Inequalities

Manuel Kauers
RISC-Linz

I. What?

II. How?

III. Why?

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II. How?

III. Why?

THE AMERICAN MATHEMATICAL MONTHLY

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Some Recent Monthly Problems

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$$f(a, b, c) + f(b, c, a) + f(c, a, b) \geq 0.$$

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Find the minimum value of $E(a,b,c)$ on the set D consisting of all positive triples (a,b,c), other than $(2,2,2)$, at which $abc = a+b+c+2$.

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- Its applicability extends far beyond Monthly problems.
- It is not as widely known as it deserves.

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A **polynomial inequality** is an expression of the form

$$f(x_1, x_2, \ldots, x_n) \diamondsuit g(x_1, x_2, \ldots, x_n)$$

where

- $\diamondsuit$ is one of $=, \neq, <, >, \leq, \geq$
- f and g are polynomials in $x_1, x_2, \ldots, x_n$ with coefficients in $\mathbb{Q}$.

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Examples: $x > 0$, $x^2 + y^2 < 1$, $\sqrt{1 - x^2} < \sqrt[3]{y}$

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$(-1 \leq x \wedge y \leq 1) \Rightarrow (x+y)^2 > \frac{1}{2} \vee x \neq y$,
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Examples involving shorthand notation:

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Examples:
The formula $x^2 + 1 = 0$ is always false.
The formula $x^2 - 2 = 0$ may be true or false.
The formula $x^2 \geq 0$ is always true.

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Two systems $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ are **equivalent** if

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Examples:
$x^2 < 1$ and $-1 < x \wedge x < 1$ are equivalent.
$x^2 + y^2 + z^2 < 0$ and false are equivalent.
$x^2 + y^2 + z^2 \geq 0$ and true are equivalent.

Geometric Interpretation

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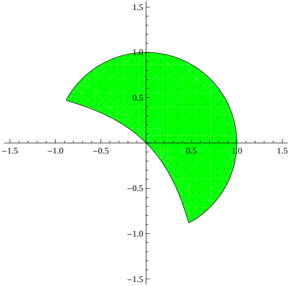
Example:
$(x-1)(y-1) > 1 \wedge x^2 + y^2 < 1$



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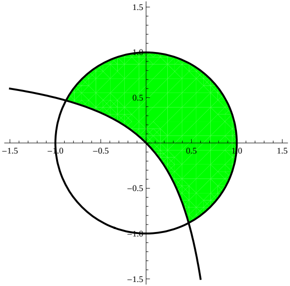

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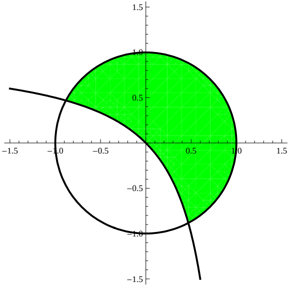

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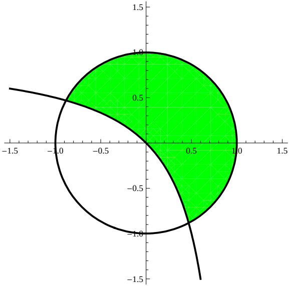

Sets defined by systems of polynomial inequalities are called **semialgebraic sets.**

"Given a semialgebraic set" means
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Then

$$\max\{x, y, z\} = x, \quad \max\{a, b, c\} = a,$$
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For geometric reasons, we have

$$\begin{aligned} a + b &\geq c \geq 0 \\ a + c &\geq b \geq 0 \\ b + c &\geq a \geq 0. \end{aligned}$$

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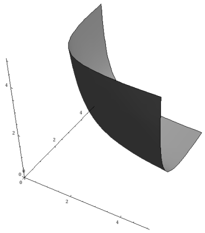

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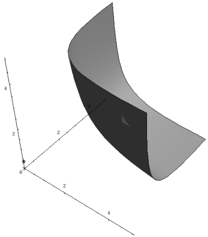

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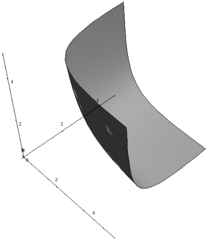

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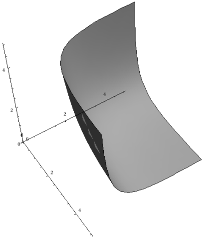

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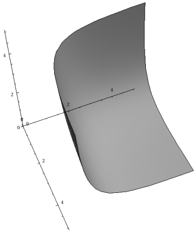

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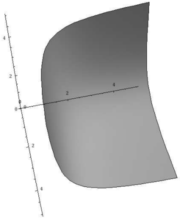

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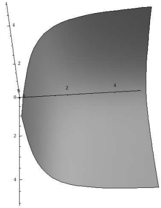

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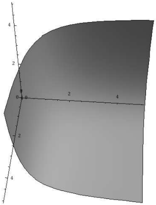

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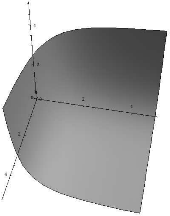

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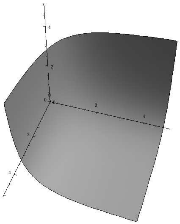

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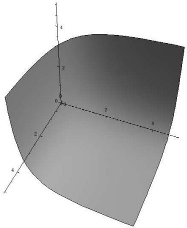

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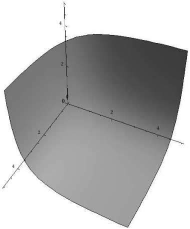

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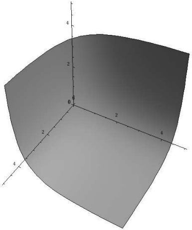

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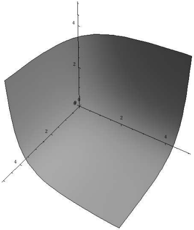

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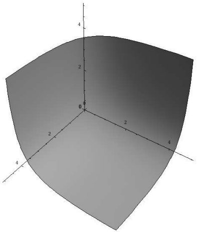

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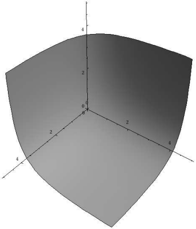

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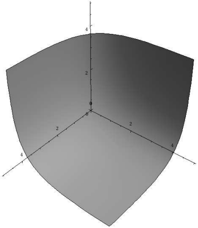

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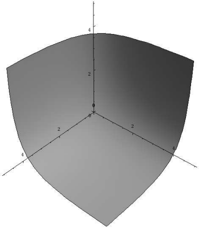

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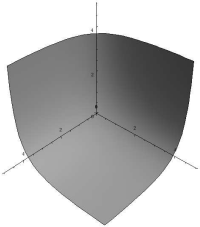

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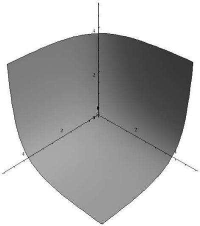

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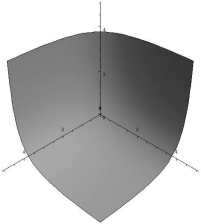

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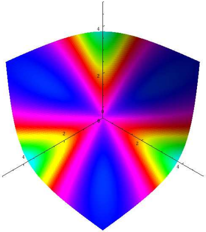

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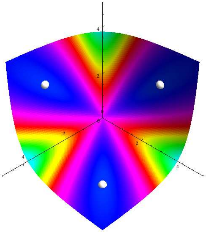

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(Lagrange multipliers + Gröbner bases would have worked as well.)

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\begin{aligned}
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The boxes represent some formulas involving a, b, c, e which are guaranteed to be satisfiable.

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In general, CAD brings a system of polynomial inequalities into the following recursive format:

$$\cdots \quad \vee \quad \blacksquare < x_1 < \blacksquare \wedge \boxed{} \quad \vee \quad x_1 = \blacksquare \wedge \boxed{} \quad \vee \quad \cdots$$

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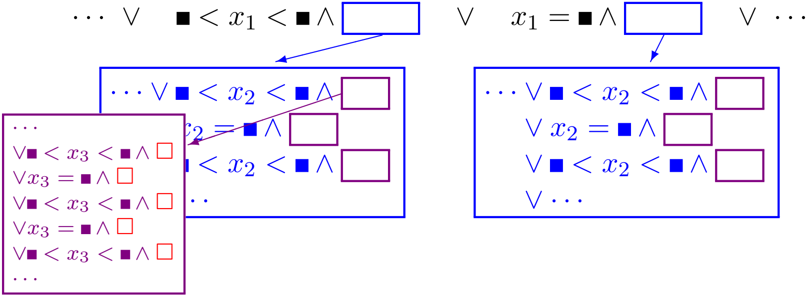

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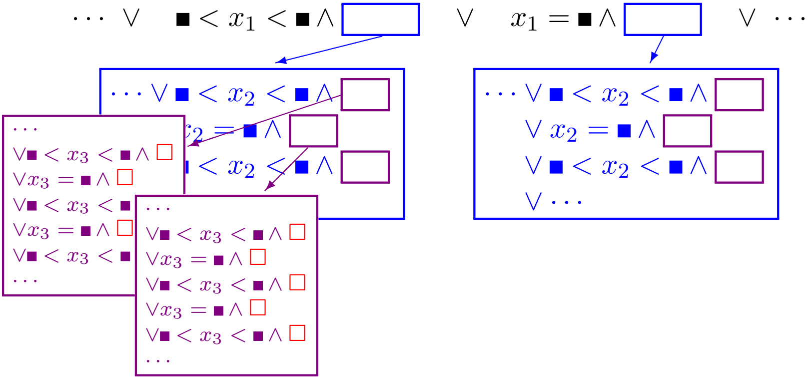

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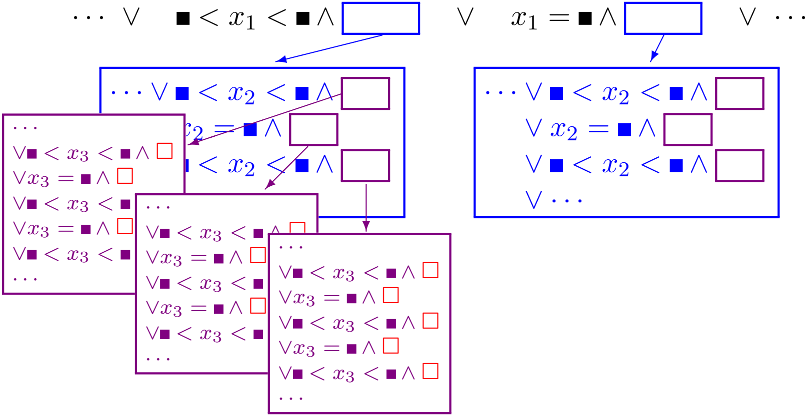

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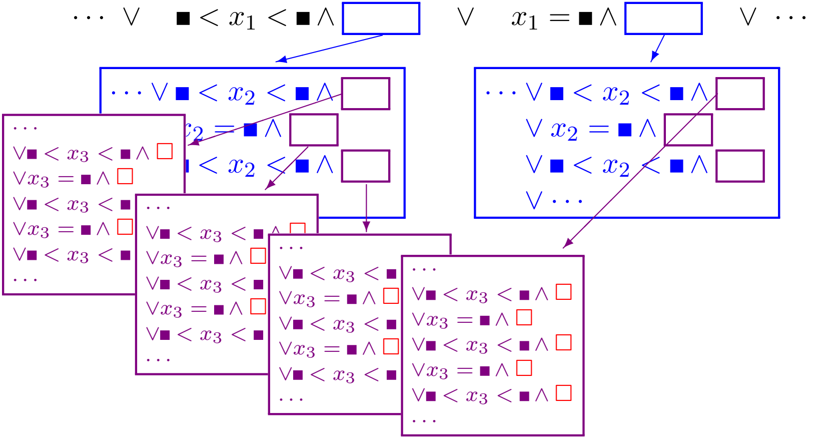

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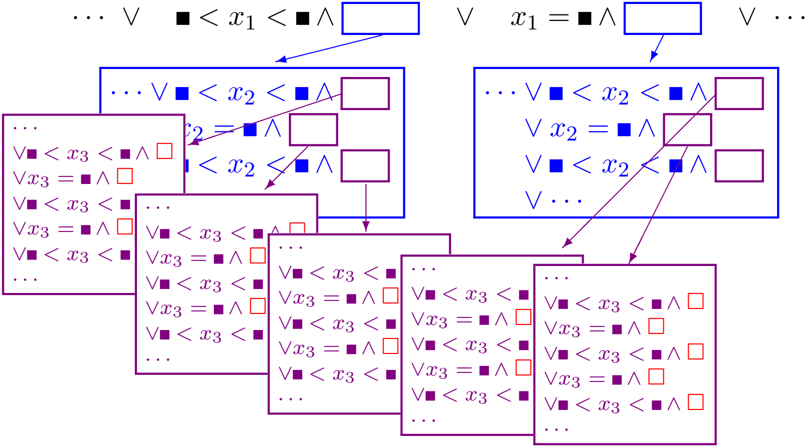

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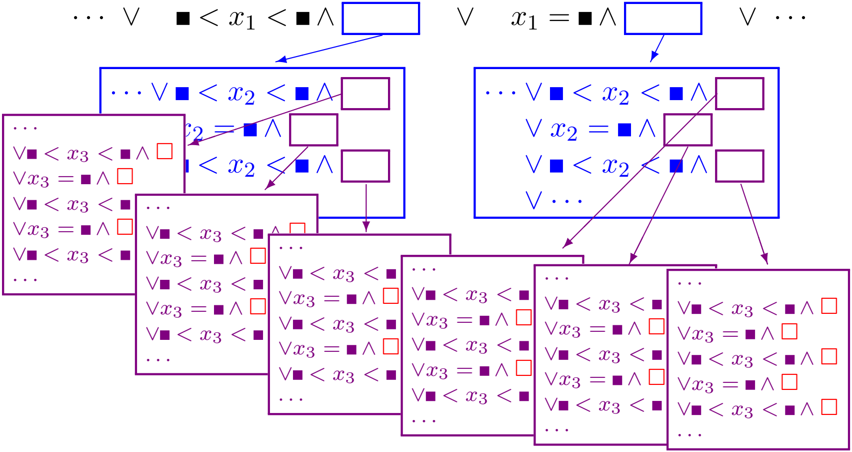

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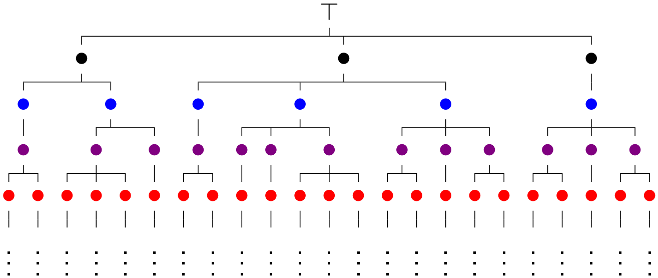

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- *1 variable:* A system of polynomial inequalities is called a **CAD** in x if it is of the form

$$\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_m$$

  where each Φ_k is of the form $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers α, β, γ $(\alpha < \beta)$ and any two Φ_k are mutually inconsistent.

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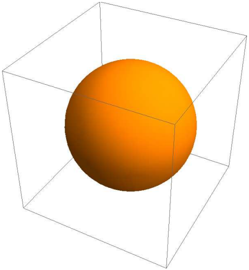
  where the Φ_k are such that $\Phi_1 \vee \cdots \vee \Phi_k$ is a CAD in x_1 and the Ψ_k are CADs in $x_2, \ldots, x_n$ whenever x_1 is replaced by a real algebraic number satisfying Φ_k.

Example

Here is a CAD for the unit sphere:



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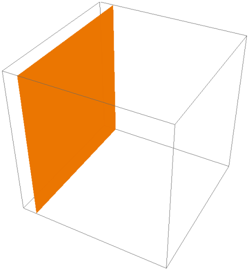

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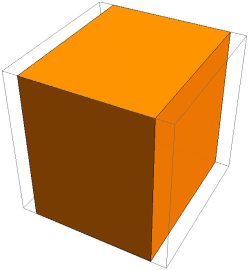

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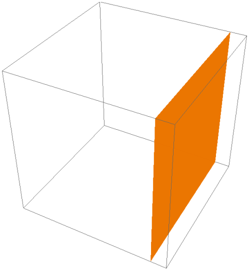

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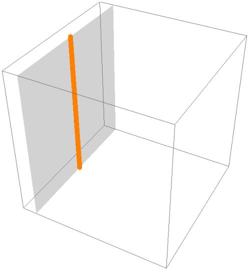

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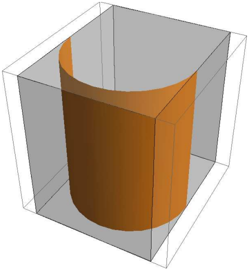

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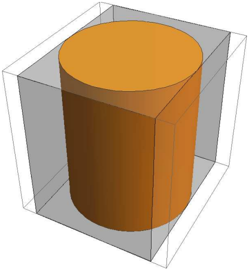

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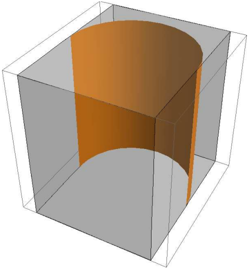

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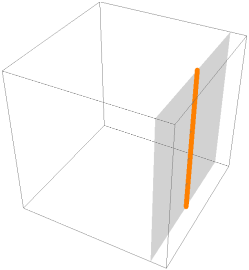

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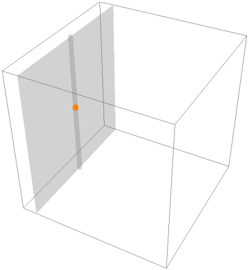

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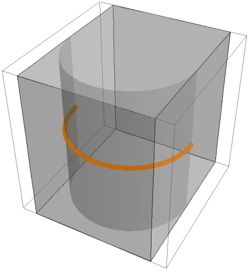

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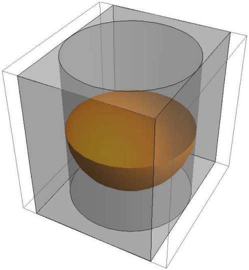

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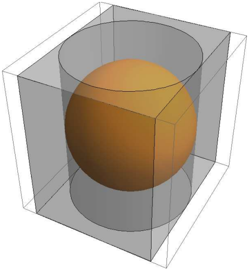

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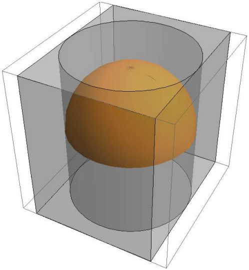

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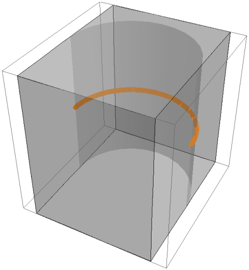

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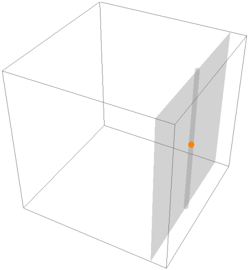

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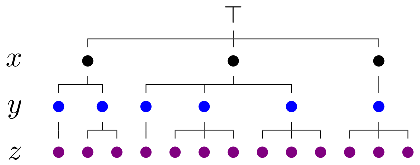

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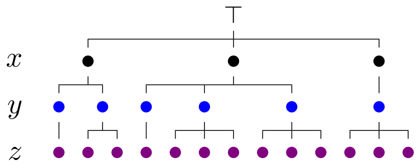

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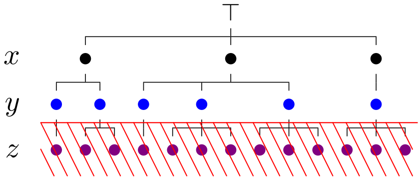

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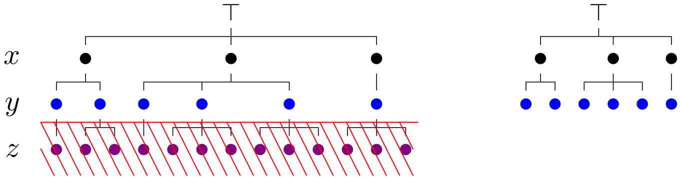

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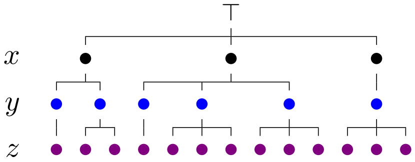

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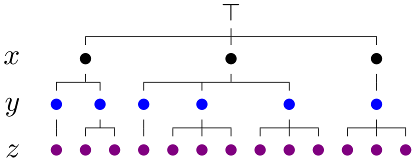

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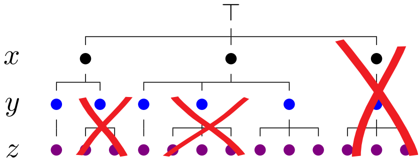

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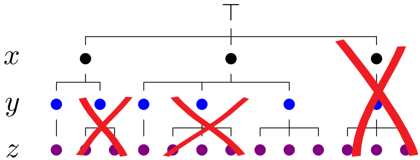

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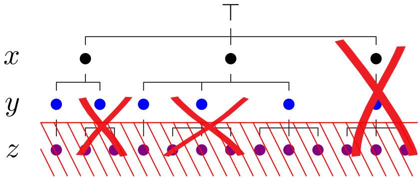

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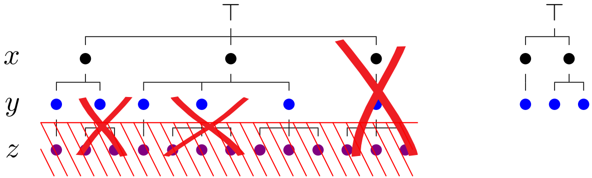

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INPUT: a system of polynomial inequalities over the reals

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- is provably equivalent to the system given as input, and
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Often, CAD computations in such applications are feasible only after some appropriate preprocessing.

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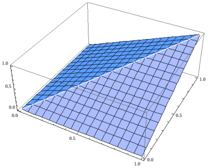

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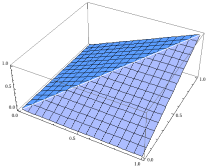

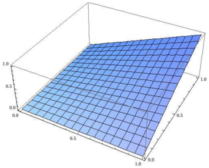

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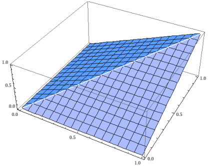
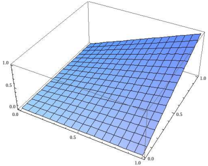
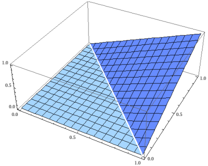

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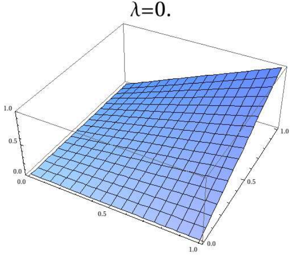

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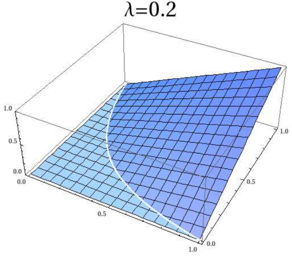

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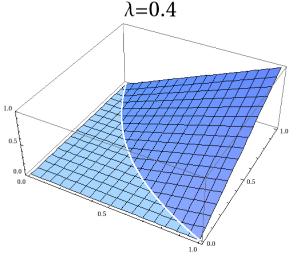

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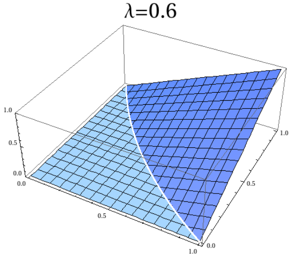

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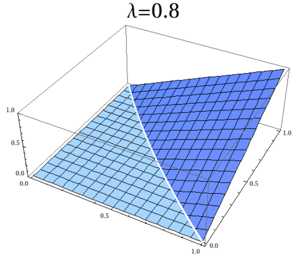

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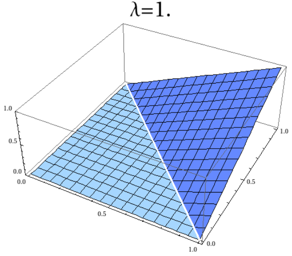

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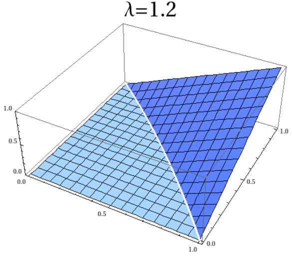

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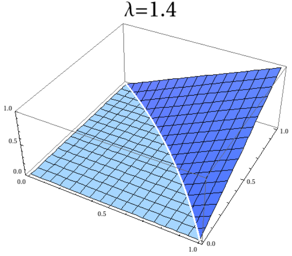

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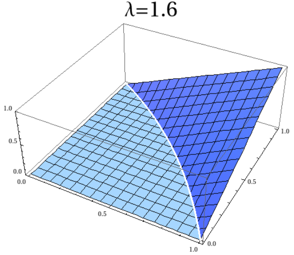

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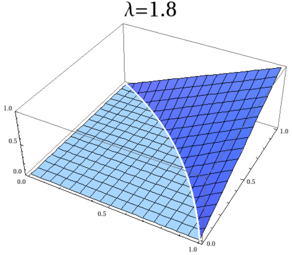

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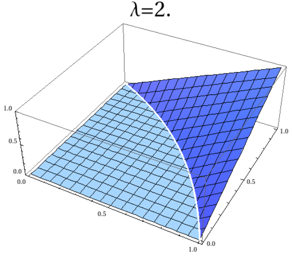

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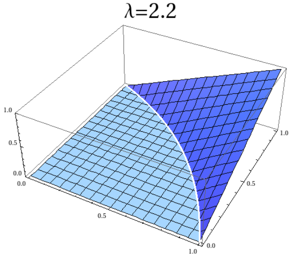

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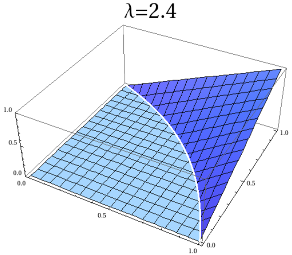

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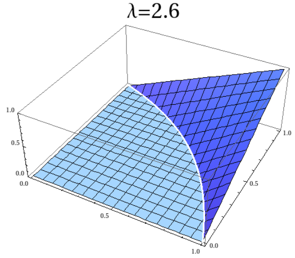

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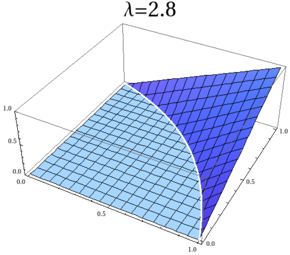

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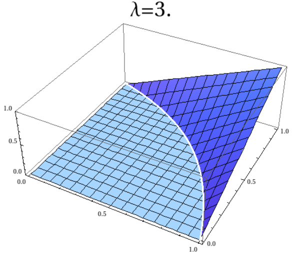

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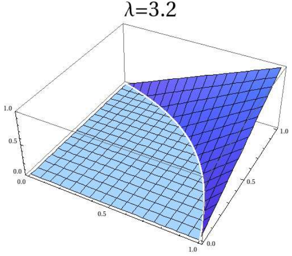

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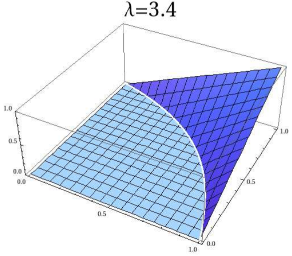

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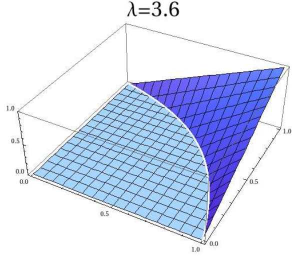

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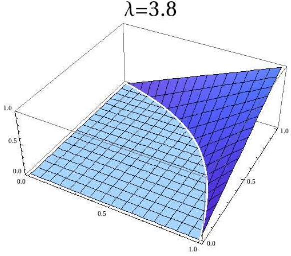

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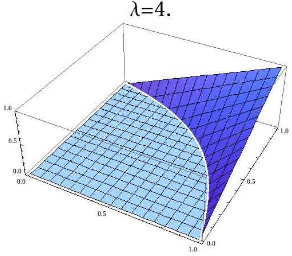

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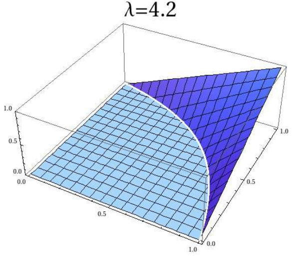

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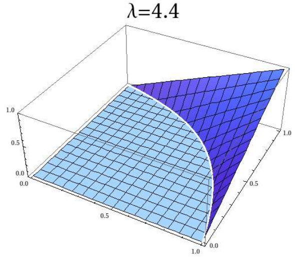

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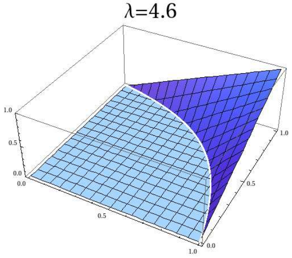

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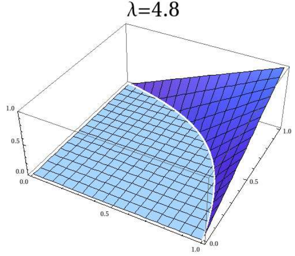

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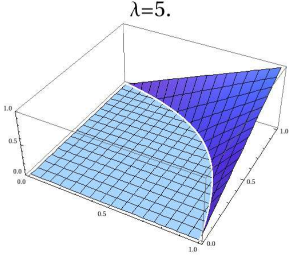

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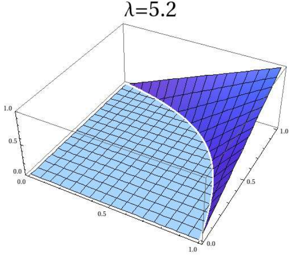

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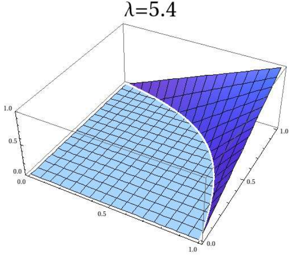

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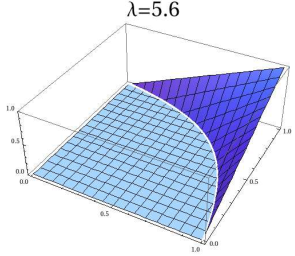

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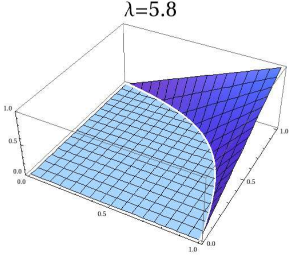

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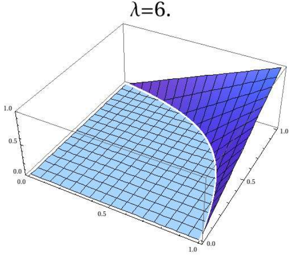

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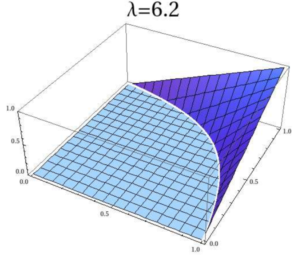

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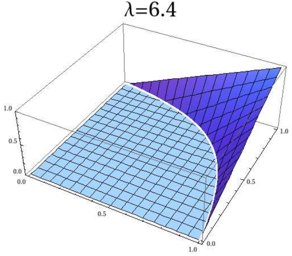

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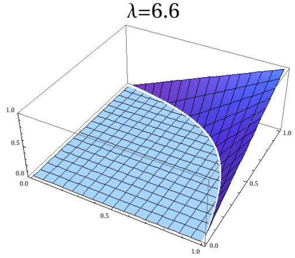

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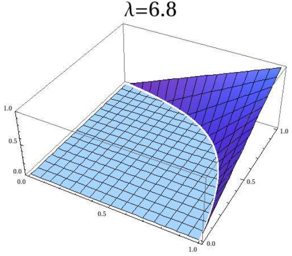

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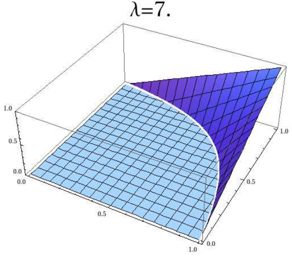

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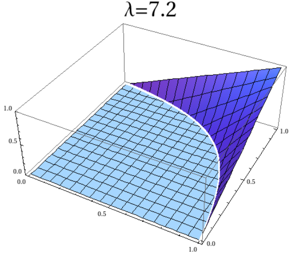

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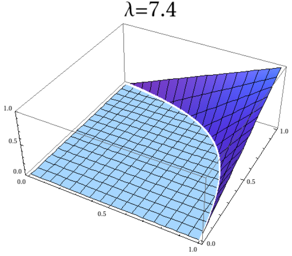

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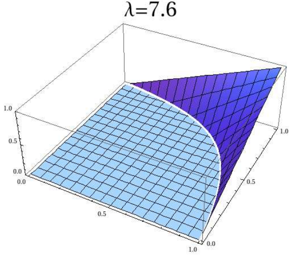

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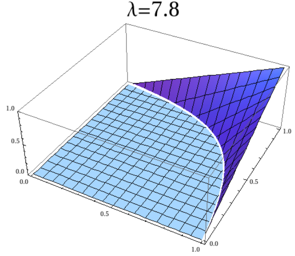

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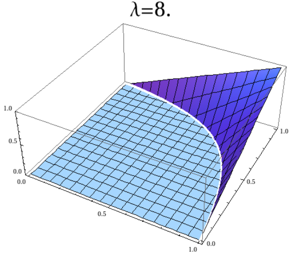

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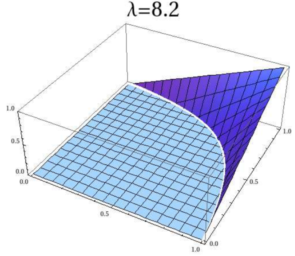

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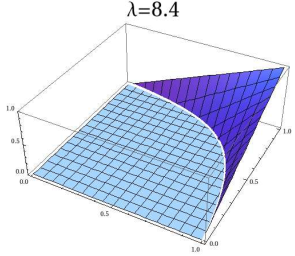

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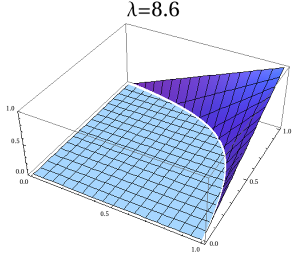

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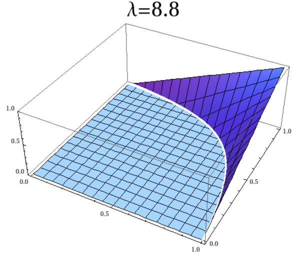

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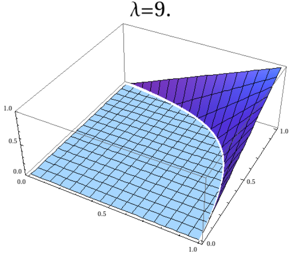

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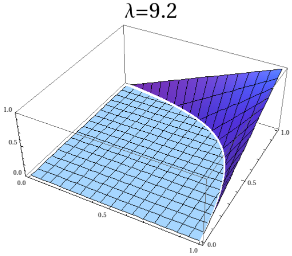

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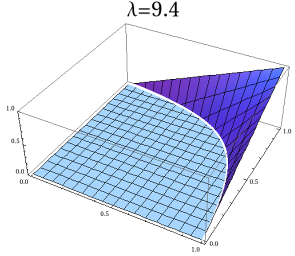

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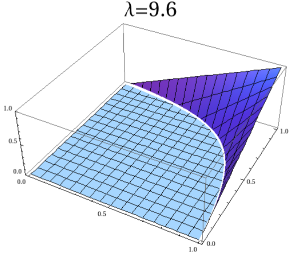

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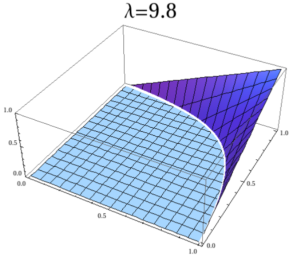

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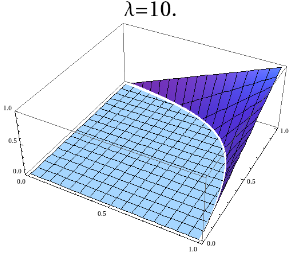

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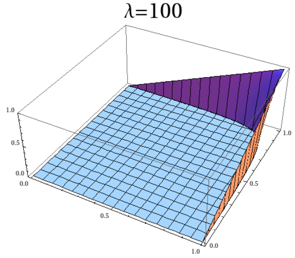

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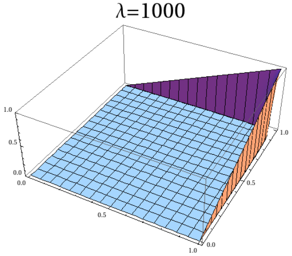

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A norm T is said to *dominate* a norm T' if

$$T(T'(u,v),T'(x,y)) \leq T'(T(u,x),T(v,y))$$

for all $x,y,u,v \in [0,1]$.

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Question: What are the $\lambda,\mu \geq 0$ such that the Sugeno-Weber norm T_λ dominates the Sugeno-Weber norm T_μ?

Theorem (Kauers, Pillwein, Saminger-Platz, 2010)
T_λ dominates T_μ if and only if (a) $\lambda = \mu$ or (b) $0 \leq \lambda \leq \mu \leq 17 + 12\sqrt{2}$ or (c) $\mu < 17 + 12\sqrt{2}$ and $0 \leq \lambda \leq (\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}})^2$.

A nontrivial Example

Just use CAD to eliminate the quantifiers from the formula

$$
\begin{aligned}
&\forall\ x, y, u, v \in [0,1] : \\
&\quad \max\bigl(0, (1-\lambda)\max(0, (1-\mu)uv + \mu(u+v-1)) \\
&\qquad\quad \times \max(0, (1-\mu)xy + \mu(x+y-1))\bigr) \\
&\qquad + \lambda\bigl(\max(0, (1-\mu)uv + \mu(u+v-1)) \\
&\qquad\quad + \max(0, (1-\mu)xy + \mu(x+y-1)) - 1)\bigr) \\
&\quad \geq \max\bigl(0, (1-\mu)\max(0, (1-\lambda)ux + \lambda(u+x-1)) \\
&\qquad\quad \times \max(0, (1-\lambda)vy + \lambda(v+y-1))\bigr) \\
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This is possible *in principle*, but not *in practice.*

A nontrivial Example

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7. Apply CAD to finish up

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It is “easy to see” that it suffices to consider the cases

$$0 < \lambda < \mu \qquad \text{and} \qquad x, y, u, v \in (0, 1)$$

instead of

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(Homework.)

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Apply the general equivalence

$$\max(0, A) \geq \max(0, B) \iff B \leq 0 \vee A \geq B > 0 \quad (A, B \in \mathbb{R})$$

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to obtain

$$\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\;\Rightarrow \big((1-\mu)\max(0, (1-\lambda)ux + \lambda(u+x-1))\max(0, (1-\lambda)vy + \lambda(v+y-1)) \\
&\qquad + \mu(\max(0, (1-\lambda)ux + \lambda(u+x-1)) + \max(0, (1-\lambda)vy + \lambda(v+y-1)) - 1) \leq 0 \\
&\quad \vee (1-\lambda)\max(0, (1-\mu)uv + \mu(u+v-1))\max(0, (1-\mu)xy + \mu(x+y-1)) \\
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If $\Phi(X)$ is any formula depending on a real variable X, then

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For a formula in several variables, we have

$$\begin{aligned}\Phi(\max(0, X_1), \max(0, X_2)) \iff \big(&X_1 \leq 0 \wedge X_2 \leq 0 \wedge \Phi(0, 0)\\ \vee &X_1 > 0 \wedge X_2 \leq 0 \wedge \Phi(X_1, 0)\\ \vee &X_1 \leq 0 \wedge X_2 > 0 \wedge \Phi(0, X_2)\\ \vee &X_1 > 0 \wedge X_2 > 0 \wedge \Phi(X_1, X_2)\big)\end{aligned}$$

A nontrivial Example

3. Eliminate the inner maxima.

Writing

$$X_1 := (1-\lambda)ux + \lambda(u+x-1),$$
$$X_2 := (1-\lambda)vy + \lambda(v+y-1),$$
$$X_3 := (1-\mu)uv + \mu(u+v-1),$$
$$X_4 := (1-\mu)xy + \mu(x+y-1),$$

this turns the formula into...

A nontrivial Example

3. Eliminate the inner maxima.

$$
\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\Rightarrow \big((X_1 \le 0 \wedge X_2 \le 0 \wedge (1-\mu)0\,0 + \mu(0+0-1) \le 0 \\
&\quad \vee X_1 > 0 \wedge X_2 \le 0 \wedge (1-\mu)X_1\,0 + \mu(X_1+0-1) \le 0 \\
&\quad \vee X_1 \le 0 \wedge X_2 > 0 \wedge (1-\mu)0\,X_2 + \mu(0+X_2-1) \le 0 \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge (1-\mu)X_1X_2 + \mu(X_1+X_2-1) \le 0\big) \\
&\ \vee \big(X_1 \le 0 \wedge X_2 \le 0 \wedge X_3 \le 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda)0\,0 + \lambda(0+0-1) \ge (1-\mu)0\,0 + \mu(0+0-1) > 0 \\
&\quad \vee X_1 > 0 \wedge X_2 \le 0 \wedge X_3 \le 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda)0\,0 + \lambda(0+0-1) \ge (1-\mu)X_1\,0 + \mu(X_1+0-1) > 0 \\
&\quad \vee \cdots \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda)X_3\,0 + \lambda(X_3+0-1) \ge (1-\mu)X_1X_2 + \mu(X_1+X_2-1) > 0 \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 > 0 \\
&\qquad \wedge (1-\lambda)X_3X_4 + \lambda(X_3+X_4-1) \ge (1-\mu)X_1X_2 + \mu(X_1+X_2-1) > 0\big)\big)
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A nontrivial Example

4. Discard redundant clauses.

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This formula is of the form

$$\forall\, x, y, u, v \in \mathbb{R} : H \Rightarrow (C_1 \vee C_2 \vee \cdots \vee C_{20}).$$

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&\forall\ x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
&\qquad \wedge\ 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\ \Rightarrow \big(X_1 \le 0 \vee X_2 \le 0 \\
&\qquad \vee\ (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \le 0 \\
&\qquad \vee\ X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 > 0 \\
&\qquad\qquad \wedge\ (1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \\
&\qquad\qquad \ge (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0\big).
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Furthermore, we can prove with CAD the formulas

$$\forall\ x, y, u, v \in \mathbb{R} : H \wedge D \Rightarrow A$$
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are true.

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are true. Dropping also A and B leads us to. . .

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&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
&\quad \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge y < v < 1 + \lambda y \\
&\quad \Rightarrow \big(u((\lambda - 1)x + 1)((\mu - 1)v + 1) \\
&\qquad\quad + (\mu - 1)vx + v + x - 1 \geq 0 \\
&\qquad \vee vx(1 - (\lambda - 1)(\mu - 1)uy) \\
&\qquad\quad + y((\lambda - 1)uy((\mu - 1)x + 1) + u - x) \geq 0\big).
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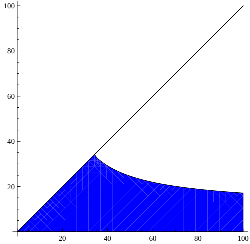

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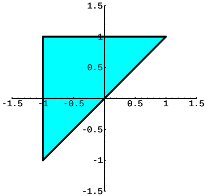
Tomorrow: How does the CAD algorithm work.

A Simple Exercise

What is the image of the triangle $(-1,-1), (-1,1), (1,1)$ under the map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (x^2+y^2, xy-1)\,?$$



$\overset{f}{\longmapsto}$?

Inequalities

Manuel Kauers
RISC-Linz

I. What?

II. How?

III. Why?

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Cylindrical Algebraic Decomposition (CAD)

INPUT: a system of polynomial inequalities over the reals

OUTPUT: a system of polynomial inequalities over the reals, which

- is provably equivalent to the system given as input, and
- has a nice structural property which allows for answering a variety of otherwise nontrivial questions merely by inspection.

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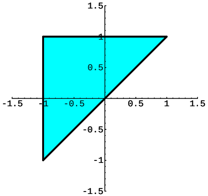

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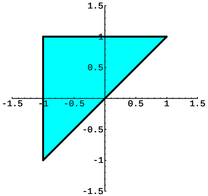

$$\overset{f}{\longmapsto} \qquad ?$$

Answer: Eliminate x, y from the formula

$$\exists\, x,y : (-1 \le x \le 1 \wedge -1 \le y \le 1 \wedge x \le y \wedge$$
$$X = x^2 + y^2 \wedge Y = xy - 1)$$

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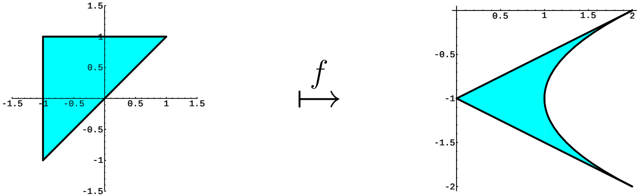

Result:

$$f(\Delta) = \{(x,y) \in \mathbb{R}^2 : \big(0 \le x \le 1 \wedge |y+1| \le \tfrac{1}{2}x\big) \\ \vee \big(1 < x \le 2 \wedge \sqrt{x-1} \le |y+1| \le \tfrac{1}{2}x\big)\}\}$$

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- *1 variable:* A system of polynomial inequalities is called a **CAD** in x if it is of the form

$$\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_m$$

where each Φ_k is of the form $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers α, β, γ ($\alpha < \beta$) and any two Φ_k are mutually inconsistent.

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  where the Φ_k are such that $\Phi_1 \vee \cdots \vee \Phi_k$ is a CAD in x_1 and the Ψ_k are CADs in $x_2, \ldots, x_n$ whenever x_1 is replaced by a real algebraic number satisfying Φ_k.

Example

Here is a CAD for the unit sphere:

$$
\begin{aligned}
& x = -1 \wedge y = 0 \wedge z = 0 \\
& \vee -1 < x < 1 \wedge \Big(y = -\sqrt{1-x^2} \wedge z = 0 \\
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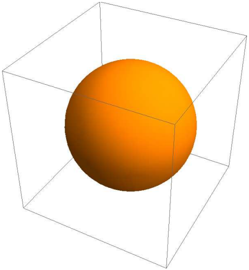

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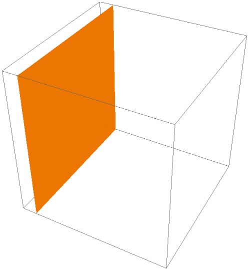

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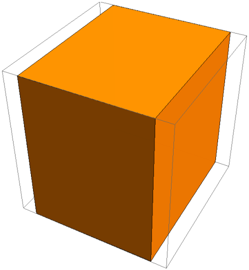

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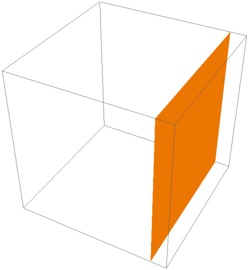

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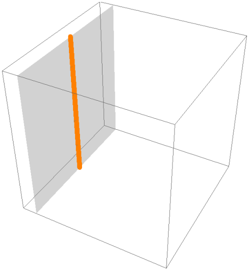

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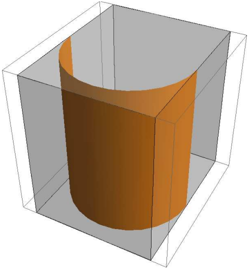

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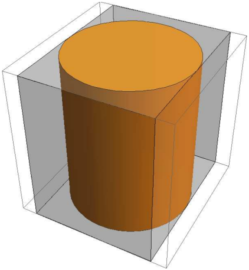

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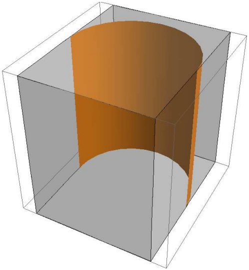

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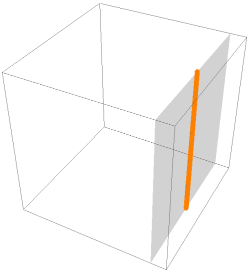

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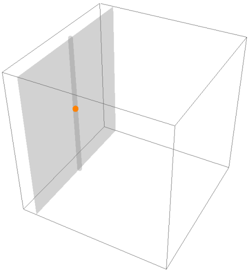

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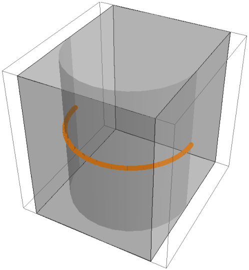

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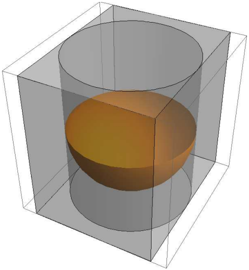

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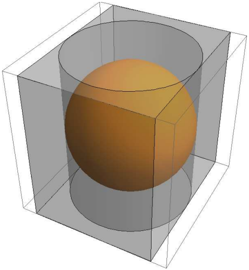

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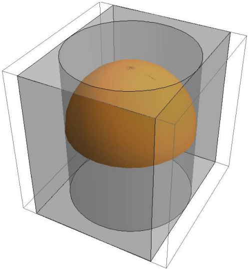

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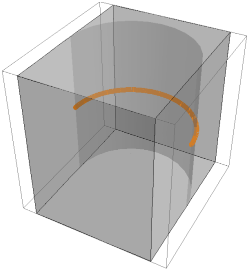

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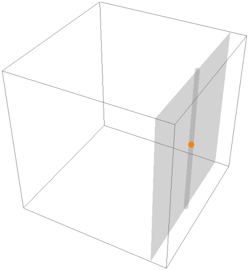

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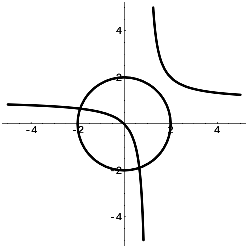

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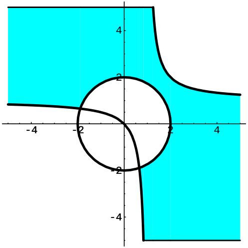

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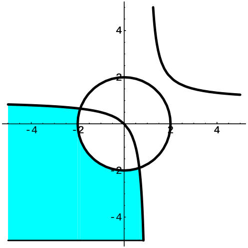

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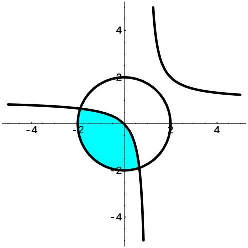

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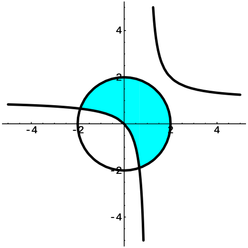

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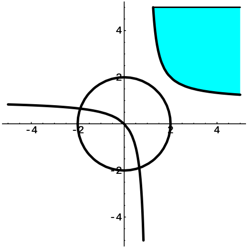

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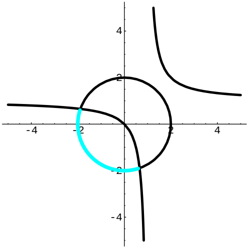

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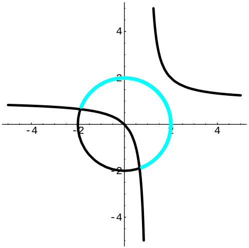

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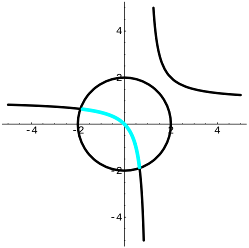

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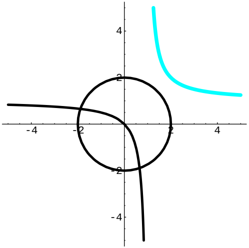

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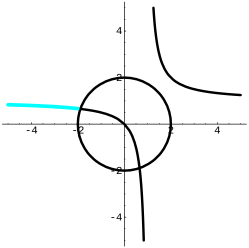

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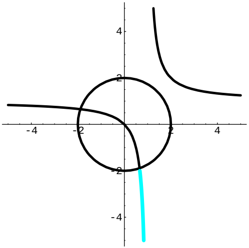

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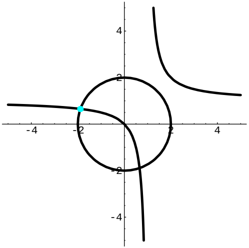

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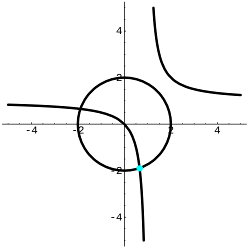

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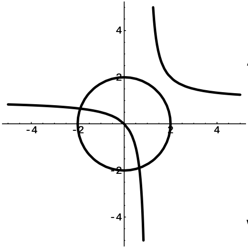

Precise Definition:
A **cell** in the algebraic decomposition of

$$\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$$

is a *maximal connected* subset of $\mathbb{R}^n$ on which all the p_i are *sign invariant.*

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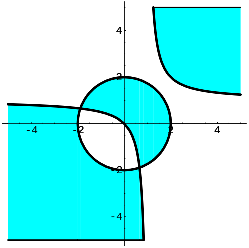

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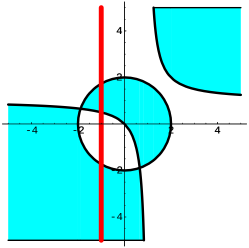
Obviously, each vertical line $x = \alpha$ intersects one of those cells nontrivially. The $\forall x \exists y$ claim follows.



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Observation: It does not hurt if we change from a decomposition for $\{p_1, \ldots, p_m\}$ to a decomposition for $\{p_1, \ldots, p_m, q_1, \ldots, q_k\}$ for some polynomials $q_1, \ldots, q_k \in \mathbb{Q}[x_1, \ldots, x_n]$.

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This motivates the following definition.

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For $n \in \mathbb{N}$, let

$$\pi_n \colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \qquad (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

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Definition: Let $p_1, \ldots, p_m \in \mathbb{Q}[x_1, \ldots, x_n]$. The algebraic decomposition of $\{p_1, \ldots, p_m\}$ is called **cylindrical,** if

- For any two cells C, D of the decomposition, the images $\pi_n(C), \pi_n(D)$ are either identical or disjoint.
- The algebraic decomposition of $\{p_1, \ldots, p_m\} \cap \mathbb{Q}[x_1, \ldots, x_{n-1}]$ is cylindrical.

CAD: Geometric Definition

For $n \in \mathbb{N}$, let

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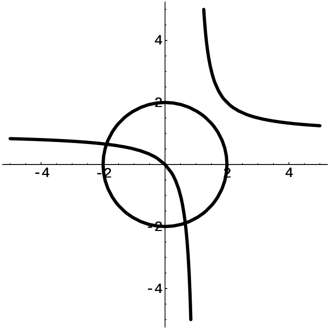
Base case: Any algebraic decomposition of $\mathbb{R}^1$ is cylindrical.

Example

Consider again $\{x^2 + y^2 - 4, (x-1)(y-1) - 1\} \subseteq \mathbb{Q}[x, y]$



This is not a CAD. Why not?

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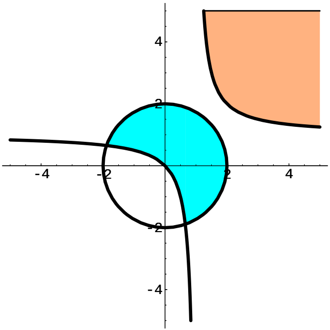

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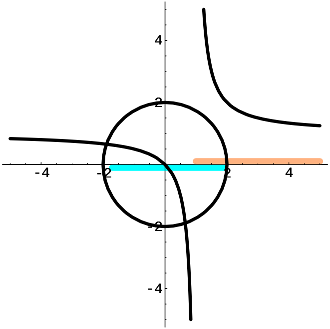

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Consider the two shaded cells.

Their projection to the real line is neither disjoint nor identical.

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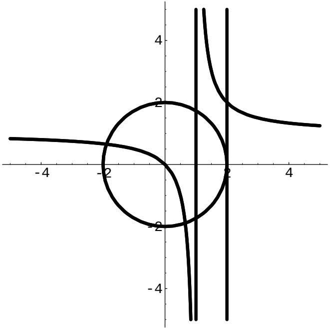

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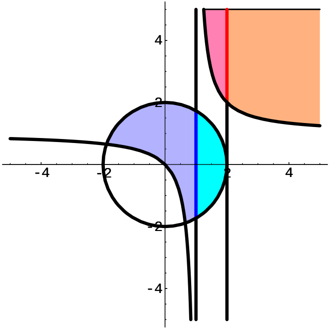

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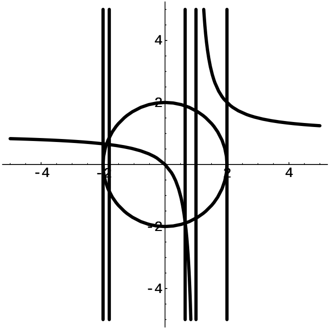

This is not a CAD. Why not?

Consider the two shaded cells.

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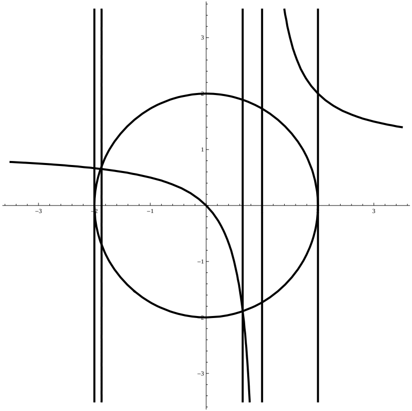
Fix: Insert two vertical lines.

Proceed analogously for all other cell pairs. The result is a CAD.

Example

In a CAD, we can construct a *sample point* for each cell.



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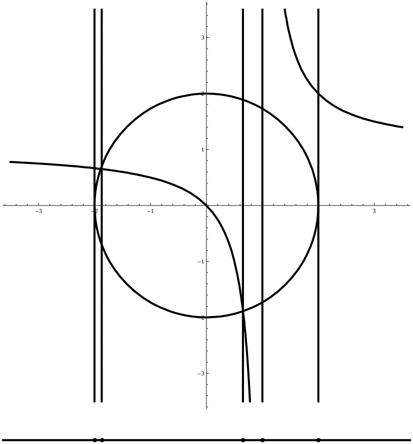

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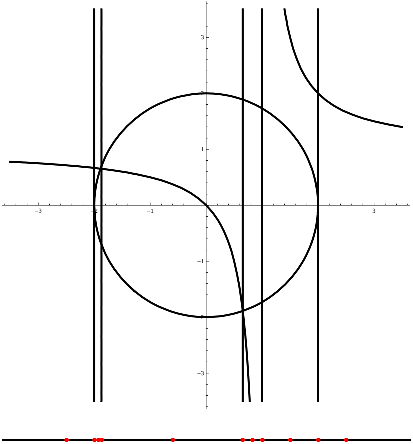

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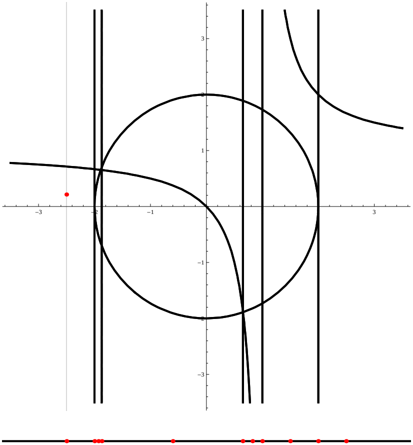

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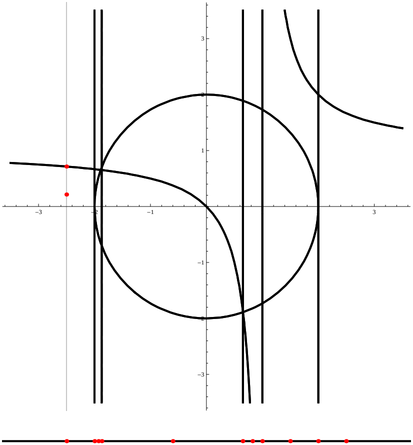

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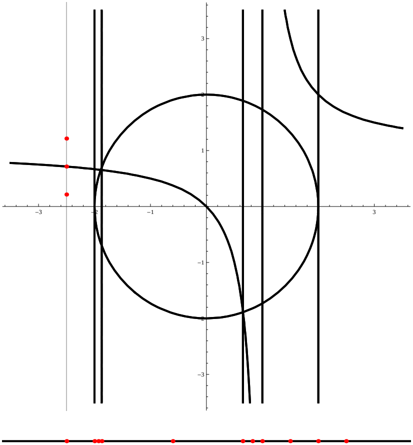

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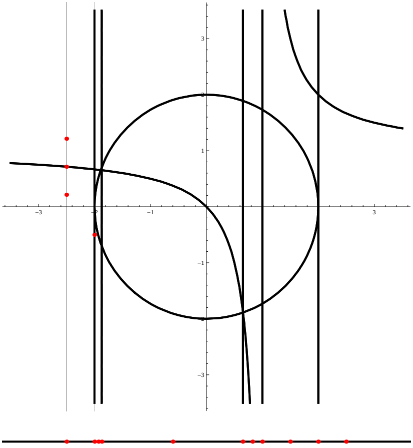

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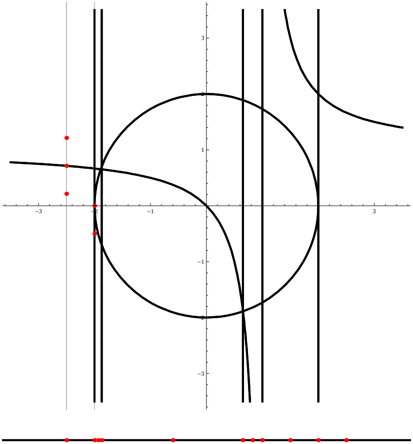

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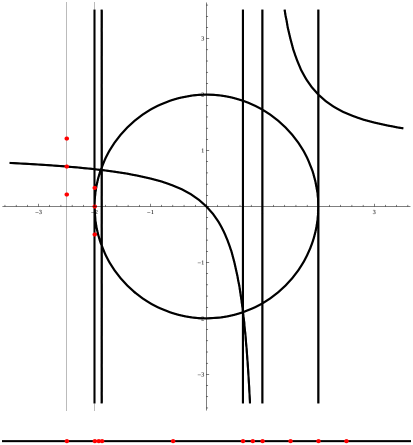

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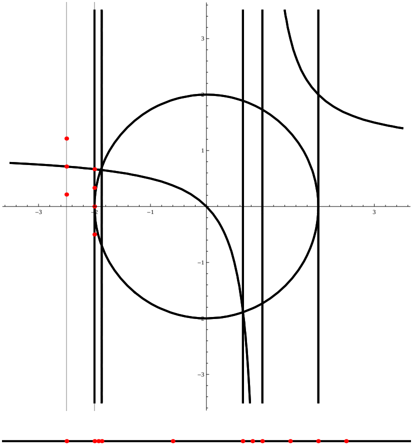

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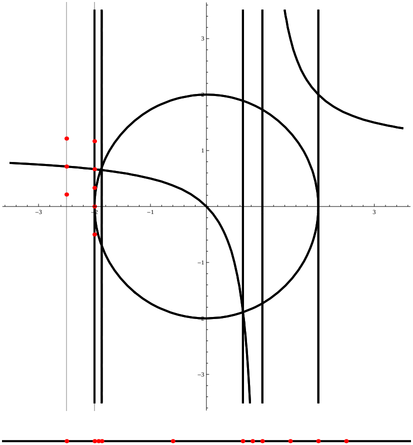

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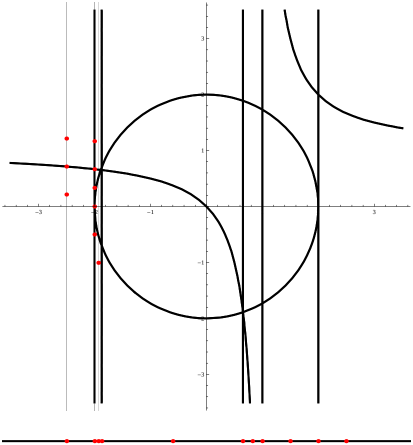

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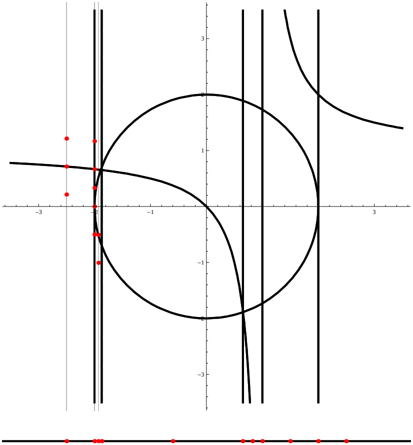

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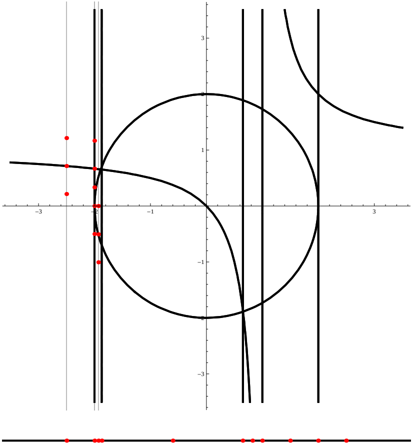

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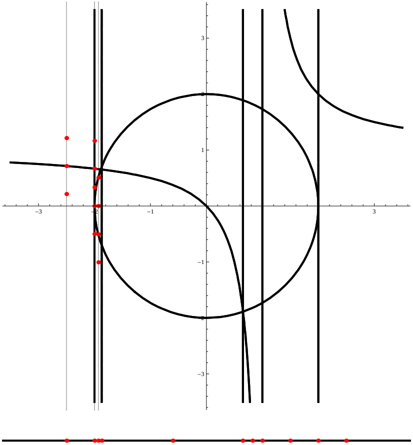

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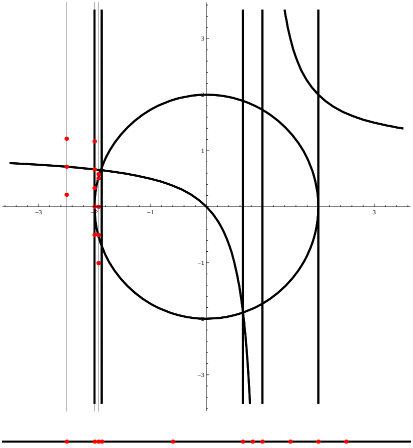

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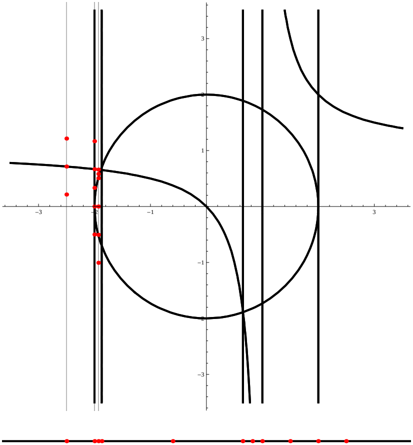

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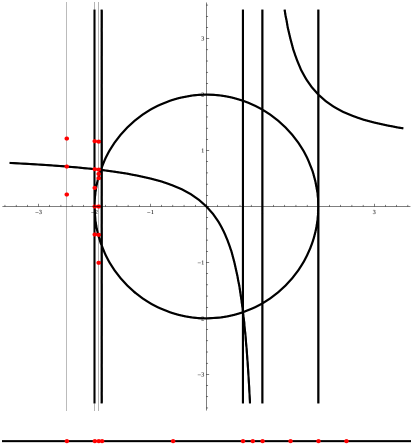

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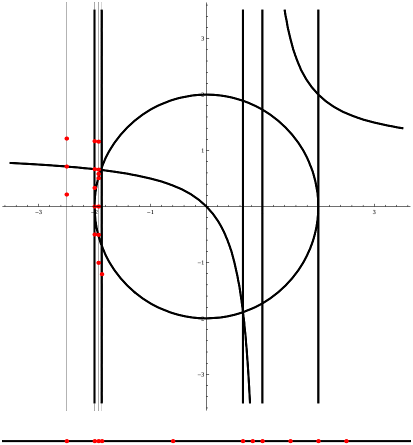

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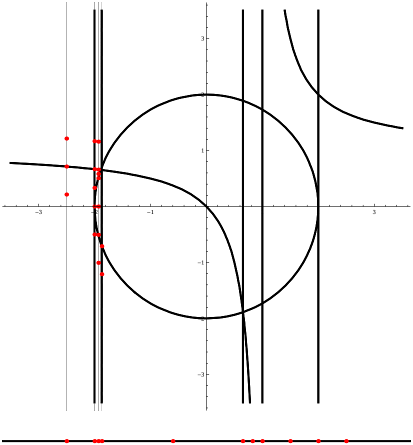

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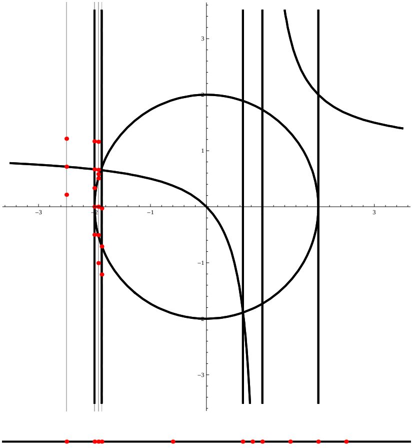

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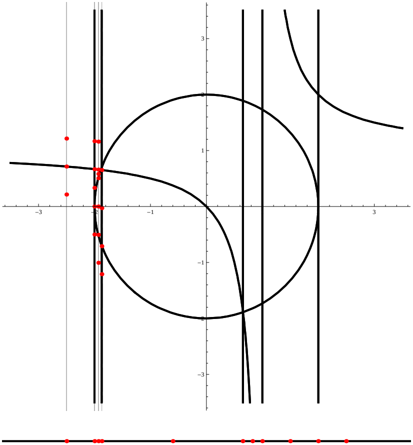

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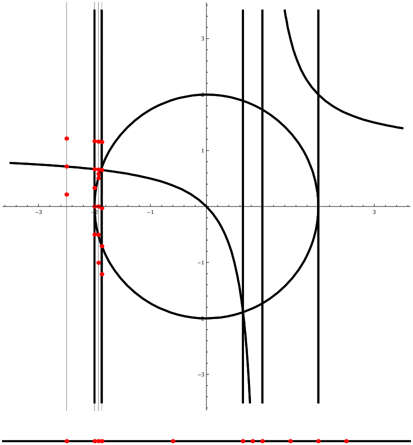

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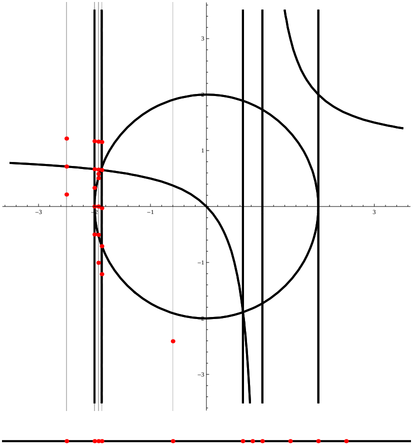

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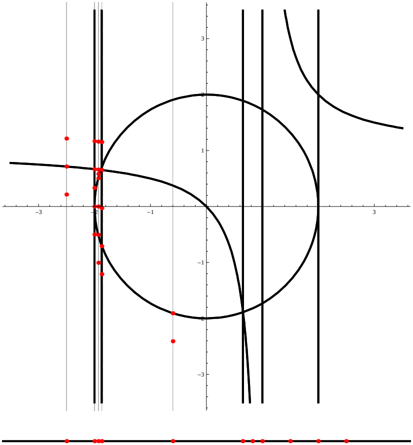

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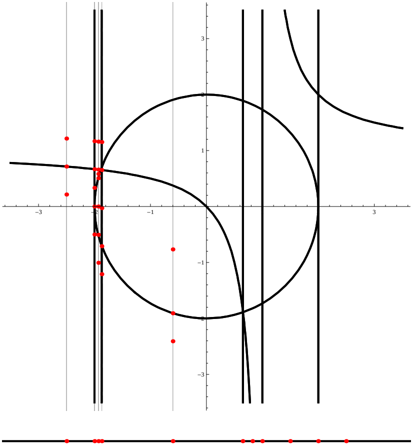

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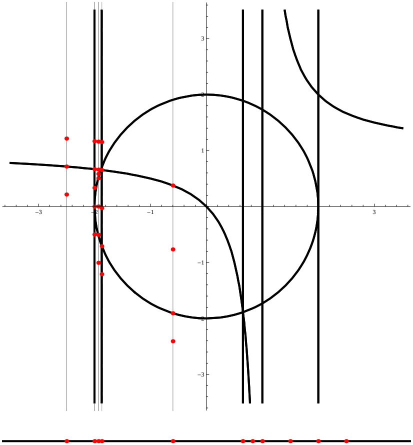

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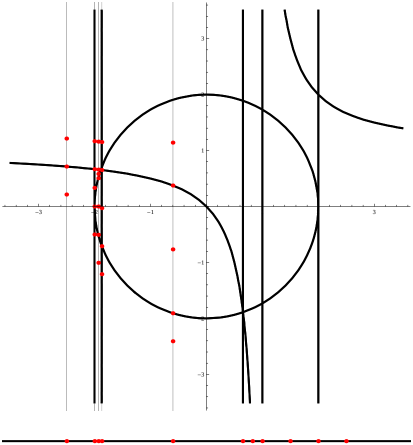

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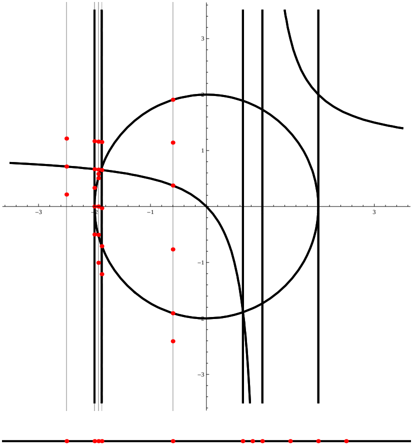

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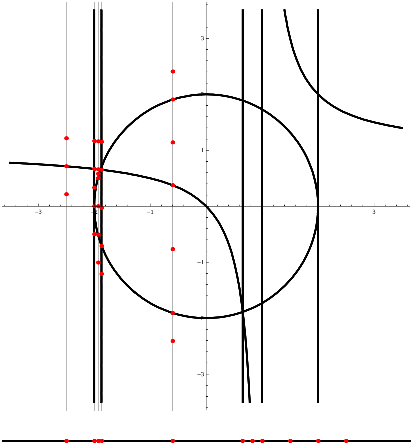

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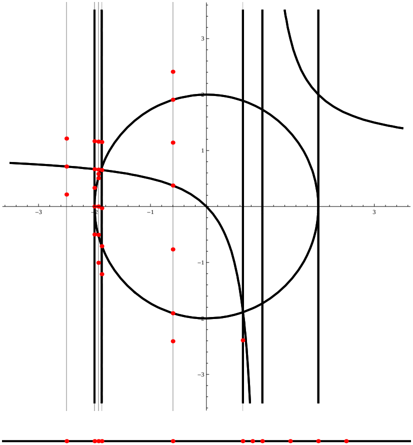

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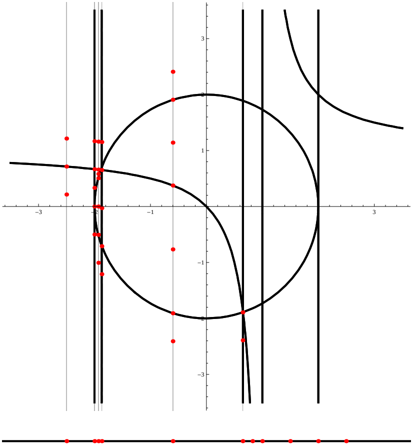

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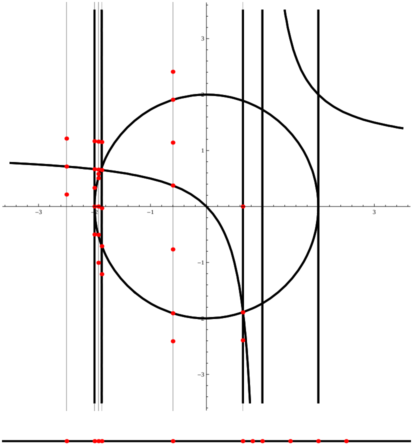

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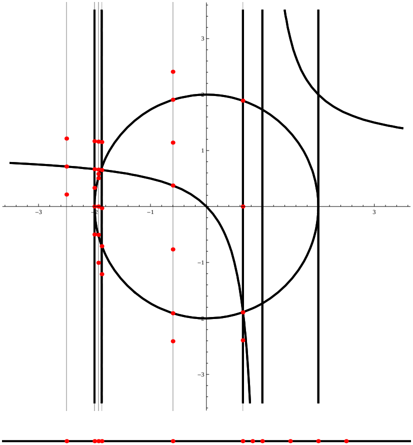

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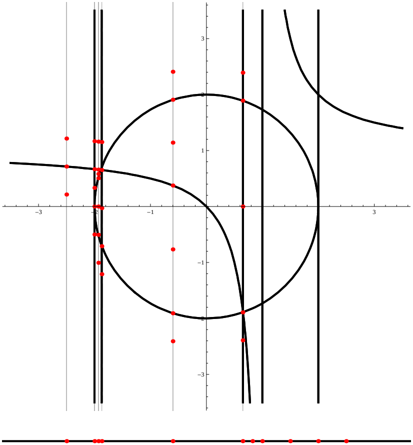

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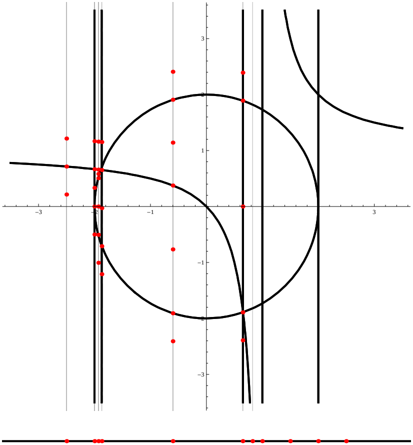

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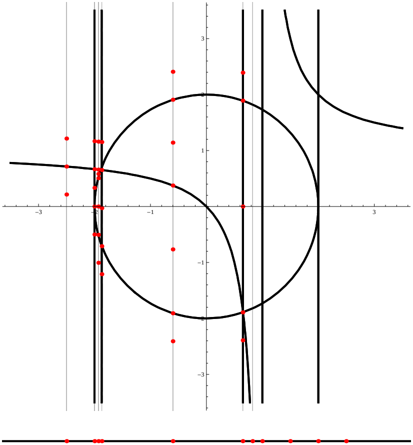

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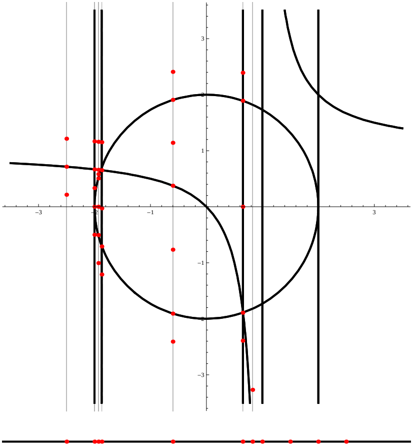

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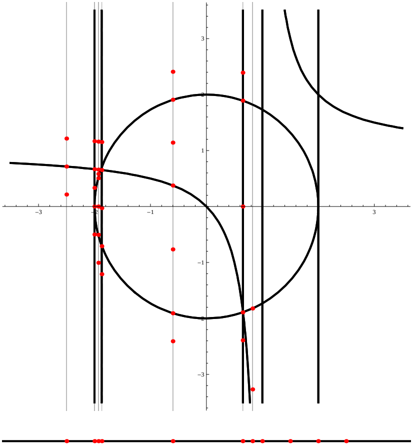

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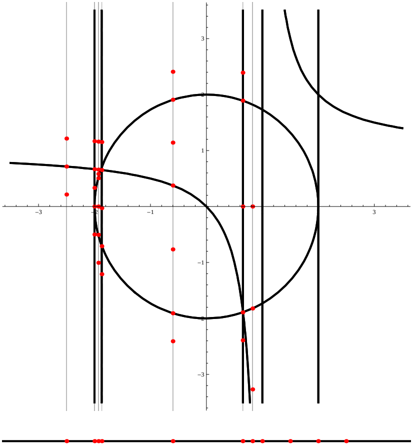

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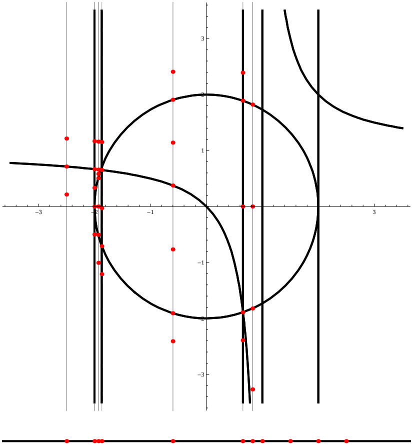

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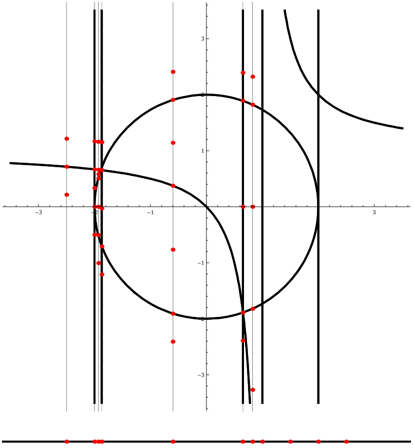

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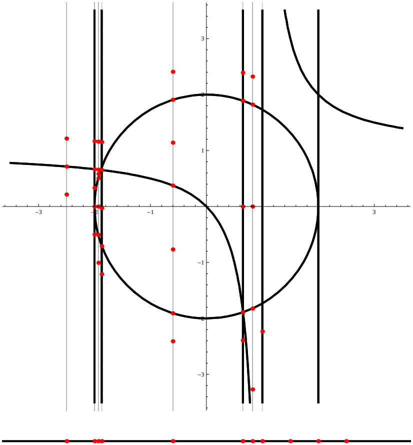

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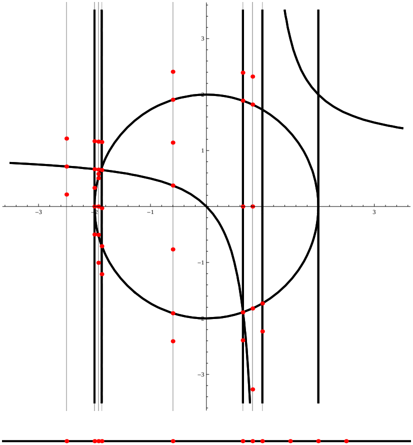

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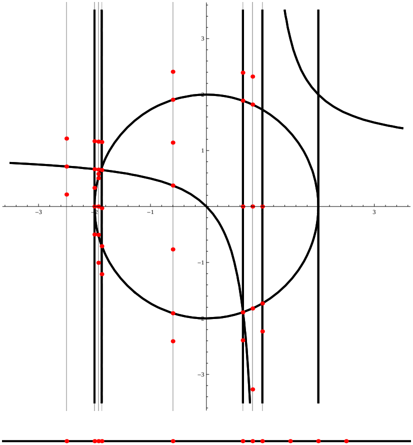

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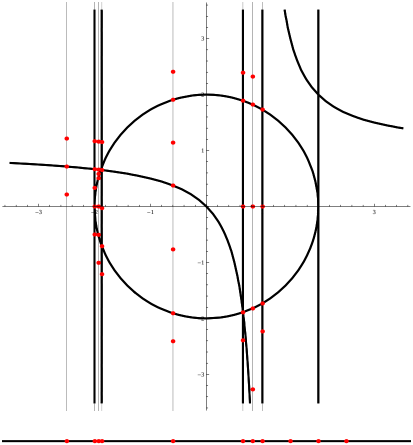

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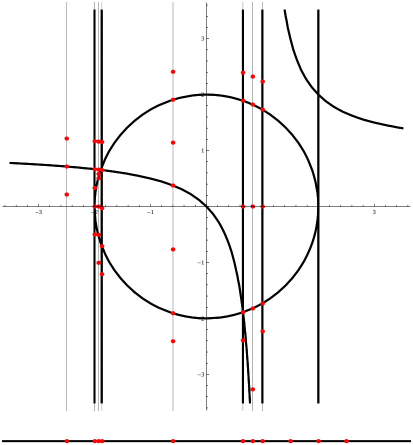

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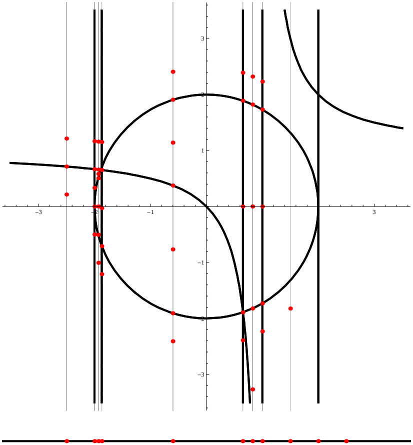

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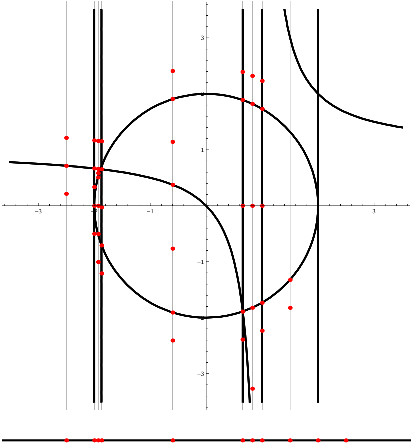

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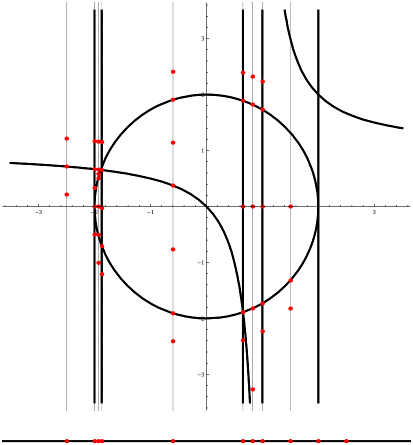

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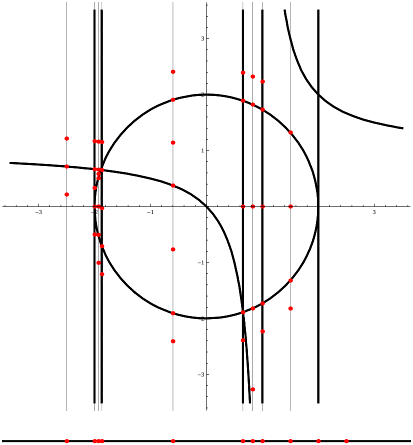

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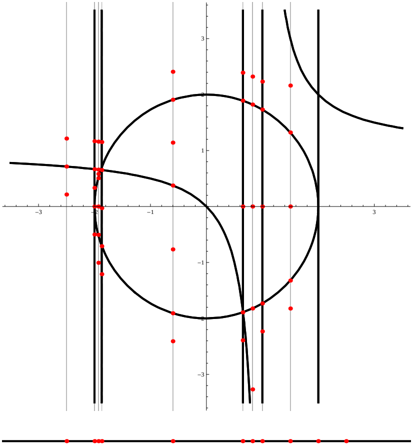

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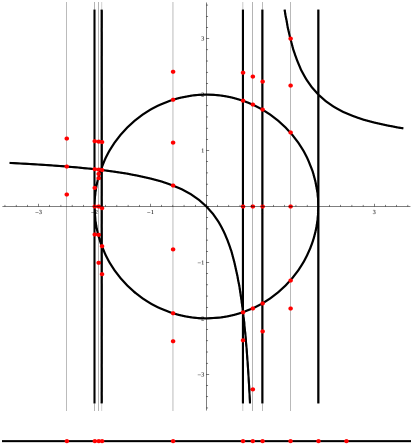

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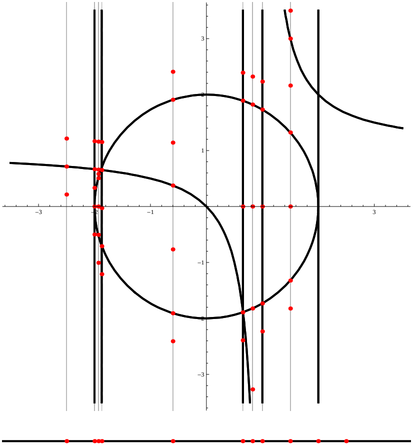

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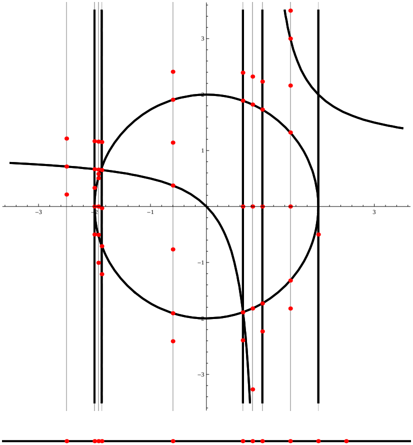

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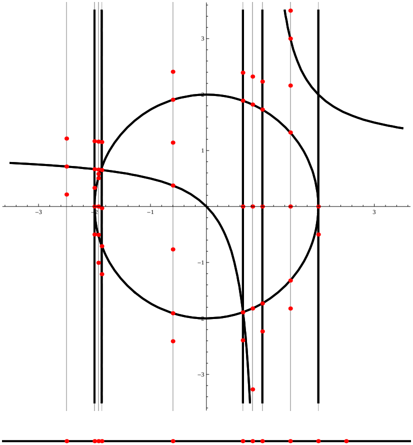

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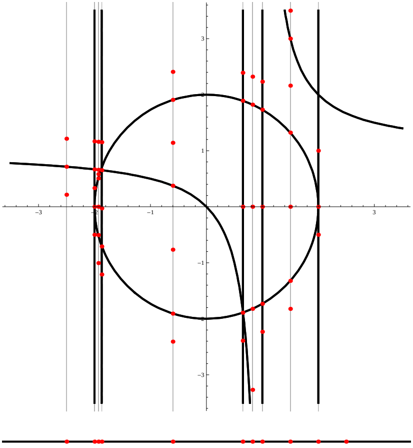

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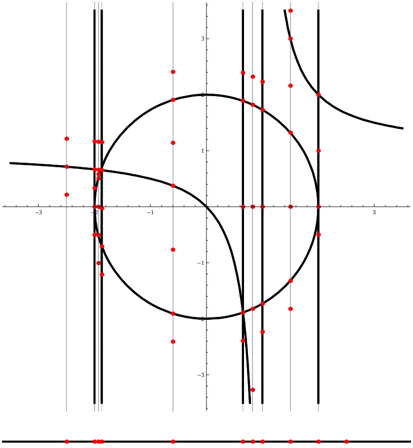

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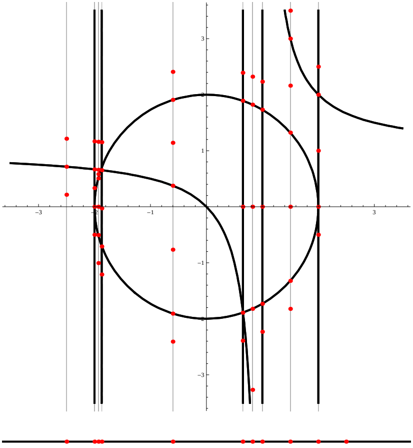

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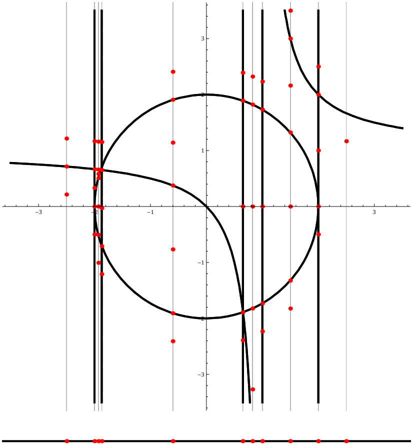

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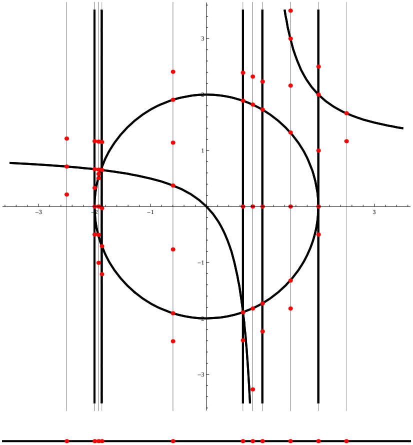

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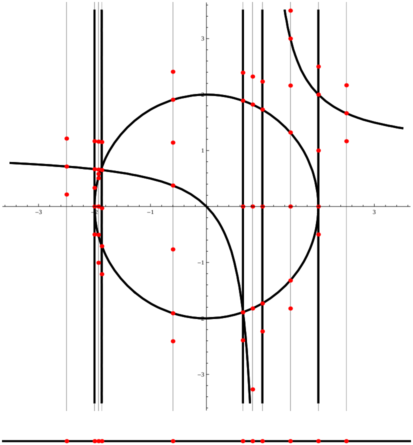

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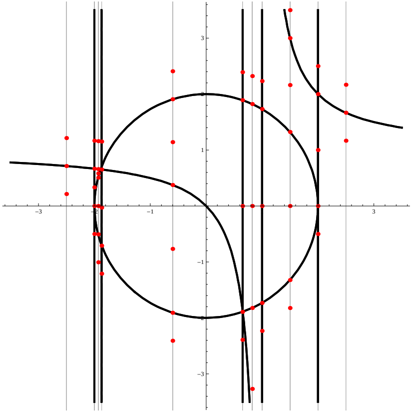

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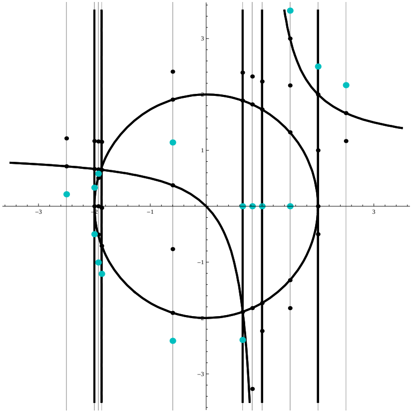

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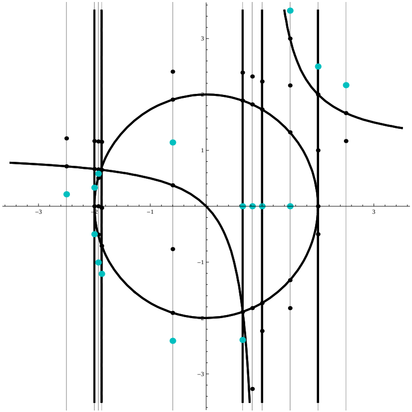

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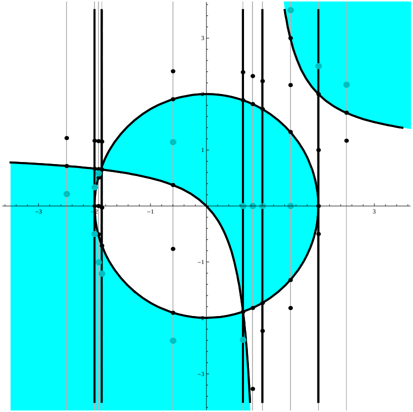

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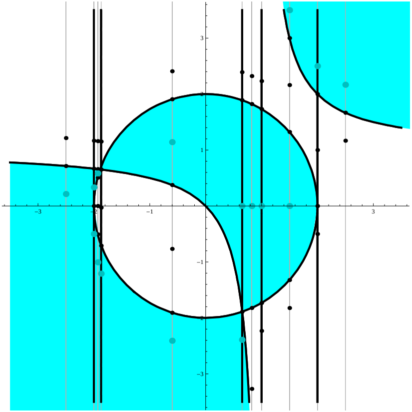
From this, we can extract a *solution formula.*



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Beginning with x_n, we handle one variable after the other.

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A **projection operator** is a function

$$\begin{array}{ccc} A & \longmapsto & P_n(A) \\ \cap & & \cap \\ \mathbb{R}[x_1, \ldots, x_n] & & \mathbb{R}[x_1, \ldots, x_{n-1}] \end{array}$$

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If B is a CAD of $P_n(A)$ in $\mathbb{R}[x_1, \ldots, x_{n-1}]$
then $B \cup A$ is a CAD of A in $\mathbb{R}[x_1, \ldots, x_{n-1}]$.

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Here is one of several known projection operators:

$$P_n(A) := \bigcup_{p \in A} \operatorname{coeffs}_{x_n}(p) \cup \bigcup_{p \in A} \{\operatorname{disc}_{x_n}(p)\} \cup \bigcup_{p,q \in A} \{\operatorname{res}_{x_n}(p, q)\}.$$

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$$:= \begin{vmatrix} * & & & * & & \\ * & * & & * & * & \\ * & * & * & * & * & * \\ * & * & * & & * & * & * \\ & * & * & & & * & * \\ & & * & & & & * \end{vmatrix}$$

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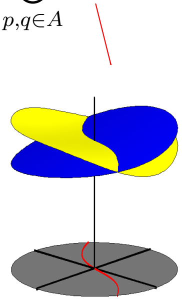

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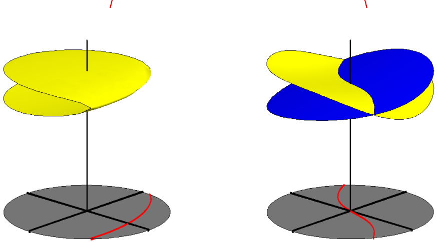

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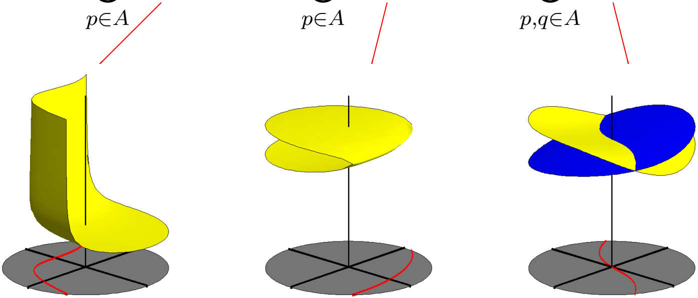

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1. Projection.

The projection algorithm:

INPUT: $A \subseteq \mathbb{Q}[x_1, \ldots, x_n]$
OUTPUT: $C \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ such that $A \subseteq C$ and C is a CAD.

1. $C := A$
2. for $k = n$ down to 2 do
3. $\quad C := C \cup P_k(C \cap \mathbb{Q}[x_1, \ldots, x_k])$
4. return C

The CAD algorithm

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The case of one variable: $p_1(x), p_2(x), \ldots, p_m(x) \in (\bar{\mathbb{Q}} \cap \mathbb{R})[x]$.

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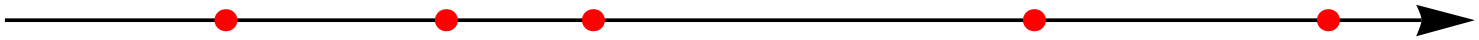

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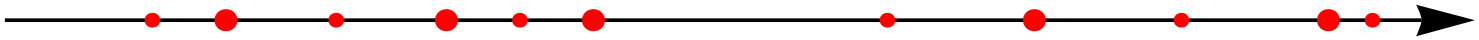

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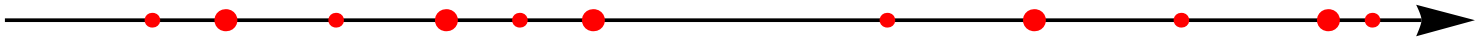

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- The sample points are $\rho_0, \xi_1, \rho_1, \xi_2, \ldots, \rho_{k-1}, \xi_k, \rho_k$.

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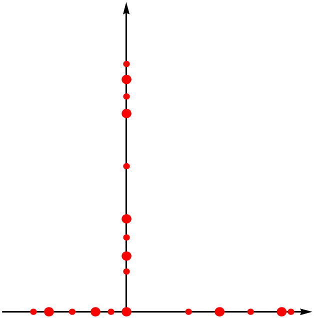

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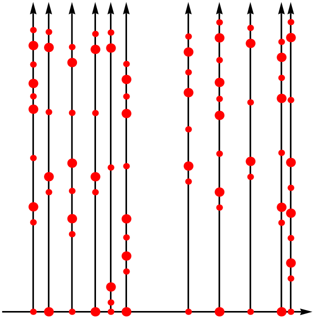

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- The sample points are then $(\sigma_i,\sigma_{i,j}) \in (\bar{\mathbb{Q}} \cap \mathbb{R})^2$.

The CAD algorithm

2. Lifting.

The lifting algorithm:

INPUT: a CAD $C \subseteq \mathbb{Q}[x_1, \ldots, x_n]$
OUTPUT: a set of sample points $\sigma \in (\bar{\mathbb{Q}} \cap \mathbb{R})^n$ for C

1. $S_1 :=$ sample points for $C \cap \mathbb{Q}[x_1]$
2. for $k = 2$ to n do
3. $\quad C_k := C \cap \mathbb{Q}[x_1, \ldots, x_k]$
4. $\quad S_k = \bigcup_{\sigma \in S_{k-1}} \{\sigma\} \times$ sample points for $C_k\big|_{(x_1,\ldots,x_k)=\sigma}$
5. return S_n

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Given $p \in (\bar{\mathbb{Q}} \cap \mathbb{R})[x]$; $\varepsilon > 0$
Find $\xi_1^- < \xi_1^+ < \cdots < \xi_k^- < \xi_k^+ \in \mathbb{Q}$
such that

- $\xi_i^+ - \xi_i^- < \varepsilon \ (i = 1, \ldots, k)$
- every real root of p is contained in exactly one interval (ξ_i^-, ξ_i^+)

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- Simplification is a software engineering challenge, but not problematic in theory.

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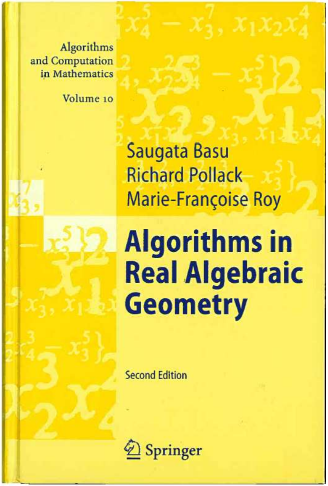
Further Reading



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- *Redlog:* by Andreas Dolzmann, Andreas Seidl, et. al.; Package for the CA-system Reduce; http://www.fmi.uni-passau.de/~redlog/

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Implementations of CAD:

- *Qepcad:* by Hoon Hong, Chris Brown, et. al.; Standalone program; http://www.cs.usna.edu/~qepcad/B/QEPCAD.html
- *Redlog:* by Andreas Dolzmann, Andreas Seidl, et. al.; Package for the CA-system Reduce; http://www.fmi.uni-passau.de/~redlog/
- *Mathematica:* part of the standard distribution from Version 5 on. Command names:
  - `CylindricalDecomposition` (raw CAD) and
  - `Resolve` (quantifier elimination)

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CADable *in theory* $\not\Longrightarrow$ CADable *in practice*

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- $n \ldots$ number of variables (hyper critical!)
- $d \ldots$ maximum degree of input polynomials
- $m \ldots$ number of input polynomials
- $b \ldots$ maximum bitsize of the rational numbers in the input

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- internal improvements (for the programmer of CAD)
- external improvements (for the user of CAD)

Internal Improvements

- Use the most efficient algorithms for computing with real algebraic numbers.

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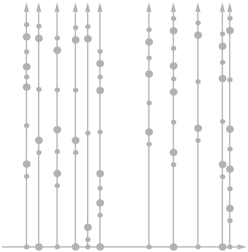
Consider a $\forall\, x\ \exists\, y : \Phi(x, y)$ formula.

Under favorable circumstances, only a small part of the expensive lifting phase has to be carried out in order to decide whether this formula is true or false.



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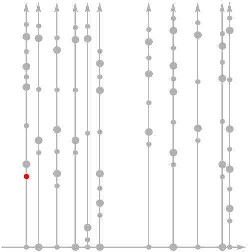

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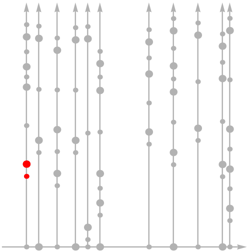

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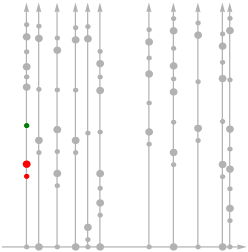

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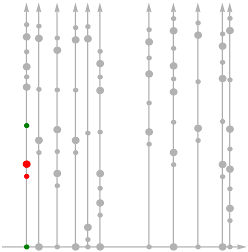

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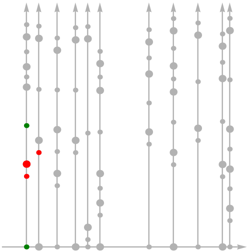

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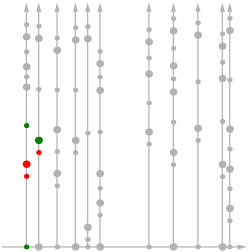

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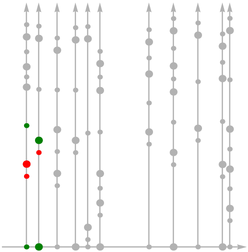

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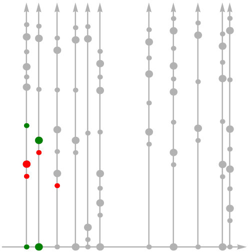

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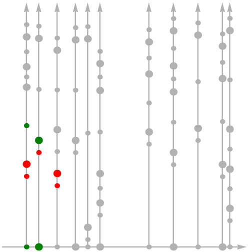

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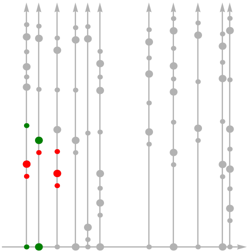

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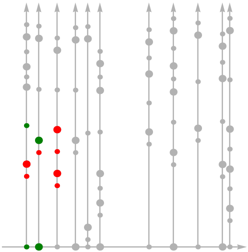

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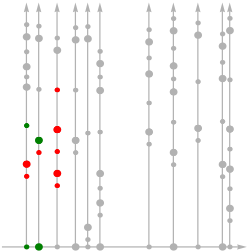

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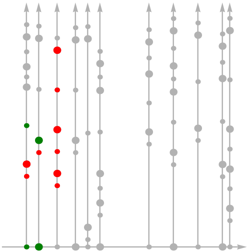

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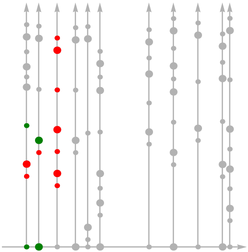

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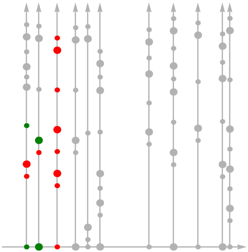

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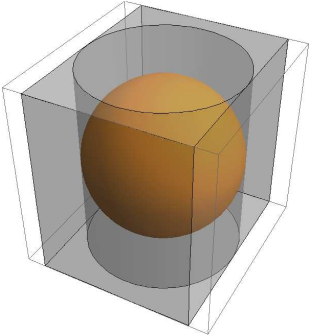
Example: The CAD of the unit sphere has 25 cells.

Only 7 of them are full dimensional.

Only arithmetic in $\mathbb{Q}$ is needed to find them.



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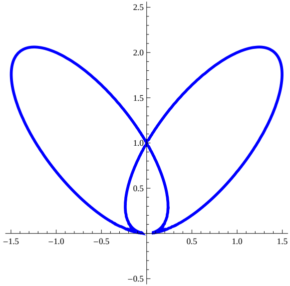
Tomorrow: Applications of CAD to special function inequalities.

A Simple Exercise

What is (pictorially) the CAD of the tacnode polynomial

$$p(x,y) = 2x^4 - 3x^2y + y^4 - 2y^3 + y^2$$

- with respect to x, y?
- with respect to y, x?



Inequalities

Manuel Kauers
RISC-Linz

I. What?

II. How?

III. Why?

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III. Why?

Cylindrical Algebraic Decomposition (CAD)

INPUT: a system of polynomial inequalities over the reals

OUTPUT: a system of polynomial inequalities over the reals, which

- is provably equivalent to the system given as input, and
- has a nice structural property which allows for answering a variety of otherwise nontrivial questions merely by inspection.

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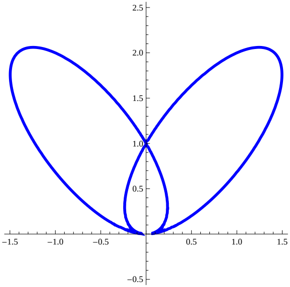

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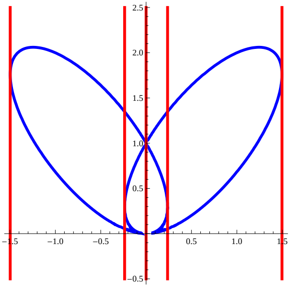

Discriminant of $p(x, y)$ wrt. y:

$$x^6(2048x^6 - 4608x^4 + 37x^2 + 12)$$

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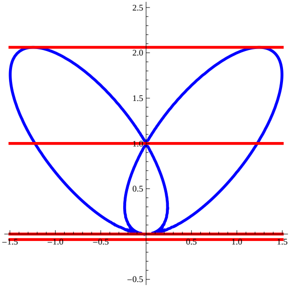

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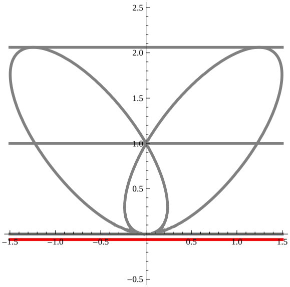
The quadratic factor introduces an unnecessary case distinction.



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Some Recent Monthly Problems

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11033. *Proposed by M. N. Deshpande and R. M. Welukar, Institute of Science, Nagpur, India.* Let

$$P(m,n,r) = \sum_{k=0}^{r} (-1)^k \binom{m+n-2(k+1)}{n} \binom{r}{k}.$$

Let m, n, and r be integers such that $0 \le r \le n \le m-2$. Show that $P(m,n,r)$ is positive and that $\sum_{r=0}^{n} P(m,n,r) = \binom{m+n}{n}$.

Some Recent Monthly Problems

11442. *Proposed by José Díaz-Barrero and José Gibergans-Báguena, Universidad Politécnica de Cataluña, Barcelona, Spain.* Let $\langle a_k \rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \leq \frac{1}{4}\left(\frac{a_1+a_n}{\sqrt{a_1 a_n}}\right).$$

Some Recent Monthly Problems

11445. *Proposed by H. A. ShahAli, Tehran, Iran.* Given $a, b, c > 0$ with $b^2 > 4ac$, let $\langle \lambda_n \rangle$ be a sequence of real numbers, with $\lambda_0 > 0$ and $c\lambda_1 > b\lambda_0$. Let $u_0 = c\lambda_0$, $u_1 = c\lambda_1 - b\lambda_0$, and for $n \geq 2$ let $u_n = a\lambda_{n-2} - b\lambda_{n-1} + c\lambda_n$. Show that if $u_n > 0$ for all $n \geq 0$, then $\lambda_n > 0$ for all $n \geq 0$.

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Today's topic:

- How can CAD be helpful for such problems.

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Bernoulli's inequality:

$$\forall\ n \in \mathbb{N}\ \forall\ x \geq -1 : (x+1)^n \geq 1 + nx.$$

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- View $(x+1)^n - (1+nx)$ as a sequence of polynomials.
- View Bernoulli's inequality as a sequence of polynomial inequalities.

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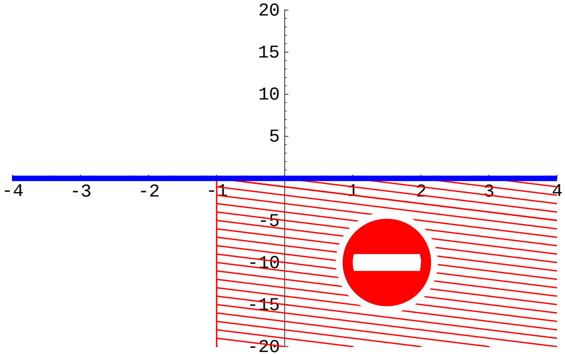

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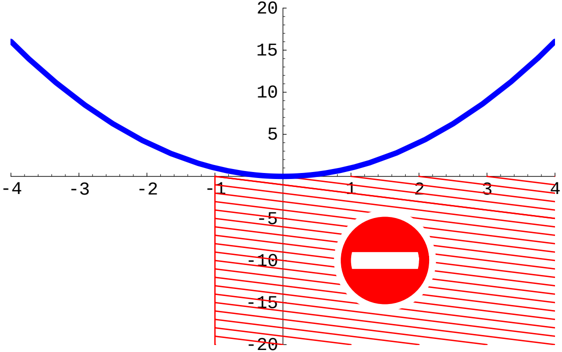

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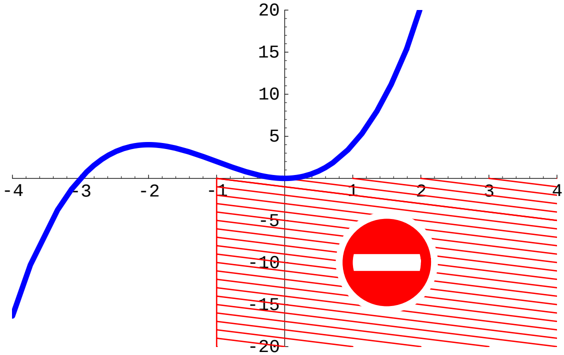

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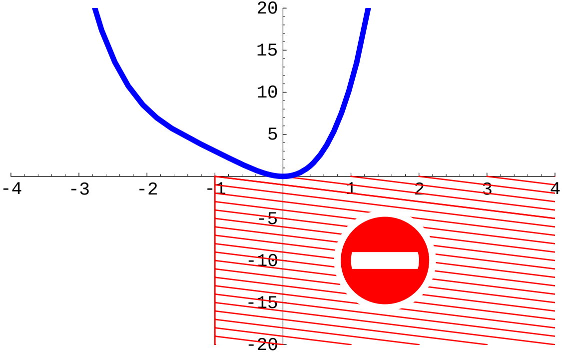

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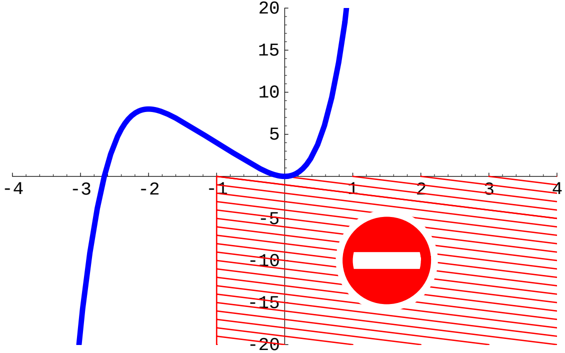

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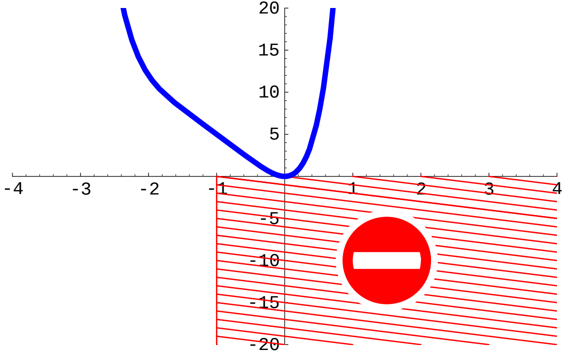

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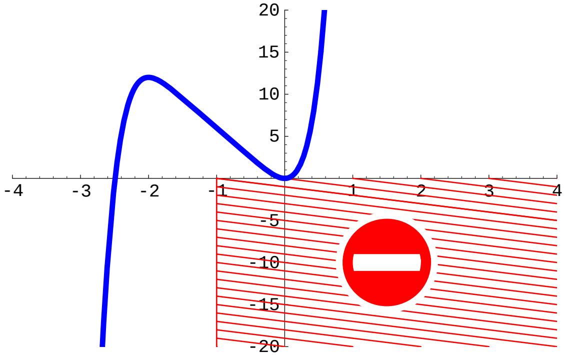

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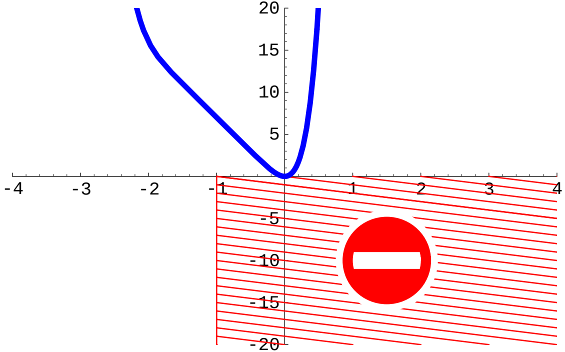

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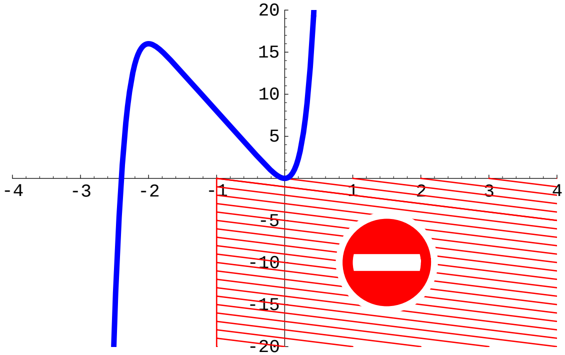

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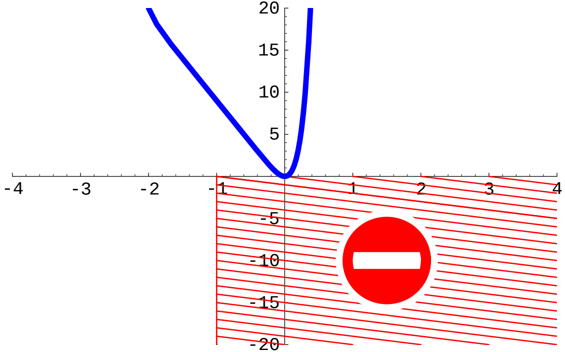

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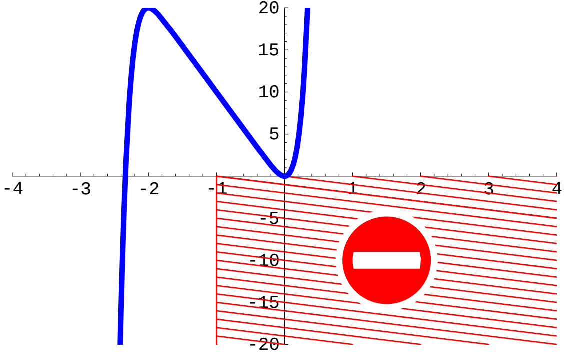

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- The resulting formula is indeed *true.*

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- This completes the proof. ■

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- Done.

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This condition is sufficient but not necessary.

What if it is not true?

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Bernoulli's inequality reloaded:

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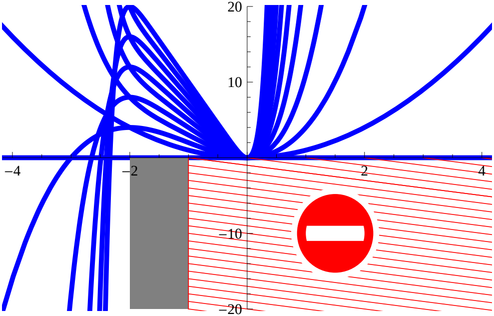

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New idea: Instead of $\Phi(n) \Rightarrow \Phi(n+1)$, try

$$\Phi(n) \wedge \Phi(n+1) \Rightarrow \Phi(n+2)$$

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The extended induction step formula:

$$\forall\ n \geq 0\ \forall\ y\ \forall\ x \geq -2 : y \geq 1 + nx \wedge (x+1)y \geq 1 + (n+1)x$$
$$\Rightarrow (x+1)^2 y \geq 1 + (n+2)x$$

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Check *two* initial values:

$$n = 1: \quad x \geq -2 \Rightarrow (x+1) \geq 1 + 1x \quad ✓$$
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The truth of the inequality follows.

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- Also this does not work for every inequality.
- In general, you have to *experiment!*
- Claim: Finding a CADable reformulation of a conjectured inequality can be much easier than finding a CAD-free proof.

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11033. *Proposed by M. N. Deshpande and R. M. Welukar, Institute of Science, Nagpur, India.* Let

$$P(m,n,r) = \sum_{k=0}^{r} (-1)^k \binom{m+n-2(k+1)}{n} \binom{r}{k}.$$

Let m, n, and r be integers such that $0 \le r \le n \le m-2$. Show that $P(m,n,r)$ is positive and that $\sum_{r=0}^{n} P(m,n,r) = \binom{m+n}{n}$.

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Summation software finds the recurrence

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Sometimes you have got to be lucky…

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(Side remark: The identity can of course also be done by computer algebra.)

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$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \leq \frac{1}{4}\left(\frac{a_1 + a_n}{\sqrt{a_1 a_n}}\right).$$

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Because of

$$\forall\ a > 1 : \tfrac{1}{2}(a^2 + 1) > a,$$

the sequence a_n is increasing.

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Square the claim to get $s_1(n)s_2(n) \leq \frac{(3+a_n)^2}{48a_n}$ where $s_1(n)$ and $s_2(n)$ are the first and the second sum, respectively.

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Besides the defining recurrence of a_n, we have

$$s_1(n) = s_1(n-1) + \tfrac{a_n}{1+a_n}, \quad s_2(n) = s_2(n-1) + \tfrac{1}{a_n(1+a_n)}.$$

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Since a_n is positive and increasing, so are $s_1(n)$ and $s_2(n)$, hence

$$a_n \ge a_1 = 3, \qquad s_1(n) \ge s_1(1) = \tfrac{3}{4}, \qquad s_2(n) \ge s_2(1) = \tfrac{1}{15}.$$

Back to the Monthly Problems

11442. *Proposed by José Díaz-Barrero and José Gibergans-Báguena, Universidad Politécnica de Cataluña, Barcelona, Spain.* Let $\langle a_k \rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \leq \frac{1}{4}\left(\frac{a_1 + a_n}{\sqrt{a_1 a_n}}\right).$$

Since a_n is positive and increasing, so are $s_1(n)$ and $s_2(n)$, hence

$$a_n \geq a_1 = 3, \qquad s_1(n) \geq s_1(1) = \tfrac{3}{4}, \qquad s_2(n) \geq s_2(1) = \tfrac{1}{15}.$$

For $n \geq 3$, we can even assume

$$a_n \geq 13, \qquad s_1(n) \geq \tfrac{211}{84}, \qquad s_2(n) \geq \tfrac{667}{5460}.$$

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CAD proves the induction step formula

$$\forall\, a, s_1, s_2 : \left(a \geq 13 \wedge s_1 \geq \tfrac{211}{84} \wedge s_2 \geq \tfrac{667}{5460} \wedge s_1 s_2 \leq \tfrac{(a+3)^2}{48a}\right)$$
$$\Rightarrow \frac{(a^2(s_1+1) + 3s_1 + 1)((a^4 + 4a^2 + 3)s_2 + 4)}{(a^2+1)(a^2+3)^2} \leq \frac{(a^2+7)^2}{96(a^2+1)}.$$

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Now the problem is solved by checking the inequality for $n = 1, 2, 3$.

Back to the Monthly Problems

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For $n \mapsto n + 1$, we use CAD:

$$\forall\ a, b, c, \lambda, \lambda', \lambda'' : \Big(a > 0 \wedge b > 0 \wedge c > 0 \wedge b^2 > 4ac \\ \wedge\ a\lambda - b\lambda' + c\lambda'' > 0 \wedge \lambda' > \frac{b}{2c}\lambda > 0\Big) \Rightarrow \lambda'' > \frac{b}{2c}\lambda'.$$

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Just a crazy way to solve some more Monthly Problem?

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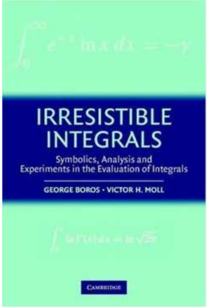

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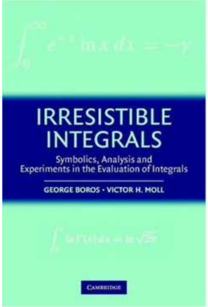

One of his absolute favorites:

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx$$

where $a > -1$ is real and $m \geq 0$ is an integer.

Moll's Conjecture

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General formula:

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi P_m(a)}{2^{m+3/2}(a+1)^{m+1/2}}$$

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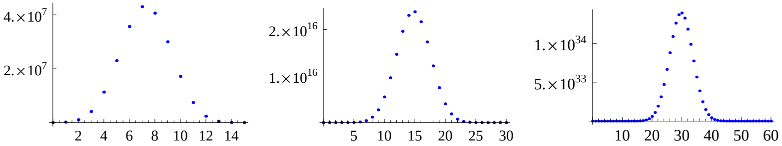

$m = 15$ $m = 30$ $m = 60$

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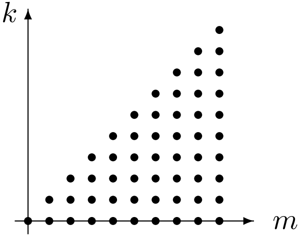
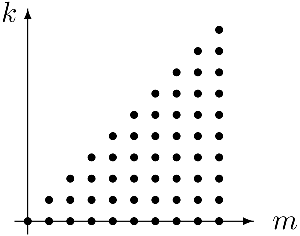
What else can we say about the $d_k(m)$?

Moll's Conjecture



Theorem (Moll) $d_k(m) > 0$

Moll's Conjecture



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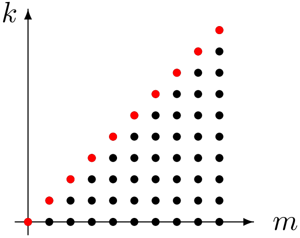

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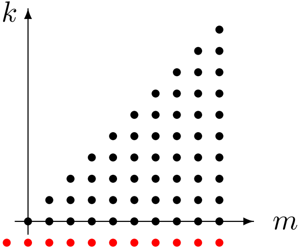

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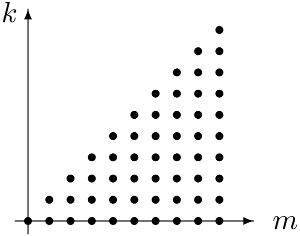

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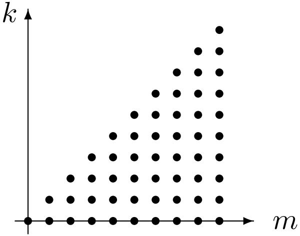

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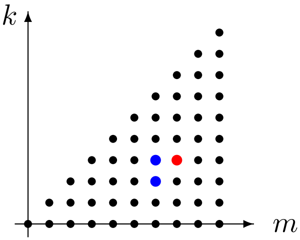

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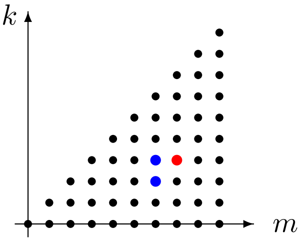

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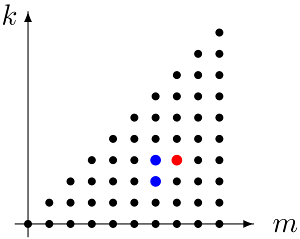

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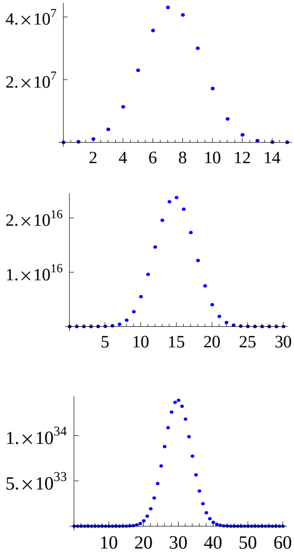

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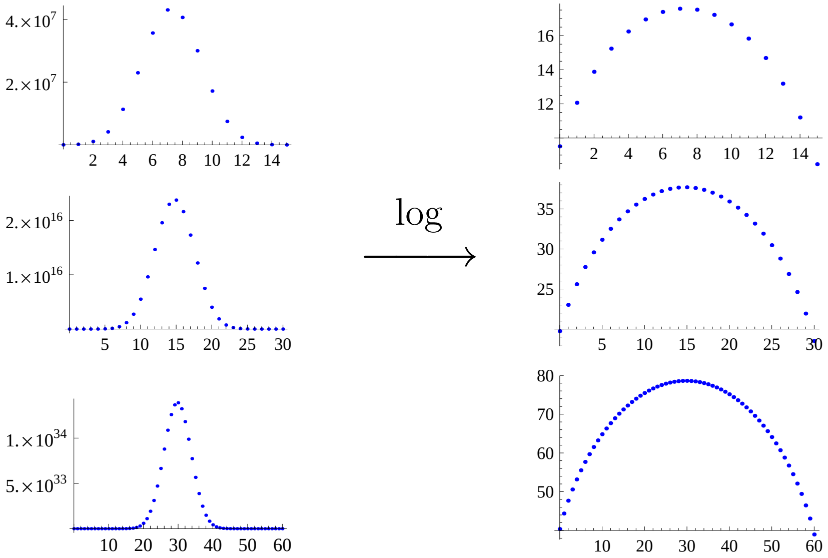

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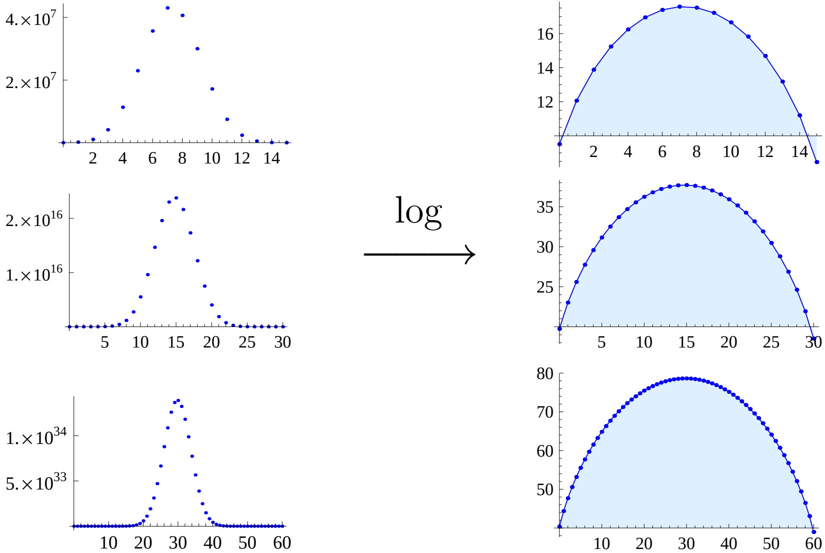

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Theorem (Kauers/Paule, 2007): That's true.

Moll's Conjecture

Proof Outline:

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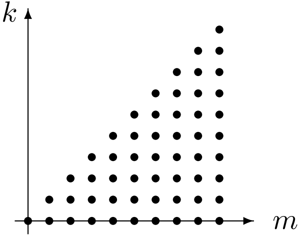

Relations between:

Moll's Conjecture

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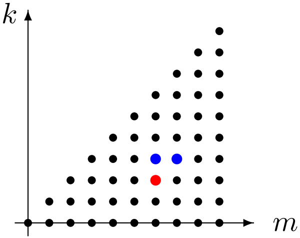

Relations between:

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Moll's Conjecture

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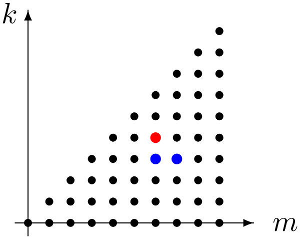

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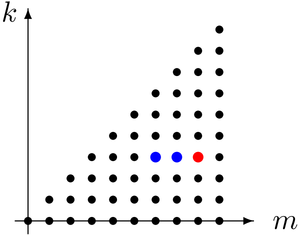

Relations between:

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(b) $d_{k+1}(m)$, $d_k(m+1)$, $d_k(m)$.

(c) $d_k(m+2)$, $d_k(m+1)$, $d_k(m)$.

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Rewrite $d_{k-1}(m)$ and $d_{k+1}(m)$ in terms of $d_k(m)$ and $d_k(m+1)$.

To show:

$$(16km^2 + 28km + 9k + 16m^3 + 40m^2 + 33m + 9)d_k(m)^2$$
$$4(m+1)(2k^2 - 4m^2 - 7m - 3)d_k(m+1)d_k(m)$$
$$-4(m+1)^2(k-m-1)d_k(m+1)^2 \geq 0$$

Moll's Conjecture

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Induction step formula:

$$\forall\, m\, \forall\, k\, \forall\, D_0\, \forall\, D_1 : \Big(0 < k < m \wedge D_0 > 0 \wedge D_1 > 0$$
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This is false.

Moll's Conjecture

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In the range of interest, this is equivalent to

$$0 < m \leq \tfrac{1}{2} + \sqrt{2} \vee 0 < k \leq \mathrm{algfun}(m)$$

for some cubic algebraic function algfun.

Moll's Conjecture

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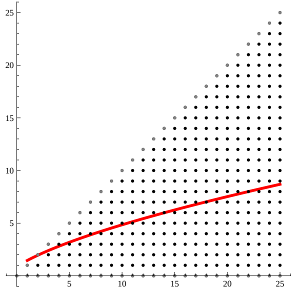

This algebraic function splits the region into two parts.

Moll's Conjecture

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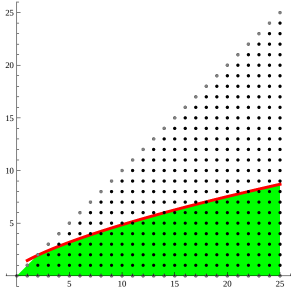

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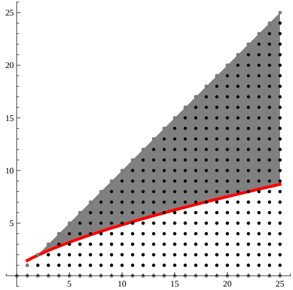

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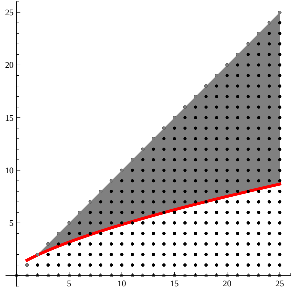

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What's going wrong there?

Moll's Conjecture

4. For these (m, k), switch to a nicer but stronger statement.

Back to the induction step formula:

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4. For these (m, k), switch to a nicer but stronger statement.

$$0 < m \leq \tfrac{1}{2} + \sqrt{2} \vee 0 < k \leq \mathrm{algfun}(m) \wedge D_0 > 0$$
$$\wedge \frac{p_1(m,k) - \sqrt{p_2(m,k)}}{p_3(m,k)} D_0 < D_1 < \frac{p_1(m,k) + \sqrt{p_2(m,k)}}{p_3(m,k)} D_0$$

for some polynomials $p_1(m, k)$, $p_2(m, k)$, $p_3(m, k)$.

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for some polynomials $p_1(m, k)$, $p_2(m, k)$, $p_3(m, k)$.

Meaning: if some (m, k) in the gray area is really a counterexample, then for this (m, k) we must have

$$d_k(m+1) < \frac{p_1(m,k) + \sqrt{p_2(m,k)}}{p_3(m,k)} d_k(m).$$

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4. For these (m, k), switch to a nicer but stronger statement.

We are done if we can prove

$$d_k(m+1) \geq \frac{p_1(m,k) + \sqrt{p_2(m,k)}}{p_3(m,k)} d_k(m).$$

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- Worse, because there is a radical.

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Idea: Introduce under the root a (small) positive polynomial $u(m, k)$ that turns $p_2(m, k) + u(m, k)$ into a square.

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Suitable polynomials $u(m, k)$ are easy to find.

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This completes the proof.

So what?

Just a crazy way to solve some more Monthly Problem?

No! This is strong enough to prove open conjectures

1. Moll's log-concavity conjecture (Kauers, Paule, 2007)
2. Alzer's conjecture (Alzer, Gerhold, Kauers, Lupas, 2007)
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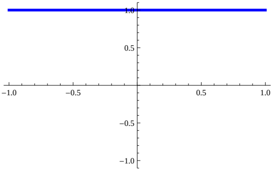

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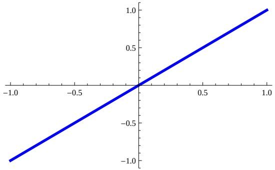

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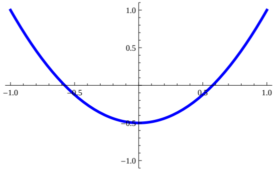

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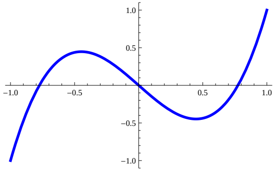

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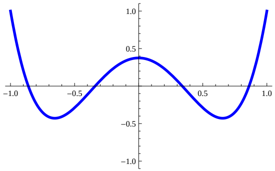

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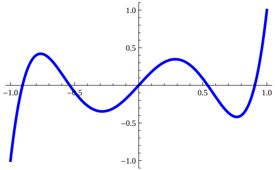

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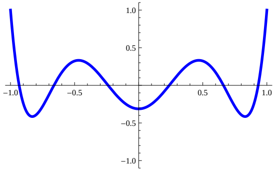

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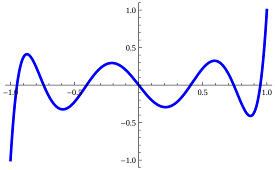

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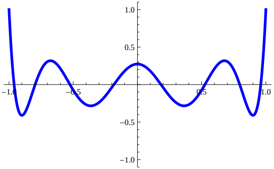
Alzer's Conjecture



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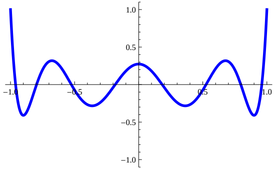
Alzer's Conjecture

This is about *Legendre Polynomials* $P_n(x)$.

These polynomials form one of the classical families of *orthogonal polynomials.*



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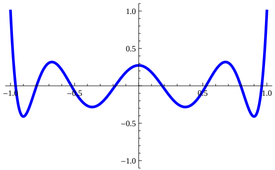

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As such, they satisfy lots of useful identities, including

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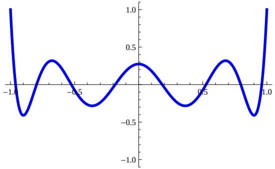

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There are also some interesting inequalities, including

$$\forall\ n \in \mathbb{N}\ \forall\ x \in [-1,1] : -1 \leq P_n(x) \leq 1.$$

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Here is another example:

$$\forall\ n \in \mathbb{N}\ \forall\ x \in [-1,1] :\ P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0$$

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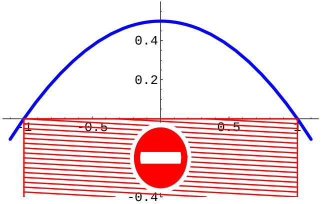

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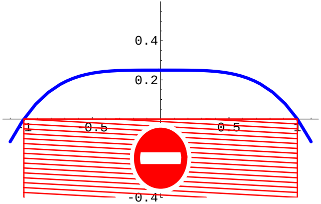

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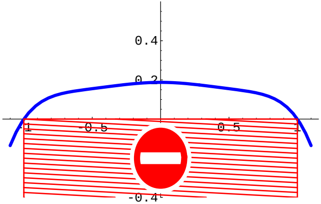

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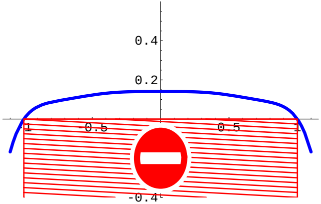

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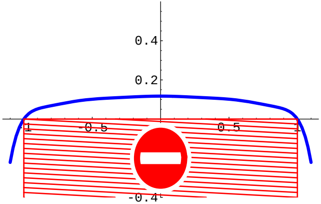

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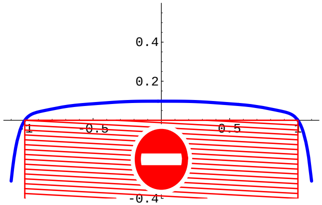

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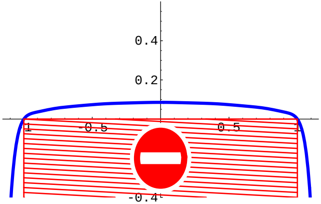

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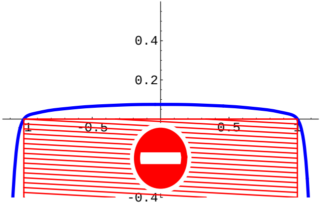

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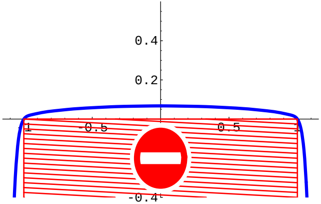

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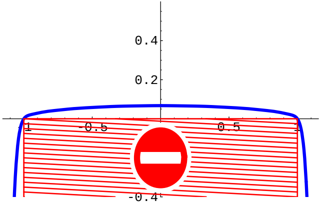

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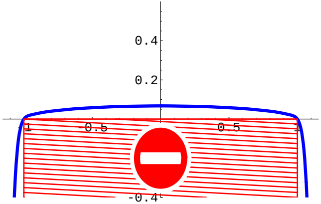

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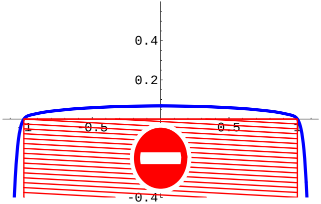

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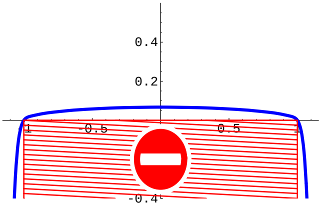

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- For specific n, it is just a polynomial inequality.
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A proof for general n can be obtained in the same way as for Bernoulli's inequality using induction, recurrences, and CAD.

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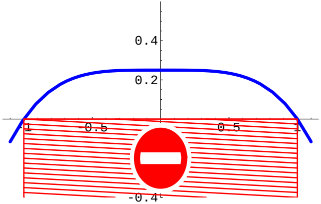
Alzer conjectured that Turan's inequality

$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0$$



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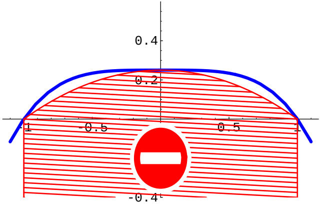

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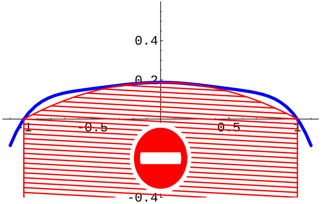

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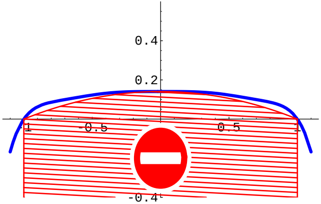

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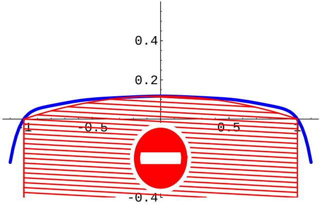

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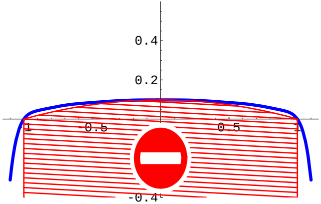

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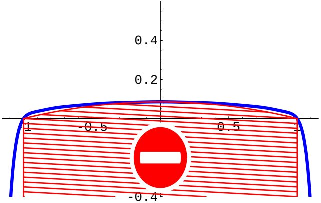

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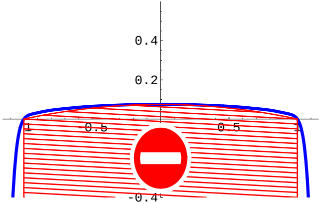

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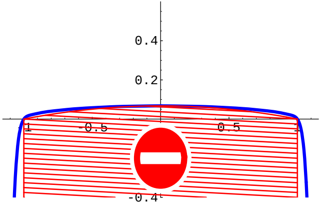

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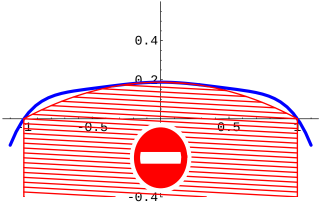

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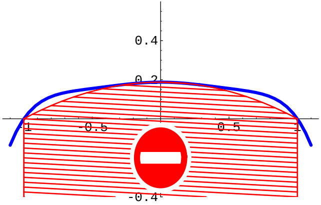

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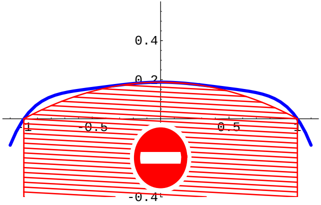

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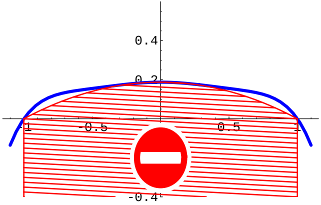

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The obvious induction step formula is *large* and *false.*

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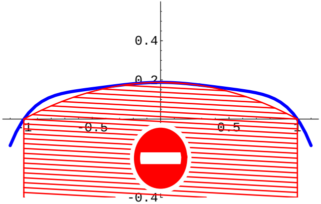

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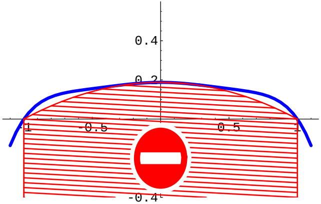

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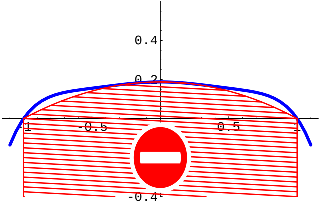

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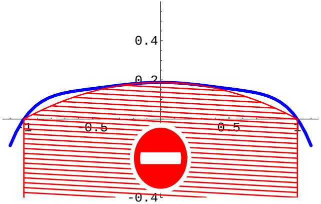

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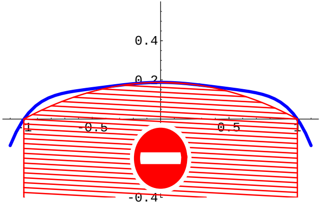

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New idea: Show that $\frac{d}{dx}\frac{\Delta_n(x)}{1 - x^2} \geq 0$

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A positivity proof for the latter expression by CAD and induction on n succeeds.

So what?

Just a crazy way to solve some more Monthly Problem?

No! This is strong enough to prove open conjectures

1. Moll's log-concavity conjecture (Kauers, Paule, 2007) ✓
2. Alzer's conjecture (Alzer, Gerhold, Kauers, Lupas, 2007)
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All three proofs depend heavily on CAD computations.

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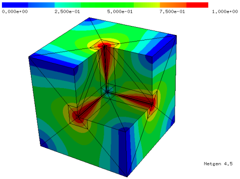

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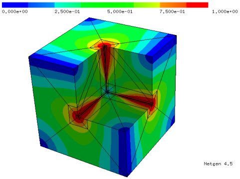

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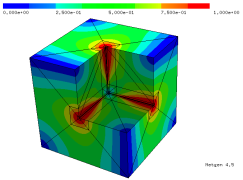

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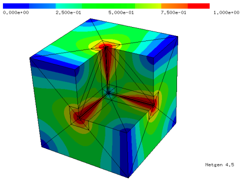

- Good basis functions have good properties.
- What a good properties are, this depends on the particular application.

Schöberl's Conjecture

- For one particular application, Schöberl chose

$$f_n(x) := \frac{1}{2x(n+1)} \sum_{k=n}^{2n} (k+1)(P_{k+1}(x)P_k(0) - P_{k+1}(0)P_k(x))$$

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- Hence was born the Schöberl conjecture.

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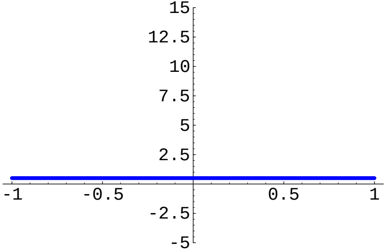
Consider

$$S_n(x) := \sum_{k=0}^{n} (4k+1)(2n-2k+1) P_{2k}(0) P_{2k}(x)$$

for $n = 0, 1, \ldots, 20$.



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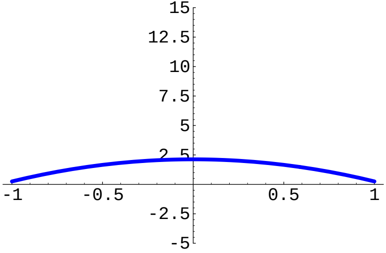

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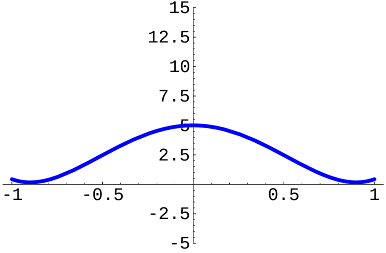

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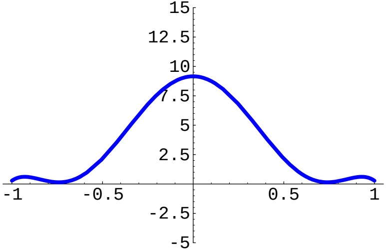

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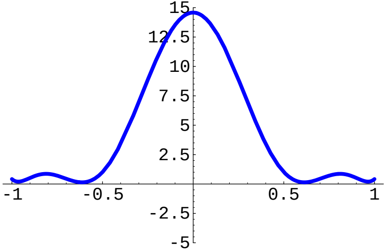

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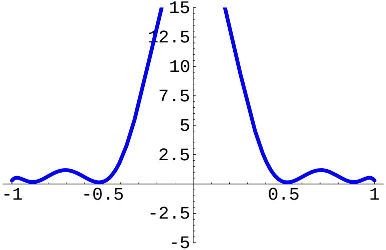

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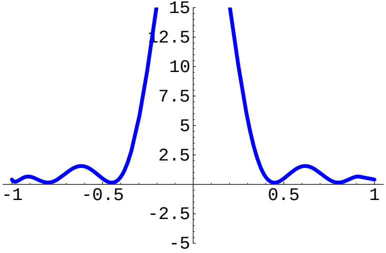

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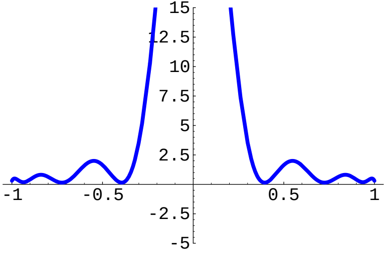

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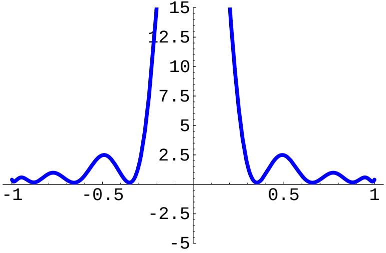

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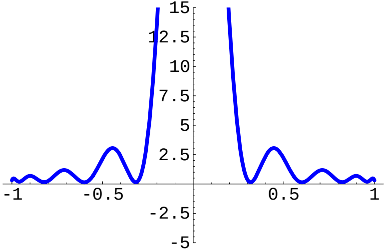

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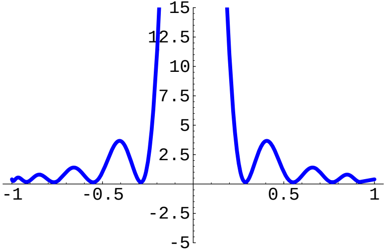

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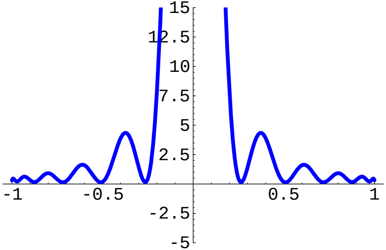

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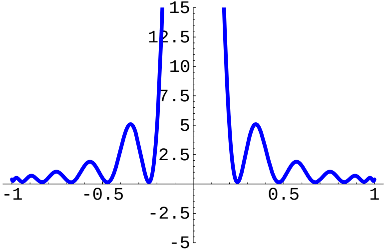

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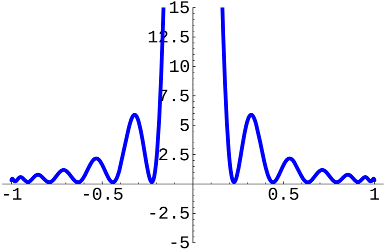

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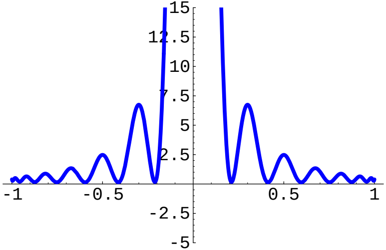

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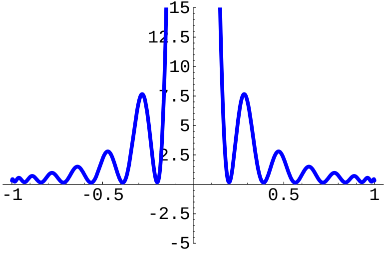

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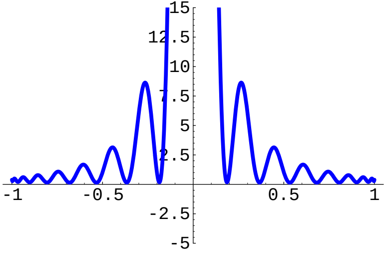

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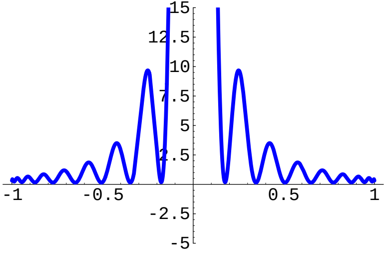

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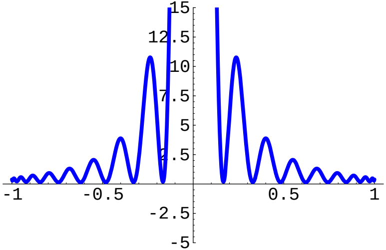

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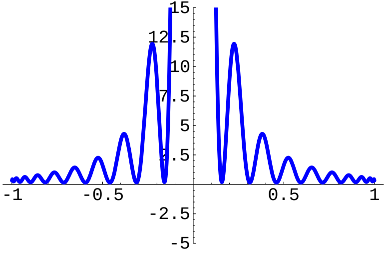

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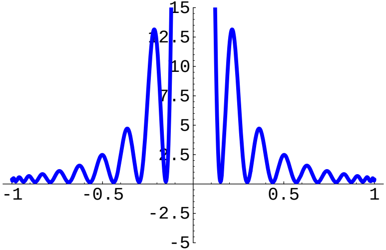

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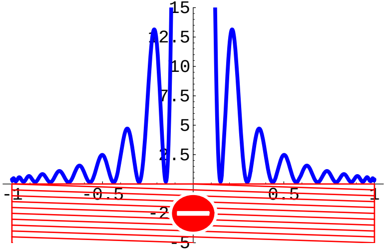

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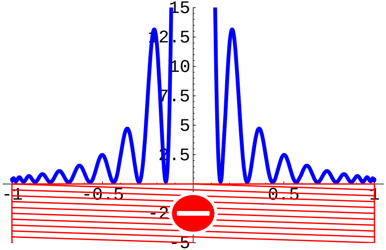

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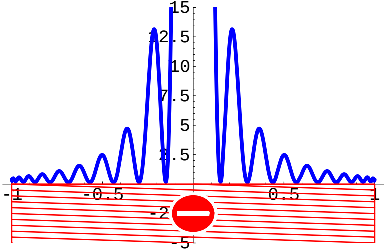

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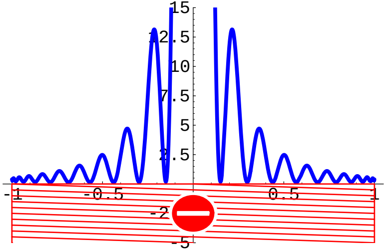

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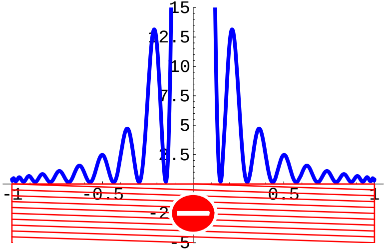

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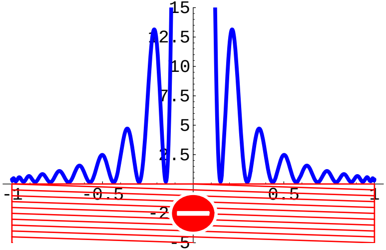

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- For "symbolic" n and x: *not easy at all!*

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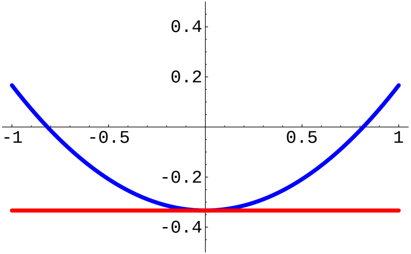

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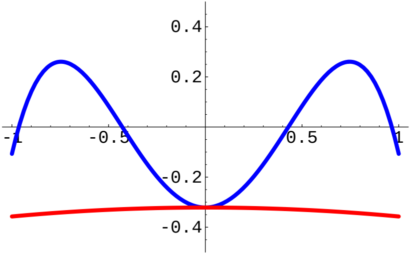

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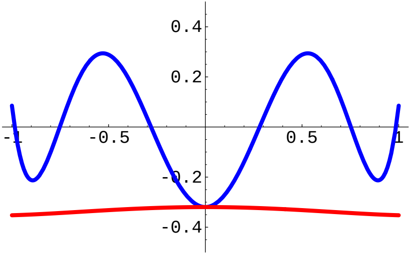

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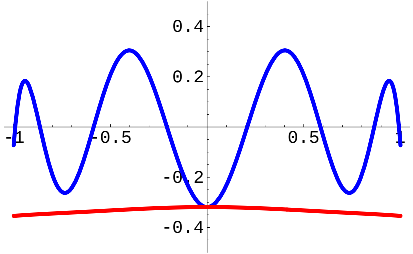

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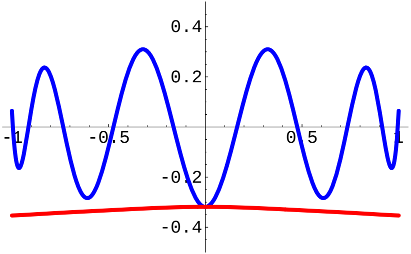

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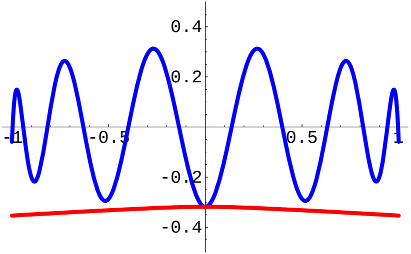

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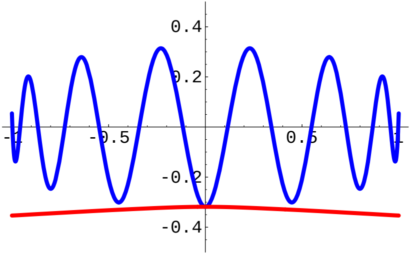

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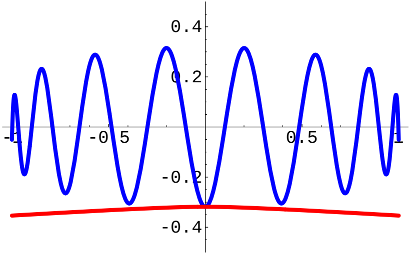

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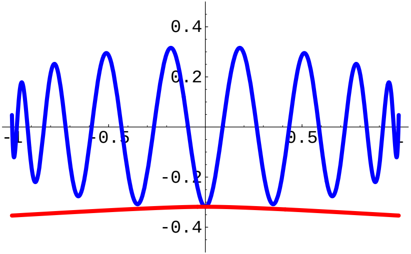

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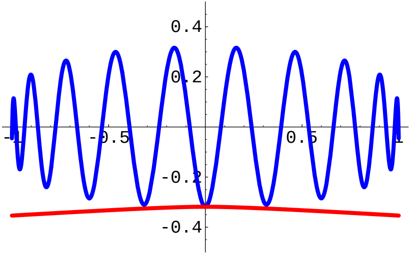

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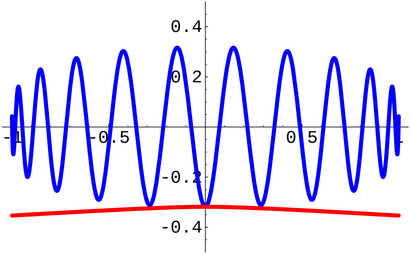

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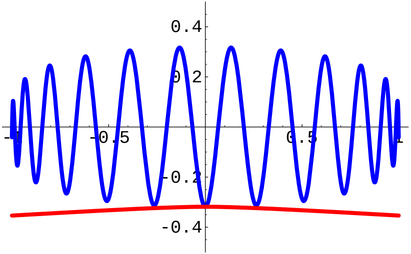

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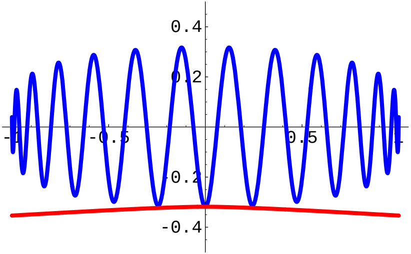

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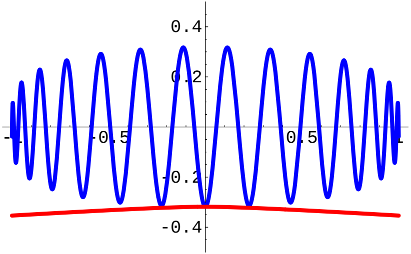

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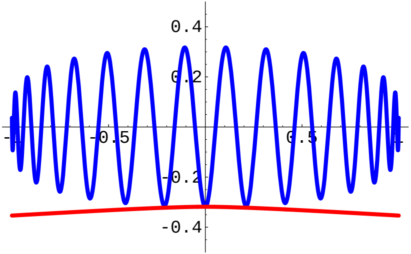

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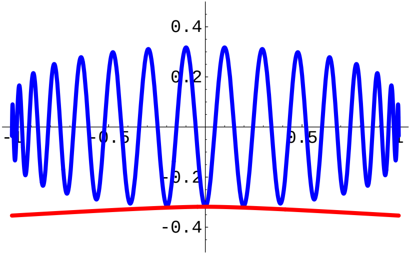

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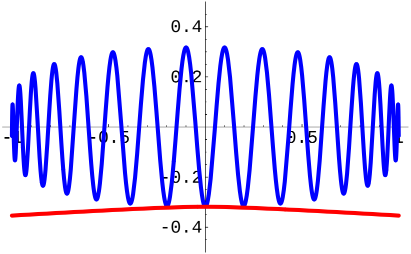
$$\underbrace{(2n+1)P_{2n}(0)\Big(xP_{2n+1}(x) - \tfrac{2(2n+1)}{4n+3}P_{2n}(x)\Big)}_{\text{no sum}} \overset{?}{\geq} \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)}.$$



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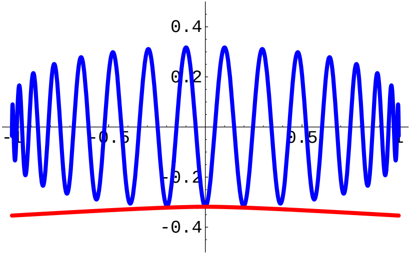

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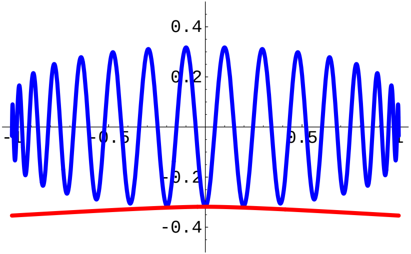

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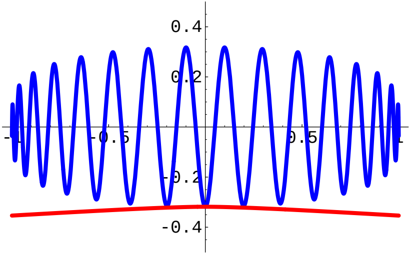

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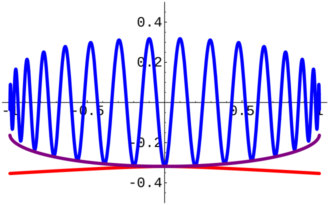

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It suffices to prove the stronger statement

$$P_{2n}(0)\big(xP_{2n+1}(x) - \tfrac{2(2n+1)}{4n+3}P_{2n}(x)\big) \overset{?}{\geq} xP_{2n}(x)P_{2n+1}(x) - \tfrac{2n+1}{4n+3}P_{2n}(x)^2 - \tfrac{2n+1}{4n+1}P_{2n+1}(x)^2 - \tfrac{2n+1}{4n+3}P_{2n}(0)^2.$$

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- *Punch line:* Both the human part and the CAD part are nontrivial.

So what?

Just a crazy way to solve some more Monthly Problem?

No! This is strong enough to prove open conjectures

1. Moll's log-concavity conjecture (Kauers, Paule, 2007) ✓
2. Alzer's conjecture (Alzer, Gerhold, Kauers, Lupas, 2007) ✓
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All three proofs depend heavily on CAD computations.

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- It's not clear a priori what "appropriate" means.

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then all $a_{n,m,k,l}$ are positive.

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A Simple Exercise

Prove, by whatever method you prefer, the following three inequalities:

- $$\sum_{k=1}^{n} \frac{L_k^2}{F_k} \geq \frac{(L_{n+2}-3)^2}{F_{n+2}-1} \qquad (n \geq 2)$$
- $$\left(\sum_{k=1}^{n} \sqrt{k}\right)^2 \leq \left(\sum_{k=1}^{n} \sqrt[3]{k}\right)^3 \qquad (n \geq 0)$$
- $$\prod_{k=1}^{n}(1-a_k) < \frac{1}{1+\sum\limits_{k=1}^{n} a_k} \qquad (n \geq 1;\ a_1, \ldots, a_k \in (0,1))$$