G \to \Gamma$ (with G, Γ groupoids) satisfying the following condition:

$$\text{if } (x, y) \in G^{(2)}, \text{ then } (\varphi(x), \varphi(y)) \in \Gamma^{(2)} \text{ and } \varphi(xy) = \varphi(x)\varphi(y)$$

It follows that $\varphi(x^{-1}) = (\varphi(x))^{-1}$ and $\varphi(G^{(0)}) \subset \Gamma^{(0)}$.

2.2 Borel groupoids and topological groupoids

We shall state some conventions and facts about measure theory (see [2], Chapter 3).

By a Borel space $(X, \mathcal{B}(X))$ we mean a space X, together with a σ-algebra $\mathcal{B}(X)$ of subsets of X, called Borel sets. A subspace of a Borel space $(X, \mathcal{B}(X))$ is a subset $S \subset X$ endowed with the relative Borel structure, namely the σ-algebra of all subsets of S of the form $S \cap E$, where E is a Borel subset of X. $(X, \mathcal{B}(X))$ is called countably separated if there is a sequence $(E_n)_n$ of sets in $\mathcal{B}(X)$ separating the points of X: i.e., for every pair of distinct points of X there is $n \in \mathbf{N}$ such that E_n contains one point but not both.

A function from one Borel space into another is called Borel if the inverse image of every Borel set is Borel. A one-one onto function Borel in both directions is called Borel isomorphism.

The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. $(X, \mathcal{B}(X))$ is called standard if it is Borel isomorphic to a Borel subset of a complete separable metric space. $(X, \mathcal{B}(X))$ is called analytic if it is countably separated and if it is the image of a Borel function from a standard space. The locally compact Hausdorff second countable spaces are analytic.

By a *measure* μ on a Borel space $(X, \mathcal{B}(X))$ we always mean a map $\mu : \mathcal{B}(X) \to \overline{\mathbf{R}}$ which satisfies the following conditions:

1. μ is positive ($\mu(A) \geq 0$ for all $A \in \mathcal{B}(X)$)

2. $\mu(\emptyset) = 0$

3. μ is countable additive (i.e. $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ for all sequences $\{A_n\}_n$ of mutually disjoint sets $A_n \in \mathcal{B}(X)$)

Let $(X, \mathcal{B}(X))$ be a Borel space. By a *finite measure* on X we mean a measure μ with $\mu(X) < \infty$ and by a *probability measure* a measure with value 1 on X. We denote by ε_x the unit point mass at $x \in X$, i.e. the probability measure on $(X, \mathcal{B}(X))$ such $\varepsilon_x(A) = 1$ if $x \in A$ and $\varepsilon_x(A) = 0$ if $x \notin A$ for any $A \in \mathcal{B}(X)$. The measure μ is σ-*finite* if there is a sequence $\{A_n\}_n$ with $A_n \in \mathcal{B}(X)$ for all n, such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$ for all n. A subset of X or a function on X is called μ-measurable (for a σ-finite measure μ) if it is measurable with respect to the completion of μ which is again denoted μ. The complement of a $\mu-$null set (a set A is $\mu-$null if $\mu(A) = 0$) is called $\mu-$*conull*.

If $(X, \mathcal{B}(X))$ and μ is a σ-finite measure on $(X, \mathcal{B}(X))$, then there is a Borel subset X_0 of X such that $\mu(X - X_0) = 0$ and such that X_0 is a standard space in its relative Borel structure. Analytic subsets of a countably separated space are universally measurable (i.e. μ-measurable for all finite measures μ).

The measures μ and λ on a Borel space $(X, \mathcal{B}(X))$ are called *equivalent measures* (and we write $\mu \sim \nu$) if they have the same null sets (i.e. $\mu(A) = 0$ iff $\nu(A) = 0$). Every measure class $[\mu] = \{\nu : \nu \sim \mu\}$ of a σ-finite measure $\mu \neq 0$ contains a probability measure. If $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ are Borel space, $p : X \to Y$ a Borel function and μ a finite measure on $(X, \mathcal{B}(X))$, then by $p_*(\mu)$ we denote the finite measure on $(Y, \mathcal{B}(Y))$ defined by $p_*(\mu)(A) = \mu(p^{-1}(A))$ for all $A \in \mathcal{B}(Y)$, and we call it the *image of μ by p*. We shall not mention explicitly the Borel sets when they result from the context (for instance, in the case of a topological space we shall always consider the σ-algebra generated by the open sets).

If X is a topological space, then by a *Borel measure* μ on X we mean a measure with the property that $\mu(K) < \infty$ for all compact subsets of X. If X is a topological space which is σ-compact (i.e. there is sequence $\{K_n\}_n$ of compact subsets K_n of X such that $X = \bigcup_{n=1}^{\infty} K_n$), then any Borel measure on X is σ-finite. A measure μ on X is called *regular measure* if for each $A \in \mathcal{B}(X)$ (with $\mu(A) < \infty$) and each $\varepsilon > 0$ there are a compact subset K of X and an open subset G of X with $K \subset A \subset G$ such that for all sets $A' \in \mathcal{B}(X)$ with $A' \subset G - K$, we have $\mu(A') < \varepsilon$.

If X is a locally compact Hausdorff space, we denote by $C_c(X)$ the space of complex-valuated continuous functions with compact support on X. A *Radon measure* on X is a linear map $L : C_c(X) \to E$ (where E is a Banach space) which is continuos with respect to the inductive limit topology on $C_c(X)$. If $E = \mathbf{C}$ (the space of complex numbers), then a Radon measure L is called *positive* if $L(f) \geq 0$ for all $f \in C_c(X)$, $f \geq 0$. Any linear map $L : C_c(X) \to \mathbf{C}$ which is positive (i.e. $L(f) \geq 0$ for all $f \in C_c(X)$, $f \geq 0$) is in fact a positive Radon measure. According to Riesz-Kakutani Theorem there is a bijective correspondence between the positive Radon measures on X (i.e. linear positive maps $L : C_c(X) \to \mathbf{C}$) and the Borel regular (positive) measures on X (the bijection is given by $L(f) = \int f(x) \, d\mu(x)$ for all real function $f \in C_c(X)$; in the sequel we shall identify L with μ).

Mostly, in this paper we shall work with locally compact Hausdorff spaces which are σ-compact. These kind of spaces are paracompact and consequently, normal. In particular, every Hausdorff locally compact second countable space is paracompact.

A *Borel groupoid* is a groupoid G such that $G^{(2)}$ is a Borel set in the product structure on $G \times G$, and the functions $(x, y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ and $x \mapsto x^{-1} \ [: G \to G]$ are Borel functions. G is an *analytic groupoid* if its Borel structure is analytic.

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

1. $x \mapsto x^{-1} \ [: G \to G]$ is continuous.

2. $(x, y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

Obviously, if G is a topological groupoid, then the inverse map $x \mapsto x^{-1} \ [: G \to G]$ is a homeomorphism and the maps r and d are continuous. Moreover, the maps r and d are identification maps, since they have the inclusion $G^{(0)} \hookrightarrow G$ as a right inverse. If G is Hausdorff then $G^{(0)}$ is closed in G, being the image of the map $x \mapsto xx^{-1} \ [: G \to G]$ whose square is itself. If u is a unit then $x \mapsto x^{-1}$ is a homeomorphism from G^u to G_u. If $u \sim v$ are two equivalent units and x is such that $r(x) = u$ and $d(x) = v$ then $y \mapsto xy$ is a homeomorphism from G^v to G^u, $y \mapsto yx$ is a homeomorphism from G_u to G_v and $y \mapsto xyx^{-1}$ is an isomorphism of topological groups from G_v^v to G_u^u.

We are concerned with topological groupoids whose topology is Hausdorff and locally compact. We call them *locally compact Hausdorff groupoids*. It was shown in [17] that measured groupoids (in the sense of Definition 2.3./p. 6 [10]) may be assume to have locally compact topologies, with no loss in generality.

A subset A of a locally compact groupoid G is called *r-(relatively) compact* iff $A \cap r^{-1}(K)$ is (relatively) compact for each compact subset K of $G^{(0)}$. Similarly, one may define *d-(relatively) compact* subsets of G. A subset of G which is *r*-(relatively) compact and *d*-(relatively) compact is said *conditionally-(relatively) compact*. If the unit space $G^{(0)}$ is paracompact, then there exists a fundamental system of conditionally-(relatively) compact neighborhoods of $G^{(0)}$ (see the proof of Proposition II.1.9/p.56 [19]). Let us also recall that a groupoid is *proper* iff the map $G \ni x \mapsto (r(x), d(x)) \ \left[: G \to G^{(0)} \times G^{(0)}\right]$ is proper (in the realm of locally compact Hausdorff spaces it means that inverse images of compact sets are compact).

A *locally transitive groupoid* is a topological groupoid G which satisfies the condition that the map r_u is open, where $r_u : G_u \to G^{(0)}$ is defined by

$$r_u(x) = r(x) \ \text{for all } x \in G_u.$$

If G is a locally transitive groupoid, then its orbits $[u] = r_u(G_u)$ are open subsets of $G^{(0)}$ and consequently, each unit u has an open neighborhood $U = [u]$ in $G^{(0)}$

such that for any $v \in U$ there is $x \in G$ such that $r(x) = u$ and $d(x) = v$. Also if G is a locally transitive groupoid, then G is topologically, as well as algebraically, the disjoint union of its transitivity components. According to a result of Muhly, Renault and Williams ([13], p. 7-8), if G is a locally compact Hausdorff transitive groupoid whose topology is second countable, then the map r_u is open for one (and hence for all) $u \in G^{(0)}$. Therefore any locally compact Hausdorff second countable groupoid having open orbits is locally transitive.

A locally compact Hausdorff groupoid G is *r-discrete* if its unit space $G^{(0)}$ is an open subset of G (Definition I.2. 6/p.18 [19]). If G is an r-discrete groupoid, then for any $u \in G^{(0)}$, G^u and G_u are discrete spaces (indeed, let $x \in G^u$ and let $v = d(x)$; since $\{v\} = G^{(0)} \cap G^v$, it follows that $\{v\}$ is an open set in G^v, and since $y \mapsto xy$ is a homeomorphism from G^v to G^u, it follows that $\{x\}$ is open in G^u).

2.3 Examples of structures which fit naturally into the study of groupoids

1. *Groups*: A group G is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the unit element).

2. *Group bundles*: Let $\{G_u\}_{u \in U}$ be a family of groups indexed by a set U. The disjoint union of the family $\{G_u\}_{u \in U}$ is a groupoid. Here, two elements may be compose if and only if they belong to the same group G_u and the inverse of an element $x \in G_u$ is its inverse in the group G_u (therefore $r(x) = d(x) = e_u$ the unit element of G_u). Generally, a groupoid G is group bundle if $d(x) = r(x)$ for each $x \in G$. If the groupoid G is a group bundle, then $G = G'$ (the isotropy group bundle of G).

3. *Spaces.* A space X is a groupoid letting

$$X^{(2)} = diag(X) = \{(x, x), x \in G\}$$

and defining the operations by $xx = x$, and $x^{-1} = x$.

4. *Transformation groups.* Let Γ be a group acting on a set X such that for $x \in X$ and $g \in \Gamma$, xg denotes the transform of x by g. Let $G = X \times \Gamma$, $G^{(2)} = \{((x, g), (y, h)) : y = xg\}$. With the product $(x, g)(xg, h) = (x, gh)$ and the inverse $(x, g)^{-1} = (xg, g^{-1})$ G becomes a groupoid. The unit space of G may be identified with X. If Γ is a locally compact Hausdorff group acting continuous on a locally compact Hausdorff space X, then $X \times \Gamma$, with the product topology, is a locally compact Hausdorff groupoid.

5. *Equivalence relations.* Let $\mathcal{E} \subset X \times X$ be an equivalence relation on the set X. Let $\mathcal{E}^{(2)} = \{((x_1, y_1), (x_2, y_2)) \in \mathcal{E} \times \mathcal{E} : y_1 = x_2\}$. With product $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$, $\mathcal{E}$ is a principal groupoid. $\mathcal{E}^{(0)}$ may be identified

with X. Two extreme cases deserve to be single out. If $\mathcal{E} = X \times X$, then $\mathcal{E}$ is called the *trivial groupoid* on X, while if $\mathcal{E} = diag\,(X)$, then $\mathcal{E}$ is called the *co-trivial groupoid* on X (and may be identified with the groupoid in example 3).

If G is any groupoid, then

$$R = \{(r\,(x)\,, d\,(x))\,, \ x \in G\}$$

is an equivalence relation on $G^{(0)}$. The groupoid defined by this equivalence relation is called the *principal groupoid associated with G*.

Any locally compact principal groupoid can be viewed as an equivalence relation on a locally compact space X having its graph $\mathcal{E} \subset X \times X$ endowed with a locally compact topology compatible with the groupoid structure. This topology can be finer than the product topology induced from $X \times X$. We shall endow the principal groupoid associated with a groupoid G with the quotient topology induced from G by the map

$$\theta : G \to R, \theta\,(x) = (r\,(x)\,, d\,(x))$$

This topology consists of the sets whose inverse images by θ in G are open.

2.4   Haar systems

For developing an algebraic theory of functions on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. Several generalizations of the Haar measure to the setting of groupoids were taken into considerations in the literature (see [27], [23], [10], [9], [19]). We use the definition adopted by Renault in [19]. The analogue of Haar measure in the setting of groupoids is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". More precisely, a (left) Haar system on a locally compact Hausdorff groupoid G is a family of positive Radon measures (or equivalently, Borel regular measures) on G, $\nu = \{\nu^u,\ u \in G^{(0)}\}$, such that

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.

2) For all $f \in C_c\,(G)$,

$$u \mapsto \int f\,(x)\,d\nu^u\,(x) \ \ \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f \in C_c\,(G)$ and all $x \in G$,

$$\int f\,(y)\,d\nu^{r(x)}\,(y) \ = \ \int f\,(xy)\,d\nu^{d(x)}\,(y)$$

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Surveys in Mathematics and its Applications **1** (2006), 71 – 98
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Unlike the case for locally compact groups, Haar systems on groupoids need not exist. Also, when a Haar system does exist, it need not be unique. The continuity assumption 2) has topological consequences for G. It entails that the range map $r : G \to G^{(0)}$, and hence the domain map $d : G \to G^{(0)}$ is an open (Proposition I.4 [26]. We shall see later that (according to a result of Seda [25]) the continuity assumption 2) is essential for construction of the C^*-algebra of G in the sense of Renault [19].

A. K. Seda has established sufficient conditions for the existence of Haar systems. He has proved that if a locally compact Hausdorff groupoid G is locally transitive, then the continuity assumption 2) follows from the left invariance assumption 3) (Theorem 2/p. 430 [24]). Thus he has proved that locally transitive locally compact Hausdorff groupoids admit Haar system. At the opposite case of totally intransitive groupoids, Renault has established necessary and sufficient conditions. More precisely, Renault has proved that a locally compact Hausdorff groupoid G which is a group bundle (a groupoid with the property that $r(x) = d(x)$ for all x) admits a Haar system if and only if r is open (Lemma 1.3/p. 6 [20]). In [5] we have considered a locally compact Hausdorff groupoid G having paracompact unit space and a family of positive Radon measures on G, $\{\nu^u, u \in G^{(0)}\}$, satisfying condition 1) and 3) in the definition of Haar system and for each $f \in C_c(G)$ we have denoted by $F_f : G \to \mathbf{R}$ the map defined by

$$F_f(x) = \int f(y) \, d\nu^{d(x)}(y) - \int f(y) \, d\nu^{r(x)}(y) \quad (\forall) \; x \in G$$

We have established that F_f is a groupoid homomorphism continuous at every unit $u \in G^{(0)}$. We also proved that if G has open range map, if F_f is continuous on G for all $f \in C_c(G)$ and if there is a function $h : G \to [0,1]$, universally measurable on each transitivity component $G|_{[u]}$, with $\nu^u(h) = 1$ for all $u \in G^{(0)}$, then $\{\nu^u, u \in G^{(0)}\}$ is a Haar system (Theorem 4 [5]).This result generalizes Lemma 1.3 [20] (for group bundle $F_f = 0$) and Theorem 2/page 430 [24] (for locally transitive locally compact Hausdorff groupoids, if a groupoid homomorphism F is continuous at every unit, then F is continuous everywhere).

Also for the case of r-discrete groupoids Renault has established necessary and sufficient conditions for the existence of Haar systems. If G is an r-discrete groupoid, then G admits a Haar system if and only if r (and hence d) is a local homeomorphism. A Haar system on an r-discrete groupoid is essentially the counting measures system -each measure in the Haar system is multiple of counting measure on the corresponding fiber. (Lemma I.2.7/p. 18 and Proposition I.2.8/p. 19 [19]).

If $\nu = \{\nu^u, u \in G^{(0)}\}$ is a (left) Haar system on G, then for each $u \in G^{(0)}$ we denote by ν_u the image of ν^u by the inverse map $x \to x^{-1}$:

$$\int f(x) \, d\nu_u(x) = \int f(x^{-1}) \, d\nu(x), \text{ for all } f \in C_c(G)$$

Then $\left\{\nu_u,\, u \in G^{(0)}\right\}$ is a right Haar system on G, that is a family of positive Radon measures on G such that

1) For all $u \in G^{(0)}$, $supp(\nu_u) = G_u$.

2) For all $f \in C_c(G)$,

$$u \mapsto \int f(x)\, d\nu_u(x) \;\; \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f \in C_c(G)$ and all $x \in G$,

$$\int f(y)\, d\nu_{d(x)}(y) \;=\; \int f(yx)\, d\nu_{r(x)}(y)$$

We shall work only with left Haar systems.

Examples

1. If G is locally compact Hausdorff group, then G (as a groupoid) admits an essentially unique (left) Haar system $\{\lambda\}$ where λ is a Haar measure on G.

2. If Γ is a locally compact Hausdorff group acting continuous on a locally compact Hausdorff space X, then $X \times \Gamma$ (as a groupoid) admits a distinguished (left) Haar system $\{\varepsilon_x \times \lambda,\; x \in X\}$ where λ is a Haar measure on Γ and ε_x is the unit point mass at x.

3. If X is a locally compact Hausdorff space and if λ is a positive Radon measure on X with full support (i.e. $supp(\lambda) = X$), then $\{\varepsilon_x \times \lambda,\; x \in X\}$ is a Haar system on $X \times X$ (as a trivial groupoid) where ε_x is the unit point mass at x. Conversely, any Haar system on $X \times X$ may be written in this form (for a positive Radon measure λ).

4. If X is a locally compact Hausdorff space, then $\{\varepsilon_x,\; x \in X\}$ is a Haar system on X (as a co-trivial groupoid, Examples 3 Subsection 2.3).

Let $\nu = \left\{\nu^u,\, u \in G^{(0)}\right\}$ be a Haar system on a locally compact Hausdorff groupoid G.

If μ is a positive Radon measure on $G^{(0)}$, then the measure $\nu^\mu = \int \nu^u d\mu(u)$, defined by

$$\int f(y)\, d\nu^\mu(y) = \int \left(\int f(y)\, d\nu^u(y)\right) d\mu(u), \quad f \in C_c(G)$$

is called the measure on G induced by μ. The image of ν^μ by the inverse map $x \to x^{-1}$ is denoted $(\nu^\mu)^{-1}$. The measure μ is said to be quasi-invariant (with respect to ν) if its induced measure ν^μ is equivalent to its inverse, $(\nu^\mu)^{-1}$. A measure

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Surveys in Mathematics and its Applications **1** (2006), 71 – 98

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belongings to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant. If G is σ-compact and μ is a positive Radon measure on $G^{(0)}$, the we can choose a probability measure λ^μ in the class of ν^μ. Then λ^μ is quasi-invariant, and μ is quasi-invariant if and only if $\mu \sim d_* (\lambda^\mu)$ (Proposition 3.6/p.24 [19]).

If μ is a quasi-invariant measure on $G^{(0)}$ and ν^μ is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\nu^\mu}{d(\nu^\mu)^{-1}}$ is called the modular function of μ.

For a positive Radon measure on $G^{(0)}$ μ , let us denote by $\nu^{2,\mu}$ the measure on $G^{(2)}$ defined by

$$\int f(x,y)\, d\nu^{2,\mu}(x,y) = \int \left(\int \int f(x,y)\, d\nu_u(x)\, d\nu^u(y) \right) d\mu(u), \quad f \in C_c\left(G^{(2)}\right).$$

According to Proposition I.3.3/p.23 [19], if μ is a quasi-invariant measure and if Δ is its modular function then

1. $\Delta(xy) = \Delta(x)\Delta(y)$ for $\nu^{2\mu}$-a.a. $(x,y) \in G^{(2)}$

2. $\Delta\left(x^{-1}\right) = \Delta(x)^{-1}$ for ν^μ-a.a. $x \in G$.

Property 1 means that Δ is a ν^μ-a.e. homomorphism. If the locally compact Hausdorff groupoid G is second countable, then there is a μ-conull Borel subset U of $G^{(0)}$ and a Borel function $\Delta_0 : G \to \mathbf{R}$ such that $\Delta_0 = \Delta$ a.e. and the restriction of Δ_0 to $G|_U$ is a homomorphism (this is a consequence of the proofs of Theorem 5.1 and Lemma 5.2 [16]).

2.5   The decomposition of a Haar system over the principal groupoid

Let us present some results on the structure of the Haar systems, as developed by J. Renault in Section 1 of [20] and also by A. Ramsay and M.E. Walter in Section 2 of [18].

In Section 1 of [20] Jean Renault constructed a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally compact support which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1.

Renault defined $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)\, d\beta_v^v(y)$ as usual). If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$. Renault also defined a 1-cocycle δ on G such that for every $u \in G^{(0)}$, $\delta|_{G_u^u}$ is the modular function for β_u^u. δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G. If the restriction of the range map to G' is an open map, then δ is a continuos function.

Let
$$R = (r,d)(G) = \{(r(x), d(x)), \ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. This R is the image of G under the homomorphism (r,d), so it is a σ-compact groupoid. With this apparatus in place, Renault described a decomposition of the Haar system $\{\nu^u, u \in G^{(0)}\}$ for G over the equivalence relation R (the principal groupoid associated to G). He proved that there is a unique Borel Haar system α for R with the property that

$$\nu^u = \int \beta_t^s d\alpha^u(s,t) \quad \text{for all } u \in G^{(0)}.$$

In Section 2 of [18] A. Ramsay and M.E. Walter proved that

$$\sup_u \alpha^u((r,d)(K)) < \infty, \text{ for all compact } K \subset G$$

For each $u \in G^{(0)}$ the measure α^u is concentrated on $\{u\} \times [u]$. Therefore there is a measure μ^u concentrated on $[u]$ such that $\alpha^u = \varepsilon_u \times \mu^u$, where ε_u is the unit point mass at u. Since $\{\alpha^u, u \in G^{(0)}\}$ is a Haar system, we have $\mu^u = \mu^v$ for all $(u,v) \in R$, and the function

$$u \mapsto \int f(s)\, \mu^u(s)$$

is Borel for all $f \geq 0$ Borel on $G^{(0)}$. For each u the measure μ^u is quasi-invariant (see Section 2 of [18]). Therefore μ^u is equivalent to $d_*(v^u)$ [16, Lemma 4.5/p. 277].

If η is a quasi-invariant measure for $\{\nu^u, u \in G^{(0)}\}$, then η is a quasi-invariant measure for $\{\alpha^u, u \in G^{(0)}\}$. Also if Δ_R is the modular function associated to $\{\alpha^u, u \in G^{(0)}\}$ and η, then $\Delta = \delta\Delta_R \circ (r,d)$ can serve as the modular function associated to $\{\nu^u, u \in G^{(0)}\}$ and η.

Since $\mu^u = \mu^v$ for all $(u,v) \in R$, the system of measures $\{\mu^u\}_u$ may be indexed by the elements of the orbit space $G^{(0)}/G$.

Definition 1. *We shall call the pair of systems of measures*

$$\left(\{\beta_v^u\}_{(u,v) \in R}, \ \{\mu^{\dot u}\}_{\dot u \in G^{(0)}/G} \right)$$

(described above) the decomposition of the Haar system $\{\nu^u, u \in G^{(0)}\}$ over the principal groupoid associated to G. Also we shall call δ the 1-cocycle associated to the decomposition.

Let us note that up to trivial changes in normalization, the system of measures $\{\beta_v^u\}$ and the 1-cocycle in the preceding definition are unique. They do not depend on the Haar system, but only on the continuous function F_0.

Another decomposition of the Haar system $\{\nu^u, u \in G^{(0)}\}$ can be founded in [6] for the particular case when R is closed in the product topology coming from $G^{(0)} \times G^{(0)}$. That decomposition starts from the further decomposition of a Haar system established in [10].

2.6   Groupoid representations

While groups are represented on Hilbert spaces, groupoids are represented on Hilbert bundles. Let us recall the notion of Borel Hilbert bundle.

Let $\mathcal{H} = \{\mathcal{H}(s)\}_{s \in S}$ be a family of Hilbert spaces indexed by a set S. Let us form the disjoint union

$$S * \mathcal{H} = \{(s, \xi) : \xi \in \mathcal{H}(s)\},$$

and let $p : S * \mathcal{H} \to S$ be the natural projection, $p(s, \xi) = s$. A pair $(S * \mathcal{H}, p)$ is called *Hilbert bundle* over S. For each $s \in S$, the space $\mathcal{H}(s)$, which can be identified with $p^{-1}(\{s\}) = \{s\} \times \mathcal{H}(s)$, is called the fibre over s. A *section of the bundle* is a function $f : S \to S * \mathcal{H}$ such that $p(f(s)) = s$ for all $s \in S$. Given a section f, we may write $f(s) = \left(s, \hat{f}(s)\right)$, for a uniquely determined element

$$\hat{f} \in \prod_{s \in S} \mathcal{H}(s) = \left\{\phi : S \to \bigcup_{s \in S} \mathcal{H}(s), \, \phi(s) \in \mathcal{H}(s) \text{ for all } s\right\},$$

and given an element $\hat{f} \in \prod_{s \in S} \mathcal{H}(s)$ we may define a section $f(s) = \left(s, \hat{f}(s)\right)$. Because of this link between sections of $S * \mathcal{H}$ and elements of $\prod_{s \in S} \mathcal{H}(s)$ we shall often abuse notation and write $f(s)$ instead of $\hat{f}(s)$. A *Borel Hilbert bundle* is a Hilbert bundle $(S * \mathcal{H}, p)$ where $S * \mathcal{H}$ is endowed with a Borel structure such that the following axioms are satisfied:

1. A subset E is Borel if and only if $p^{-1}(E)$ is Borel.

2. There is a sequence $\{f_n\}_n$ of sections, called a *fundamental sequence*, such that

   a) each function $\tilde{f}_n : S * \mathcal{H} \to \mathbf{C}$, defined by $\tilde{f}_n(s, \xi) = (f_n(s), \xi)_{\mathcal{H}(s)}$, is Borel.

   b) for each pair of fundamental sections, f_n and f_m, the function

   $$s \to (f_n(s), f_m(s))_{\mathcal{H}(s)}$$

   is Borel.

   c) the functions $\left\{\tilde{f}_n\right\}_n$ and p separate the points of $S * \mathcal{H}$.

If the Borel structure on $S * \mathcal{H}$ is analytic then the Hilbert bundle $(S * \mathcal{H}, p)$ is called analytic Borel Hilbert bundle and if the Borel structure on $S * \mathcal{H}$ is standard then the Hilbert bundle $(S * \mathcal{H}, p)$ is called standard Borel Hilbert.

If $(S * \mathcal{H}, p)$ is a Borel Hilbert bundle with the fundamental sequence $\{f_n\}_n$, then a section $f : S \to S * \mathcal{H}$ is Borel if and only if

$$s \mapsto \langle f(s), f_n(s) \rangle_{\mathcal{H}(s)} \quad [: S \to \mathbf{C}]$$

is Borel for each n. We denote by $B\left(S * \mathcal{H}\right)$ the space of all Borel section of $\left(S * \mathcal{H}, p\right)$. If μ is a measure on S, then we denote

$$\int_S^\oplus \mathcal{H}\left(s\right) d\mu\left(s\right) = \left\{ f \in B\left(S * \mathcal{H}\right) : \int \|f\left(s\right)\|_{\mathcal{H}(s)}^2 d\mu\left(s\right) < \infty \right\}$$

and we call this space the *direct integral* of $S * \mathcal{H}$ or the *space of square integrable sections* of $S * \mathcal{H}$ with respect to μ. Obviously, $\int_S^\oplus \mathcal{H}\left(s\right) d\mu\left(s\right)$ is a Hilbert space:

$$\langle f, g \rangle = \int \langle f\left(s\right), g\left(s\right) \rangle_{\mathcal{H}(s)} d\mu\left(s\right).$$

Let G be a locally compact Hausdorff groupoid and $\{\nu^u, \ u \in G^{(0)}\}$ be a Haar system on G. Let $G^{(0)} * \mathcal{H}$ be a Borel Hilbert bundle. We write $Iso\left(G^{(0)} * \mathcal{H}\right)$ for

$$\{(u, L, v) \mid \ L : \mathcal{H}(v) \to \mathcal{H}(u) \ \text{is a Hilbert space isomorphism}\}$$

endowed with the weakest Borel structure so that the maps

$$(u, L, v) \to \left(L f_n\left(v\right), f_m\left(u\right)\right)$$

are Borel for every n and m, where $\left(f_n\right)_n$ is fundamental sequence for $G^{(0)} * \mathcal{H}$.

$Iso\left(G^{(0)} * \mathcal{H}\right)$ is a groupoid in the operations:

$$
\begin{aligned}
(u, L_1, v)\,(v, L_2, w) &= (u, L_1 L_2, w) \\
(u, L, v)^{-1} &= \left(v, L^{-1}, u\right)
\end{aligned}
$$

A unitary representation of G (Definition 3.20/p.68 [14]) consists of a triple $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$ where μ is a quasi-invariant measure on $G^{(0)}$, $G^{(0)} * \mathcal{H}$ is a Borel Hilbert bundle, and L is a Borel map

$$L : G|_U \to Iso\left(G^{(0)} * \mathcal{H}|_U\right)$$

where U is a μ-conull subset of $G^{(0)}$ and $G^{(0)} * \mathcal{H}|_U$ is the restriction of $G^{(0)} * \mathcal{H}$ to U, such that

1. $L\left(x\right) = \left(d\left(x\right), \hat{L}\left(x\right), r\left(x\right)\right)$ and $\hat{L}\left(x\right) : \mathcal{H}(d\left(x\right)) \to \mathcal{H}(r\left(x\right))$ is a Hilbert space isomorphism for $\nu^\mu = \int \nu^u d\mu\left(u\right)$-a.a. $x \in G|_U$.

2. $\hat{L}\left(u\right) = I_u$, the identity operator on $\mathcal{H}(u)$, for μ-a.a. $u \in U$.

3. $\hat{L}\left(x\right) \hat{L}\left(y\right) = \hat{L}\left(xy\right)$ for $\nu^{2\mu} = \int \left(\nu^u \times \nu_u\right) d\mu\left(u\right)$-a.e. $\left(x, y\right) \in G^{(2)}$.

4. $\hat{L}\left(x^{-1}\right) = \hat{L}\left(x\right)^{-1}$ for $\nu^\mu = \int \nu^u d\mu\left(u\right)$-a.a. x.

In other words L is a.e. groupoid homomorphism from G to $Iso\left(G^{(0)} * \mathcal{H}\right)$.

If $(G^{(0)} * \mathcal{H}, L, \mu)$ is a representation of the groupoid G, we abuse notation and write $L(x)$ instead of $\hat{L}(x)$ $\left(L(x) = \left(d(x), \hat{L}(x), r(x)\right)\right)$. For any representation $(G^{(0)} * \mathcal{H}, L, \mu)$ of a locally compact Hausdorff second countable groupoid G on an analytic Borel Hilbert bundle $G^{(0)} * \mathcal{H}$, there is a Borel homomorphism $L_0 : G \to Iso\left(G^{(0)} * \mathcal{H}\right)$ that preserves the unit space $G^{(0)}$ in the sense that

$$L_0(x) = \left(d(x), \hat{L}_0(x), r(x)\right),$$

where $\hat{L}_0(x) : \mathcal{H}(d(x)) \to \mathcal{H}(r(x))$ is a Hilbert space isomorphism, such that L_0 agrees with L a.e.-$\nu^\mu = \int \nu^u d\mu(u)$.

3 Groupoid C*-algebras

3.1 Convolution algebras

Let $\nu = \left\{\nu^u, u \in G^{(0)}\right\}$ be a Haar system on the locally compact Hausdorff groupoid G. For $f, g \in C_c(G)$ the *convolution* is defined by:

$$\begin{aligned}
f * g(x) &= \int f(xy) g\left(y^{-1}\right) d\nu^{d(x)}(y) \\
&= \int f(y) g\left(y^{-1}x\right) d\nu^{r(x)}(y)
\end{aligned}$$

and the *involution* by

$$f^*(x) = \overline{f\left(x^{-1}\right)}.$$

Under these operations, $C_c(G)$ becomes a topological $*$-algebra (Proposition II.1.1/p. 48 [19]).

A.K.Seda has proved in [25] that if we consider a system of positive Radon measures $\left\{\nu^u, u \in G^{(0)}\right\}$ satisfying the conditions 1) and 3) ("left invariance") in the definition of the Haar system and if we require that $f * g \in C_c(G)$ for every $f, g \in C_c(G)$, then $\left\{\nu^u, u \in G^{(0)}\right\}$ should satisfy condition 2) ("continuity") in the definition of the Haar system.

Let us note that the involutive algebraic structure on $C_c(G)$ defined above depends on the Haar system $\nu = \left\{\nu^u, u \in G^{(0)}\right\}$. When it will be necessary to emphasis the role of ν in this structure, we shall write $C_c(G, \nu)$.

Muhly, Renault and Williams proved that if G has paracompact unit space, then $C_c(G, \nu)$ has a two-sided approximate identity with respect to the inductive limit topology (Corollary 2.11/p.12 [13]). In order to prove that they constructed for each triple (K, U, ε) consisting of a compact subset K of $G^{(0)}$, an open d-relatively compact neighborhood U of $G^{(0)}$ and a positive number a nonnegative function $e_{K,U,\varepsilon} \in C_c(G)$ such that

****

Surveys in Mathematics and its Applications **1** (2006), 71 – 98
http://www.utgjiu.ro/math/sma

1. $e_{K,U,\varepsilon} = e_{K,U,\varepsilon}^*$

2. $\mathrm{supp}(e_{K,U,\varepsilon}) \subset U$

3. $\left| \int e_{K,U,\varepsilon}(x) \, d\nu^u(x) - 1 \right| \leq \varepsilon$ for all $u \in K$.

Then the family $\{e_{K,U,\varepsilon}\}$, where the set $\{(K,U,\varepsilon)\}$ is directed by increasing K, decreasing U, and decreasing ε is a two-sided approximate identity in $C_c(G,\nu)$.

For each $f \in C_c(G)$, let us denote by $\|f\|_I$ the maximum of $\sup\limits_u \int |f(x)| \, d\nu^u(x)$ and $\sup\limits_u \int |f(x)| \, d\nu_u(x)$. A straightforward computation shows that $\|\cdot\|_I$ is a norm on $C_c(G)$ and

$$
\begin{aligned}
\|f\|_I &= \|f^*\|_I \\
\|f * g\|_I &\leq \|f\|_I \, \|g\|_I
\end{aligned}
$$

for all $f, g \in C_c(G)$.

Let us denote by $\mathcal{B}_c(G)$ the space of complex-valued Borel bounded functions with compact support on G and $\mathcal{B}_I(G)$ the space of complex-valued Borel functions $f : G \to \mathbf{C}$ with $\|f\|_I < \infty$. Under convolution and involution defined at the beginning of this subsection, $\mathcal{B}_c(G)$ and $\mathcal{B}_I(G)$ become $*$-algebras.

Examples

1. If G is locally compact Hausdorff group, then G (as a groupoid) admits an essentially unique (left) Haar system $\{\lambda\}$ where λ is a Haar measure on G. The convolution of $f, g \in C_c(G)$ (G is seen as a groupoid) is given by:

$$
f * g(x) = \int f(xy) \, g(y^{-1}) \, d\lambda(y) \text{ (usual convolution on } G)
$$

and the involution by

$$
f^*(x) = \overline{f(x^{-1})}.
$$

The involution defined above is slightly different from the usual involution on groups. In fact if Δ is the modular function of the Haar measure λ, then $f \mapsto \Delta^{1/2} f$ is $*$-isomorphism from $C_c(G)$ for G seen as a groupoid to $C_c(G)$ for G seen as a group.

2. If Γ is a locally compact Hausdorff group, endowed with a Haar measure λ, acting continuous on a locally compact Hausdorff space X, $\{\varepsilon_x \times \lambda, \ x \in X\}$ is Haar system on $X \times \Gamma$ (seen as the groupoid described in Examples 4 Subsection

2.3). The convolution of f, $g \in C_c (X \times \Gamma)$ is given by:

$$
\begin{aligned}
f * g (x, \gamma) &= \int f \left((x, \gamma) (y, \gamma')\right) g \left(\left(y, \gamma'\right)^{-1}\right) d (\varepsilon_{x\gamma} \times \lambda) (y, \gamma') \\
&= \int f (x, \gamma\gamma') g (x\gamma\gamma', \gamma'^{-1}) d\lambda (\gamma') \\
&= \int f (x, \gamma') g (x\gamma', \gamma'^{-1}\gamma) d\lambda (\gamma')
\end{aligned}
$$

and the involution by

$$
f^* (x, \gamma) = \overline{f (x\gamma, \gamma^{-1})}.
$$

3. If X is a locally compact Hausdorff space and if λ is a positive Radon measure on X with full support (i.e. $supp (\lambda) = X$), then $\{\varepsilon_x \times \lambda, \ x \in X\}$ is a Haar system on $X \times X$ (as a trivial groupoid). The convolution of f, $g \in C_c (X \times X)$ is given by:

$$
\begin{aligned}
f * g (x, y) &= \int f \left((x, y) (t, z)\right) g \left((t, z)^{-1}\right) d (\varepsilon_y \times \lambda) (t, z) \\
&= \int f (x, z) g (z, y) d\lambda (z)
\end{aligned}
$$

and the involution by

$$
f^* (x, y) = \overline{f (y, x)}.
$$

4. If X is a locally compact Hausdorff space, then $\{\varepsilon_x, \ x \in X\}$ is a Haar system on X (seen as a co-trivial groupoid identified with the groupoid described in Examples 3 Subsection 2.3). The convolution of f, $g \in C_c (G)$ is given by:

$$
\begin{aligned}
f * g (x) &= \int f (xy) g \left(y^{-1}\right) d\varepsilon_x (y) \\
&= f (xx) g \left(x^{-1}\right) \\
&= f (x) g (x)
\end{aligned}
$$

and the involution by

$$
f^* (x) = \overline{f (x)}.
$$

Let $\nu = \left\{\nu^u, \ u \in G^{(0)}\right\}$ be a Haar system on the locally compact Hausdorff groupoid G. A *representation of* $C_c (G, \nu)$ is a $*$-homomorphism L from the topological $*$-algebra $C_c (G, \nu)$ into $\mathcal{B} (H)$, for some Hilbert space H, that is continuous with respect to the inductive limit topology on $C_c (G)$ and the weak operator topology on $\mathcal{B} (H)$. The representation L is said *non-degenerate* if the linear span of

$$
\{L (g) \xi : \ g \in C_c (\Gamma), \xi \in H\}
$$

is dense in H.

Every representation $\left(\mu, G^{(0)} * \mathcal{H}, L\right)$ [14, Definition 3.20/p.68] of G can be integrated into a non-degenerate representation, still denoted by L, of $C_c\left(G, \nu\right)$. The relation between the two representation is:

$$\langle L\left(f\right)\xi_1, \xi_2\rangle = \int f\left(x\right)\langle L\left(x\right)\xi_1\left(d\left(x\right)\right), \xi_2\left(r\left(x\right)\right)\rangle \Delta^{-\frac{1}{2}}\left(x\right)d\nu^u\left(x\right)d\mu\left(u\right)$$

where $f \in C_c\left(G\right)$, $\xi_1, \xi_2 \in \int_{G^{(0)}}^{\oplus} \mathcal{H}\left(u\right)d\mu\left(u\right)$ and Δ is the modular function of μ.

Conversely, if G is second countable then, every non-degenerate representation of $C_c\left(G, \nu\right)$ is obtained in this fashion (see Section 3 [11], Proposition II.1.17/p. 52[19], or Proposition 3.23/p. 70, Theorem 3.29/p. 74 [14]).

Using the correspondence of groupoid representations and representations of $C_c\left(G, \nu\right)$, it can be proved that if $f \in C_c\left(G\right)$ and L is a representation of $C_c\left(G, \nu\right)$, then

$$\|L\left(f\right)\| \leq \|f\|_I.$$

3.2 The full and the reduced C^*-algebras associated to a locally compact Hausdorff groupoid

In this subsection we recall the construction of the full and reduced C^*-algebras associated to a locally compact second countable groupoid (due to Renault [19]).

Let $\nu = \left\{\nu^u, u \in G^{(0)}\right\}$ be a Haar system on the locally compact Hausdorff second countable groupoid G. For each $f \in C_c\left(G\right)$ let us define the *full norm* of f by

$$\|f\|_{full} = \sup_L \|L\left(f\right)\|$$

where L runs over all non-degenerate representations of $C_c\left(G, \nu\right)$.

Let us single out a special class of representations of $C_c\left(G, \nu\right)$ that serve as analogues of the regular representation of a group. If μ quasi-invariant measure and $\nu^\mu = \int \nu^u d\mu\left(u\right)$, then for every $f \in C_c\left(G\right)$, we denote by $Ind\mu\left(f\right)$ the operator on $L^2\left(G, \left(\nu^\mu\right)^{-1}\right)$ defined by formula

$$Ind\mu\left(f\right)\xi\left(x\right) = f * \xi\left(x\right)$$

and for every $u \in G^{(0)}$ and $f \in C_c\left(G\right)$ we denote by $Ind_u\left(f\right)$ the operator on $L^2\left(G, \nu_u\right)$ defined by formula

$$Ind_u\left(f\right)\xi\left(x\right) = f * \xi\left(x\right) = \int f\left(xy\right)\xi\left(y^{-1}\right)d\nu^u\left(y\right)$$

Then $f \mapsto Ind\mu\left(f\right)$ and $f \mapsto Ind_u\left(f\right)$ $\left(u \in G^{(0)}\right)$ are non-degenerate representations of $C_c\left(G, \nu\right)$.

For each $f \in C_c(G)$ let us define

$$\|f\|_{red} = \sup_{u \in G^{(0)}} \|Ind_u(f)\|$$

(the *reduced norm* of f)

If μ is a quasi invariant measure with $supp(\mu) = G^{(0)}$, then

$$\|f\|_{red} = \|Ind\mu(f)\|.$$

([14] p.50).

It is not hard to see that $\|\cdot\|_{full}$ and $\|\cdot\|_{red}$ are C^*-norms and $\|\cdot\|_{full} \geq \|\cdot\|_{red}$.

The full C^*algebra $C^*(G, \nu)$ ([19]) and the reduced C^*algebra $C^*_{red}(G, v)$ are defined respectively as the completion of the algebra $C_c(G, \nu)$ for the full norm $\|\cdot\|_{full}$, and the reduced norm $\|\cdot\|_{red}$. According to Proposition 6.1.8/p.146 [1], if (G, ν) is measurewise amenable (Definition 3.3.1/p. 82 [1]), then $C^*_{full}(G, v) = C^*_{red}(G, v)$.

Examples

1. If G is locally compact Hausdorff group, endowed with a Haar measure λ, and if Δ is the modular function of λ, then $f \mapsto \Delta^{1/2} f$ [: $C_c(G) \to C_c(G)$] can be extended to $*$-isomorphism from $C^*(G)$ (respectively $C^*_{red}(G)$) for G seen as a groupoid to $C^*(G)$(respectively $C^*_{red}(G)$) for G seen as a group G.

2. If X is a locally compact Hausdorff space and if λ is a positive Radon measure on X with full support (i.e. $supp(\lambda) = X$), then $\nu = \{\varepsilon_x \times \lambda, \ x \in X\}$ is a Haar system on $X \times X$ (as a trivial groupoid). Then $C^*_{full}(X \times X, v) = C^*_{red}(X \times X, v) = \mathcal{K}(L^2(X, \lambda))$ (the algebra of compact operators on $L^2(X, \lambda)$) (Proposition 2.37/p.51 and Corollary 3.30/p. 84 [14]).

3. If X is a locally compact Hausdorff space, then $\nu = \{\varepsilon_x, \ x \in X\}$ is a Haar system on X (seen as a co-trivial groupoid identified with the groupoid described in Examples 3 Subsection 2.3). Then $C^*_{full}(X, v) = C^*_{red}(X, v) = C_0(X)$ (the algebra of complex-valued continuous functions on X vanishing at infinity).

4   Morita equivalent groupoids and strongly Morita equivalent C^*-algebras

Definition 2. *Let Γ be a groupoid and X be a set. We say Γ acts (to the left) on X if there is a surjective map $\rho : X \to \Gamma^{(0)}$ (called a momentum map) and a map $(\gamma, x) \mapsto \gamma \cdot x$ from*

$$\Gamma *_\rho X = \{(\gamma, x) : d(\gamma) = \rho(x)\}$$

to X, called (left) action, such that:

1. $\rho(\gamma \cdot x) = r(\gamma)$ *for all* $(\gamma, x) \in \Gamma *_\rho X$.

2. $\rho(x) \cdot x = x$ *for all* $x \in X$.

3. *If* $(\gamma_2, \gamma_1) \in \Gamma^{(2)}$ *and* $(\gamma_1, x) \in \Gamma *_\rho X$, *then* $(\gamma_2 \gamma_1) \cdot x = \gamma_2 \cdot (\gamma_1 \cdot x)$.

If Γ *is a topological groupoid and* X *is a topological space, then we say that a left action is continuous if the map* ρ *is* continuous and open *and the map* $(\gamma, x) \mapsto \gamma \cdot x$ *is continuous, where* $\Gamma *_\rho X$ *is endowed with the relative product topology coming from* $\Gamma \times X$.

The action is called *free* if $(\gamma, x) \in \Gamma *_\rho X$ and $\gamma \cdot x = x$ implies $\gamma \in \Gamma^{(0)}$.

The continuous action is called *proper* if the map $(\gamma, x) \mapsto (\gamma \cdot x, x)$ from $\Gamma *_\rho X$ to $X \times X$ is proper (i.e. the inverse image of each compact subset of $X \times X$ is a compact subset of $\Gamma *_\rho X$).

In the same manner, we define a *right action* of Γ on X, using a continuous map $\sigma : X \to \Gamma^{(0)}$ and a map $(x, \gamma) \mapsto x \cdot \gamma$ from

$$X *_\sigma \Gamma = \{(x, \gamma) : \sigma(x) = r(\gamma)\}$$

to X.

The simplest example of proper and free action is the case when the locally compact Hausdorff groupoid Γ acts upon itself by either right or left translation (multiplication).

Definition 3. *Let* Γ_1, Γ_2 *be two groupoids and* X *be set. Let us assume that* Γ_1 *acts to the left on* X *with momentum map* $\rho : X \to \Gamma_1^{(0)}$, *and that* Γ_2 *acts to the right on* X *with momentum map* $\sigma : X \to \Gamma_2^{(0)}$. *We say that the actions commute if*

1. $\rho(x \cdot \gamma_2) = \rho(x)$ *for all* $(x, \gamma_2) \in X *_\sigma \Gamma_2$ *and* $\sigma(\gamma_1 \cdot x) = \sigma(x)$ *for all* $(\gamma_1, x) \in \Gamma_1 *_\rho X$.

2. $\gamma_1 \cdot (x \cdot \gamma_2) = (\gamma_1 \cdot x) \cdot \gamma_2$ *for all* $(\gamma_1, x) \in \Gamma_1 *_\rho X$, $(x, \gamma_2) \in X *_\sigma \Gamma_2$.

Definition 4. *Let* Γ_1, Γ_2 *be two locally compact Hausdorff groupoids having open range maps. The locally compact Hausdorff space* X *is said a* (Γ_1, Γ_2)-*Morita equivalence if the following conditions are satisfied:*

1. Γ_1 *acts to the left on* X *with momentum map* $\rho : X \to \Gamma_1^{(0)}$ *and the action is continuous free and proper.*

2. Γ_2 *acts to the right on* X *with momentum map* $\sigma : X \to \Gamma_2^{(0)}$ *and and the action is continuous free and proper*

3. *The actions commute.*

4. *If $\rho(s) = \rho(t)$, then there is $x \in \Gamma_2$ such that $s \cdot x = t$.*

5. *If $\sigma(s) = \sigma(t)$, then there is $\gamma \in \Gamma_1$ such that $\gamma \cdot s = t$.*

The groupoids Γ_1, Γ_2 are called Morita equivalent.

The notion of Morita equivalence defined above is an equivalence relation on locally compact Hausdorff groupoids having open range maps (see [13]).

Examples

1. Any locally compact groupoid Γ having open range map is a (Γ, Γ)-Morita equivalence (Examples 5.33.1 [14])

2. If Γ_1, Γ_2 are locally compact Hausdorff groupoids having open range maps, $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism and if φ is a homeomorphism then Γ_1 is a (Γ_1, Γ_2)-Morita equivalence (Γ_1 acts to the left on Γ_1 by multiplication and Γ_2 acts to the right on Γ_1 by $y \cdot x = y\varphi(x)$)(Examples 5.33.2 [14]).

3. If G is a locally compact Hausdorff *transitive* groupoid which is second countable and u is a unit in $G^{(0)}$ then G^u is a (G, G_u^u)-Morita equivalence (Theorem 2.2A, Theorem 2.2B [13]). More generally, if Γ is a locally compact Hausdorff groupoid, F is a closed subset of $\Gamma^{(0)}$ and if the restrictions of r and d to Γ_F are open, then Γ_F is a $(\Gamma, \Gamma|_F)$-Morita equivalence.

4. If R is an equivalence relation on a locally compact Hausdorff space X, such that R as a subset of $X \times X$ is a closed set, then X implements a Morita equivalence between the groupoid R (see 5 in Subsection 2.3), and the groupoid X/R (see 2 in Subsection 2.3) (Examples 5.33.5 [14]).

Definition 5. *Let A be a C^*-algebra. A pre-Hilbert A-module is a right A-module X (with a compatible $\mathbf{C}$-vector space structure) equipped with a conjugate-bilinear map (liner in the second variable) $\langle \cdot, \cdot \rangle_A : X \times X \to A$ satisfying:*

1. *$\langle x, \ y \cdot a \rangle_A = \langle x, \ y \rangle_A \cdot a$ for all $x, y \in X$, $a \in A$.*

2. *$\langle x, \ y \rangle_A^* = \langle y, \ x \rangle_A$ for all $x, y \in X$.*

3. *$\langle x, x \rangle_A \geq 0$ for all $x \in X$.*

4. *$\langle x, x \rangle_A = 0$ only when $x = 0$.*

The map $\langle \cdot, \cdot \rangle_A$ is called A-valued inner product on X.

A left pre-Hilbert A-module is defined in the same way, except that X is required to be a left A-module, the map $_A \langle \cdot, \cdot \rangle : X \times X \to A$ is required to be linear in the first variable, and the first condition above is replaced by $_A \langle a \cdot x, \ y \rangle = a \cdot _A \langle x, \ y \rangle$ for all $x, y \in X$, $a \in A$ ([15], [21]).

It can be shown that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a norm on X. If X is complete with respect to this norm, it is called a *Hilbert A-module*. If not, all the structure can be extended to its completion to turn it into a Hilbert A-module.

As a normed linear space, a right Hilbert A-module X carries an algebra of bounded linear transformations. In the following we shall denote by $\mathbb{B}(X)$ the collection of all bounded linear maps $T: X \to X$ which are module maps (this means that $T(x \cdot a) = T(x) \cdot a$ for all $x \in X$, $a \in A$) and adjointable (this means that there is another linear bounded operator T^* on X such that $\langle T(x), y \rangle_A = \langle x, T^*(y) \rangle_A$ for all $x, y \in X$). It is easy to see that if X is Hilbert space and $A = \mathbf{C}$ (the space of complex numbers), then $\mathbb{B}(X)$ is the algebra of linear bounded operators on X. The algebra of linear bounded operators on a Hilbert X has a two-sided closed non-trivial ideal $\mathcal{K}(X)$ (the compact operators). The analog of that ideal of compact operators is given in the case of a right Hilbert A-module X by the closed linear span in $\mathbb{B}(X)$ of all the "rank one" transformations on X, i.e. of all transformations of the form $x \otimes y^* : X \to X$ defined by $x \otimes y^*(z) = x \langle y, z \rangle_A$ for all $z \in X$, with $x, y \in X$. The closed linear span in $\mathbb{B}(X)$ of all transformations $x \otimes y^*$ (with $x, y \in X$) is denoted by $\mathbb{K}(X)$, it is called the *imprimitivity algebra* of X and its elements are called *compact operators* on X. If X is a left Hilbert A-module, then *imprimitivity algebra* $\mathbb{K}(X)$ is the closed linear span of all transformations of the form $x \otimes y^* : X \to X$ defined by $x \otimes y^*(z) =_A \langle z, y \rangle x$ for all $z \in X$, with $x, y \in X$.

Proposition 6. *If A is a C^*-algebra and if X is a right Hilbert A-module, then $\mathbb{B}(X)$ and $\mathbb{K}(X)$ are C^*-algebras, with $\mathbb{B}(X)$ equal to the multiplier algebra of $\mathbb{K}(X)$. Further, X becomes a left Hilbert C^*-module over $\mathbb{K}(X)$, where the $\mathbb{K}(X)$-valued inner product is defined by the formula:*

$$_{\mathbb{K}(X)}\langle x, \ y \rangle = x \otimes y^*, \text{ for all } x, y \in X,$$

and $\mathbb{K}(X)$ (X as a left Hilbert C^*-module over $\mathbb{K}(X)$) is naturally isomorphic to A though the formula

$$x \otimes y^* \to \langle x, y \rangle_A.$$

Definition 7. *Let A and B be C^*-algebras. By an (A, B)-equivalence bimodule we mean an A, B-bimodule X equipped with A and B-valued inner products with respect to which X is a right Hilbert B-module and a left Hilbert A-module such that:*

1. $_A\langle x, y \rangle z = x \langle y, z \rangle_B$ *for all $x, y, z \in X$.*

2. $\langle ax, ax \rangle_B \leq \|a\|^2 \langle x, x \rangle_B$ *for all $a \in A$, $x \in X$ and $_A\langle xb, xb \rangle \leq \|b\|_A^2 \langle x, x \rangle$ for all $b \in B$, $x \in X$.*

3. $\langle X, X \rangle_B$ *spans a dense subset of B and $_A\langle X, X \rangle$ spans a dense subset of A.*

****

Surveys in Mathematics and its Applications **1** (2006), 71 – 98
http://www.utgjiu.ro/math/sma

We call A and B *strongly Morita equivalent* if there is an (A, B)-equivalence bimodule.

Strong Morita equivalence is an equivalence relation.

If A is a C^*-algebra and if X is a Hilbert A-module, then A and $\mathbb{K}(X)$ are strongly Morita equivalence.

Proposition 8. *Let A and B be C^*-algebras and X be an (A, B)-equivalence bimodule. Then the map from $\mathbb{K}(X)$ (X as a right Hilbert C^*-module over B) to A defined by formula*

$$_A x \otimes y^* \mapsto \langle x, y \rangle$$

is a C^*-isomorphism. Similarly, the map from $\mathbb{K}(X)$ (X as a left Hilbert C^*-module over A) to A defined by formula

$$x \otimes y^* \mapsto \langle x, y \rangle_B$$

is a C^*-isomorphism.

Thus, two C^*-algebras are strongly Morita equivalent if and only if one can be realized as compact operators of a Hilbert C^*-module over the other.

Theorem 9. *If A and B are two stably isomorphic C^*-algebras (in the sense that $A \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space), then A and B are strongly Morita equivalent. Conversely, if A and B have countable approximate identities, and if A and B are strongly Morita equivalent, then A and B are stably isomorphic. ([3])*

Using the correspondence of groupoid representations and representations of C^*-algebra, Muhly, Renault and Williams proved the following theorem:

Theorem 10. *Let Γ and G be locally compact, second countable, Hausdorff groupoids endowed with the Haar systems λ and ν, respectively. If Γ and G are Morita equivalent, then $C^*(\Gamma, \lambda)$ and $C^*(G, \nu)$ are strongly Morita equivalent. (Theorem 2.8/p. 10 [13]).*

5 Results concerning the independence of the groupoid C^*-algebra of the Haar system

The definition of the groupoid C^*-algebra depends on the choice of the Haar system (the convolution is defined using a Haar system). In the group case, Haar measure is essentially unique, but for groupoids, this is no longer the case. Due to Theorem 10, *different choices of Haar system produces strongly Morita equivalent C^*-algebras.* This still leaves open the question: *are the C^*-algebras associated with two Haar systems $*$-isomorphic.* As Muhly, Renault and Williams proved, this is indeed the case for transitive groupoid:

Theorem 11. *If the locally compact, second countable, Hausdorff groupoid G is transitive, then the (full) C^*-algebra of G is isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$, where H is the isotropy group G_e^e at any unit $e \in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$. (Theorem 3.1/p.16 [13, Theorem 3.1, p. 16]).*

Consequently, the (full) C^*-algebras of a *transitive* locally compact, second countable, Hausdorff groupoid G associated with two Haar systems are $*$-isomorphic. In order to prove that result Muhly, Renault and Williams firstly established that $C^*(G)$ and $C^*(H)$ are strong Morita equivalent via a $(C^*(H), C^*(G))$ -equivalence bimodule module X_1 (because G and H are Morita equivalent groupoids). As a consequence the C^*-algebra of G is the imprimitivity algebra of X_1. Then they needed another $C^*(H)$ module X_2 (isomorphic to X_1) whose imprimitivity algebra is $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$ for a suitable measure μ.

We can obtain the isomorphism between the C^*-algebra of G and $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$ more directly. If we endow $G^{(0)} \times H \times G^{(0)}$ with the product topology, and the operations

$$
\begin{aligned}
(u,x,v)(v,y,w) &= (u,xy,w) \\
(u,x,v)^{-1} &= (v,x,u)
\end{aligned}
$$

then it becomes a locally compact second countable groupoid. The system of measures $\left\{\varepsilon_u \times \mu_e \times \nu, \ u \in G^{(0)}\right\}$, where ε_u is the unit point mass at u, μ_e is the Haar measure on $H = G_e^e$ and ν is a measure of full support on $G^{(0)}$, is Haar system on $G^{(0)} \times H \times G^{(0)}$. It is not hard to prove that the C^*-algebra of $G^{(0)} \times H \times G^{(0)}$ endowed with the Haar system $\left\{\varepsilon_u \times \mu_e \times \nu, \ u \in G^{(0)}\right\}$ is $*$-isomorphic to $C^*(H) \otimes \mathcal{K}\left(L^2(\nu)\right)$. On the other hand since G is a locally compact, second countable, Hausdorff transitive groupoid, the restriction of the domain map to G^e is an open map (see [13]). According to Mackey Lemma (Lemma 1.1 [12]), there is $\sigma : G^{(0)} \to G^e$ a regular Borel cross section of $d : G^e \to G^{(0)}$. This means that $d(\sigma(u)) = u$ for all $u \in G^{(0)}$ and $\sigma(K)$ has compact closure in G^e for each compact set K in $G^{(0)}$. Then $\phi : G \to G^{(0)} \times H \times G^{(0)}$ defined by

$$
\phi(x) = \left(r(x), \sigma(r(x)) x \sigma(d(x))^{-1}, d(x)\right)
$$

is a Borel isomorphism which carries the Haar system of G into a Haar system of $G^{(0)} \times H \times G^{(0)}$ of the form $\left\{\varepsilon_u \times \mu_e \times \nu, \ u \in G^{(0)}\right\}$, where ε_u is the unit point mass at $u \in G^{(0)}$, μ_e is a Haar measure on $H = G_e^e$, and ν is a suitable Radon measure on $G^{(0)}$ with full support:

Proposition 12. *Let G be a locally compact second countable transitive groupoid. Let e be a unit and $\sigma : G^{(0)} \to G^e$ be a regular Borel cross section of $d : G^e \to G^{(0)}$. Then $\phi : G \to G^{(0)} \times G_e^e \times G^{(0)}$ defined by*

$$
\phi(x) = \left(r(x), \sigma(r(x)) x \sigma(d(x))^{-1}, d(x)\right)
$$

is a Borel isomorphism which carries Haar system of G into a Haar system of $G^{(0)} \times G_e^e \times G^{(0)}$ of the form $\{\varepsilon_u \times \mu_e \times \lambda, u \in G^{(0)}\}$, where ε_u is the unit point mass at $u \in G^{(0)}$, μ_e is a Haar measure on G_e^e, and λ is a suitable Radon measure on $G^{(0)}$.

Using the fact that any compactly supported Borel bounded function on a *transitive* groupoid can be viewed as an element of the groupoid C^*- algebra (Proposition 4/p. 82, Proposition 5/p. 86 [4]) and we can prove that the $f \mapsto \phi \circ f$ extends to a $*$-isomorphism from $C^* \left(G^{(0)} \times H \times G^{(0)} \right)$ to $C^* (G)$. Thus $C^* (G)$ is $*$-isomorphic to $C^* (H) \otimes \mathcal{K} \left(L^2 (\nu) \right)$.

A locally compact groupoid G is *proper* if the map $(r, d) : G \to G^{(0)} \times G^{(0)}$ is proper (i.e. the inverse map of each compact subset of $G^{(0)} \times G^{(0)}$ is compact). [1, Definition 2.1.9]. In the sequel by a *groupoid with proper orbit space* we shall mean a groupoid G for which the orbit space is Hausdorff and the map

$$(r, d) : G \to R, (r, d) (x) = (r (x), d (x))$$

is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$.

Applying Lemma 1.1 of [12] to the locally compact second countable spaces $G^{(0)}$ and $G^{(0)}/G$ and to the continuous open surjection $\pi : G^{(0)} \to G^{(0)}/G$, it follows that there is a Borel set F in $G^{(0)}$ such that:

1. F contains exactly one element in each orbit $[u] = \pi^{-1} (\pi (u))$.

2. For each compact subset K of $G^{(0)}$, $F \cap [K] = F \cap \pi^{-1} (\pi (K))$ has a compact closure.

For each unit u let us define $e (u)$ to be the unique element in the orbit of u that is contained in F, i.e. $\{e (u)\} = F \cap [u]$. For each Borel subset B of $G^{(0)}$, π is continuous and one-to-one on $B \cap F$ and hence $\pi (B \cap F)$ is Borel in $G^{(0)}/G$. Therefore the map $e : G^{(0)} \to G^{(0)}$ is Borel (for each Borel subset B of $G^{(0)}$, $e^{-1} (B) = [B \cap F] = \pi^{-1} (\pi (B \cap F))$ is Borel in $G^{(0)}$). Also for each compact subset K of $G^{(0)}$, $e (K)$ has a compact closure because $e (K) \subset F \cap [K]$.

Since the orbit space $G^{(0)}/G$ is proper the map

$$(r, d) : G \to R, (r, d) (x) = (r (x), d (x))$$

is open and R is closed in $G^{(0)} \times G^{(0)}$. Applying Lemma 1.1 of [12] to the locally compact second countable spaces G and R and to the continuous open surjection $(r, d) : G \to R$, it follows that there is a *regular cross section* $\sigma_0 : R \to G$. This means that σ_0 is Borel, $(r, d) (\sigma_0 (u, v)) = (u, v)$ for all $(u, v) \in R$, and $\sigma_0 (K)$ is relatively compact in G for each compact subset K of R.

Let us define $\sigma : G^{(0)} \to G^F$ by $\sigma (u) = \sigma_0 (e (u), u)$ for all u. It is easy to note that σ is a cross section for $d : G^F \to G^{(0)}$ and $\sigma (K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. If F is closed, then σ is regular.

Replacing σ by

$$v \mapsto \sigma\left(e\left(v\right)\right)^{-1} \sigma\left(v\right)$$

we may assume that $\sigma\left(e\left(v\right)\right) = e\left(v\right)$ for all v. Let us define $q : G \to G_F^F$ by

$$q\left(x\right) = \sigma\left(r\left(x\right)\right) x \sigma\left(d\left(x\right)\right)^{-1}, \, x \in G.$$

Let $\nu = \left\{\nu^u : u \in G^{(0)}\right\}$ be a Haar system on G and let $\left(\{\beta_v^u\}, \{\mu^{\mathring{u}}\}\right)$ be its decompositions over the principal groupoid . Let δ be the 1-cocycle associated to the decomposition.

Let us denote by $\mathcal{B}_\sigma\left(G\right)$ the linear span of the functions of the form

$$x \mapsto g_1\left(r\left(x\right)\right) g\left(q\left(x\right)\right) g_2\left(d\left(x\right)\right)$$

where g_1, g_2 are compactly supported bounded Borel functions on $G^{(0)}$ and g is a bounded Borel function on G_F^F such that if S is the support of g, then the closure of S is compact in G. $\mathcal{B}_\sigma\left(G\right)$ is a subspace of $\mathcal{B}_c\left(G\right)$, the space of compactly supported bounded Borel functions on G.

If $f_1, \, f_2 \in \mathcal{B}_\sigma\left(G\right)$ are defined by

$$\begin{aligned} f_1\left(x\right) &= g_1\left(r\left(x\right)\right) g\left(q\left(x\right)\right) g_2\left(d\left(x\right)\right) \\ f_2\left(x\right) &= h_1\left(r\left(x\right)\right) h\left(q\left(x\right)\right) h_2\left(d\left(x\right)\right) \end{aligned}$$

then

$$f_1 * f_2\left(x\right) = g * h\left(q\left(x\right)\right) g_1\left(r\left(x\right)\right) h_2\left(d\left(x\right)\right) \left\langle g_2, \overline{h_1}\right\rangle_{\pi\left(r\left(x\right)\right)}$$

$$f_1^*\left(x\right) = \overline{g_2\left(r\left(x\right)\right)} \overline{g\left(q\left(x\right)^{-1}\right)} \overline{g_1\left(d\left(x\right)\right)}$$

Thus $\mathcal{B}_\sigma\left(G\right)$ is closed under convolution and involution.

Let ω be the universal representation of $C^*\left(G, \nu\right)$ the usual (full) C^*-algebra associated to a Haar system $\nu = \left\{\nu^u, u \in G^{(0)}\right\}$. Since every cyclic representation of $C^*\left(G, \nu\right)$ is the integrated form of a representation of G, it follows that ω can be also regarded as a representation of $\mathcal{B}_c\left(G\right)$, the space of compactly supported bounded Borel functions on G. Arlan Ramsay and Martin E. Walter have used the notation $M^*\left(G, \nu\right)$ for the operator norm closure of $\omega\left(\mathcal{B}_c\left(G\right)\right)$. Since ω is an $*$-isomorphism on $C^*\left(G, \nu\right)$, we can regarded $C^*\left(G, \nu\right)$ as a subalgebra of $M^*\left(G, \nu\right)$. In [7] we denoted by $M_\sigma^*\left(G, \nu\right)$ the operator norm closure of $\omega\left(\mathcal{B}_\sigma\left(G\right)\right)$.

Theorem 13. *Let G be a locally compact second countable groupoid with proper orbit space. Let $\left\{\nu_i^u, u \in G^{(0)}\right\}$, $i = 1, 2$ be two Haar systems on G. Let F be a Borel subset of $G^{(0)}$ containing only one element $e\left(u\right)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d : G^F \to G^{(0)}$ with $\sigma\left(e\left(v\right)\right) = e\left(v\right)$ for all $v \in G^{(0)}$ and such that $\sigma\left(K\right)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Then the C^*-algebras $M_\sigma^*\left(G, \nu_1\right)$ and $M_\sigma^*\left(G, \nu_2\right)$ are $*$-isomorphic (Theorem 9 [7]).*

Theorem 14. *Let G be a locally compact second countable groupoid with proper orbit space. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar systems on G. Let F_i, $i = 1, 2$, be two Borel subsets of $G^{(0)}$ containing only one element $e_i(u)$ in each orbit $[u]$. For each $i = 1, 2$, let $\sigma_i : G^{(0)} \to G^{F_i}$ be a cross section for $d_{F_i} : G^{F_i} \to G^{(0)}$, $d_{F_i}(x) = d(x)$, satisfying the conditions*

   1. $\sigma_i(e_i(v)) = e_i(v)$ for all $v \in G^{(0)}$

   2. $\sigma_i(K)$ is relatively compact in G for all compact sets $K \subset G^{(0)}$.

   Then the C^*-algebras $M_{\sigma_1}^*(G, \nu)$ and $M_{\sigma_2}^*(G, \nu)$ are $*$-isomorphic. (Theorem 6 [8]).

Thus $M_\sigma^*(G, \nu)$ is a C^*-algebra which does not depend on the choice of the Haar system ν and also does not depend on the choice of cross section σ.

Theorem 15. *Let G be a locally compact second countable* locally transitive *groupoid endowed with a Haar system $\{\nu^u, u \in G^{(0)}\}$. Let F be a subset of $G^{(0)}$ containing only one element $e(u)$ in each orbit $[u]$. Let $\sigma : G^{(0)} \to G^F$ be a regular cross section of d_F. Then*
$$C^*(G, \nu) = M^*(G, \nu) = M_\sigma^*(G, \nu).$$

(Proposition 18 [7])

Theorem 16. *Let G be a locally compact second countable* principal proper *groupoid. Let F be a Borel subset of $G^{(0)}$ meeting each orbit exactly once. Let $\sigma : G^{(0)} \to G^F$ be a cross section for $d : G^F \to G$ such that $\sigma(K)$ is relatively compact in G for all compact $K \subset G^{(0)}$. Let $\{\nu^u, u \in G^{(0)}\}$ be a Haar system on G. Then*

$$C^*(G, \nu) \subset M_\sigma^*(G, \nu) \subset M^*(G, \nu).$$

(Corollary 23 [7]).

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