A Pfaffian analogue of the q-Catalan Hankel determinant [1]

Masao Ishikawa†

†Department of Mathematics
Tottori University

65th Séminaire Lotharingien de Combinatoire
September 12–15, 2010, Strobl (Wolfgangsee), Austria.

[1] Joint work with Hiroyuki Tagawa and Jiang Zeng

Abstract

We give a Pfaffian analogue of $\det\left(\frac{(a;q)_{i+j-2}}{(abq^2;q)_{i+j-2}}\right)$ and its proof.

$\mu_n = \frac{(a;q)_n}{(abq^2;q)_n}$ are the moments of the little q-Jacobi polynomials and we regard it as a q-generalization of Catalan numbers C_n.
Hence our main result is

$$\mathrm{Pf}\left((q^{i-1}-q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\le i,j\le 2n}$$
$$= a^{n(n-1)}q^{n(n-1)(4n+1)/3+n(n-1)r}$$
$$\times\prod_{k=1}^{n-1}(bq;q)_{2k}\prod_{k=1}^{n}\frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

We use LU type decomposition for skew-symmetric matrices arising from Plücker relations for Pfaffians.

References

In this talk

1. Macdonald's book = I. G. Macdonald, *Symmetric Functions and Hall Polynomials (2nd ed.)*, Oxford Univ. Press, (1995).
2. Gapser-Rahman's book = G. Gasper and M. Rahman, *Basic Hypergeometric Series (2nd ed.)*, Cambridge Univ. Press, (1990, 2004).
3. ITZ = M. Ishikawa, Hi. Tagawa and J. Zeng, A q-analogue of Catalan Hankel determinants *RIMS Kôkyûroku Bessatsu*, **B11** (2009), 19–42.
4. C. Krattenthaler, "Evaluations of Some Determinants of Matrices Related to the Pascal Triangle", *Séminaire Lotharingien de Combinatoire*, **B47g** (2002), 19 pp.

References

In this talk

1. Macdonald's book = I. G. Macdonald, *Symmetric Functions and Hall Polynomials (2nd ed.)*, Oxford Univ. Press, (1995).
2. Gapser-Rahman's book = G. Gasper and M. Rahman, *Basic Hypergeometric Series (2nd ed.)*, Cambridge Univ. Press, (1990, 2004).
3. ITZ = M. Ishikawa, Hi. Tagawa and J. Zeng, A q-analogue of Catalan Hankel determinants *RIMS Kôkyûroku Bessatsu*, **B11** (2009), 19–42.
4. C. Krattenthaler, "Evaluations of Some Determinants of Matrices Related to the Pascal Triangle", *Séminaire Lotharingien de Combinatoire*, **B47g** (2002), 19 pp.

References

In this talk

1. Macdonald's book = I. G. Macdonald, *Symmetric Functions and Hall Polynomials (2nd ed.)*, Oxford Univ. Press, (1995).
2. Gapser-Rahman's book = G. Gasper and M. Rahman, *Basic Hypergeometric Series (2nd ed.)*, Cambridge Univ. Press, (1990, 2004).
3. ITZ = M. Ishikawa, Hi. Tagawa and J. Zeng, A q-analogue of Catalan Hankel determinants *RIMS Kôkyûroku Bessatsu*, **B11** (2009), 19–42.
4. C. Krattenthaler, "Evaluations of Some Determinants of Matrices Related to the Pascal Triangle", *Séminaire Lotharingien de Combinatoire*, **B47g** (2002), 19 pp.

References

In this talk

1. Macdonald's book = I. G. Macdonald, *Symmetric Functions and Hall Polynomials (2nd ed.)*, Oxford Univ. Press, (1995).
2. Gapser-Rahman's book = G. Gasper and M. Rahman, *Basic Hypergeometric Series (2nd ed.)*, Cambridge Univ. Press, (1990, 2004).
3. ITZ = M. Ishikawa, Hi. Tagawa and J. Zeng, A q-analogue of Catalan Hankel determinants *RIMS Kôkyûroku Bessatsu*, **B11** (2009), 19–42.
4. C. Krattenthaler, "Evaluations of Some Determinants of Matrices Related to the Pascal Triangle", *Séminaire Lotharingien de Combinatoire*, **B47g** (2002), 19 pp.

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Plan of My Talk

1. Catalan Hankel determinant and its q-analogue
2. Notation
3. A Pfaffian analogue of the q-Catalan Hankel determinant
4. Pfaffian decomposition
5. A proof by Pfaffian decomposition
6. Open problems

Catalan numbers

Definition

For $n = 0, 1, 2, \ldots,$ The Catalan number C_n is defined to be

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan number C_n counts the Dyck paths from $(0, 0)$ to $(2n, 0)$.

Example

The generating function for the Catalan numbers is given by

$$\frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} C_n t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots.$$

Catalan numbers

Definition

For $n = 0, 1, 2, \ldots,$ The Catalan number C_n is defined to be

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan number C_n counts the Dyck paths from $(0, 0)$ to $(2n, 0)$.

Example

The generating function for the Catalan numbers is given by

$$\frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n \geq 0} C_n t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots .$$

Catalan Hankel determinants

Theorem (Desainte-Catherine-Viennot 1986)

For the Catalan numbers C_0, C_1, C_2,..., let

$$C_n^{(r)} = (C_{i+j+r-2})_{1 \le i,j \le n}$$

denote the Hankel matrix. Then

$$\det C_n^{(r)} = \prod_{0 \le i \le j \le r-1} \frac{i+j+2n}{i+j}.$$

holds for $r, n \ge 0$. For example,

$$\det C_n^{(0)} = \det C_n^{(1)} = 1,$$
$$\det C_n^{(2)} = n+1,$$
$$\det C_n^{(3)} = \frac{1}{6}(n+1)(n+2)(2n+3).$$

Notation

q-shifted factorials

We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

for $n = 0, 1, 2, \ldots$. $(a;q)_n$ is called the *q-shifted factorial*.
Frequently used compact natation:

$$(a_1, a_2, \ldots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

q-shifted factorials

We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

for $n = 0,1,2,\ldots$. $(a;q)_n$ is called the *q-shifted factorial*.

Frequently used compact natation:

$$(a_1,a_2,\ldots,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_r;q)_\infty,$$

$$(a_1,a_2,\ldots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n.$$

Notation

q-shifted factorials

We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

for $n = 0, 1, 2, \ldots$. $(a;q)_n$ is called the *q-shifted factorial*.

Frequently used compact natation:

$$(a_1, a_2, \ldots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

More Notation

Raising factorials

If we put $a = q^{\alpha}$, then

$$\lim_{q\to 1} \frac{(q^{\alpha};q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{(1-q^{\alpha})}{(1-q)} \frac{(1-q^{\alpha+1})}{(1-q)} \cdots \frac{(1-q^{\alpha+n-1})}{(1-q)}$$
$$= (\alpha)(\alpha+1)\cdots(\alpha+n-1).$$

We write $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha+k)$, which is called the *raising factorial*.

More Notation

Raising factorials

If we put $a = q^{\alpha}$, then

$$\lim_{q\to 1} \frac{(q^{\alpha};q)_n}{(1-q)^n} = \lim_{q\to 1} \frac{(1-q^{\alpha})}{(1-q)} \frac{(1-q^{\alpha+1})}{(1-q)} \cdots \frac{(1-q^{\alpha+n-1})}{(1-q)}$$
$$= (\alpha)(\alpha+1)\cdots(\alpha+n-1).$$

We write $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha+k)$, which is called the *raising factorial*.

Basic hypergeomrtric series

We shall define the ${}_{r+1}\phi_r$ *basic hypergeomrtric series* by

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. $$

Hypergeomrtric series

If we put $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_i}$ in the above series and let $q \to 1$, then we obtain the ${}_{r+1}F_r$ *hypergeomrtric series*

$$ {}_{r+1}F_r \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_{r+1})_n}{n! (\beta_1)_n \dots (\beta_r)_n} z^n. $$

Basic hypergeomrtric series

Basic hypergeomrtric series

We shall define the ${}_{r+1}\phi_r$ *basic hypergeomrtric series* by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. $$

Hypergeomrtric series

If we put $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_i}$ in the above series and let $q \to 1$, then we obtain the ${}_{r+1}F_r$ *hypergeomrtric series*

$$ {}_{r+1}F_r\left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_{r+1})_n}{n! (\beta_1)_n \dots (\beta_r)_n} z^n. $$

The little q-Jacobi polynomials

Definition

The *little q-Jacobi polynomials* are defined by

$$p_n(x;a,b;q) = \frac{(aq;q)_n}{(abq^{n+1};q)_n}(-1)^n q^{\binom{n}{2}} {}_2\phi_1\left[\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, xq\right],$$

which are orthogonal with respect to the *inner product* defined by

$$\langle f, g\rangle = \frac{(abq^2;q)_\infty}{(aq;q)_\infty}\sum_{k=0}^{\infty}\frac{(bq;q)_k}{(q;q)_k}(aq)^k f\left(q^k\right) g\left(q^k\right).$$

Moments of the little q-Jacobi polynomials

Moments of the little q-Jacobi polynomials

Here we consider the series

$$\mu_n = \langle x^n, 1 \rangle = \frac{(aq;q)_n}{(abq^2;q)_n} \quad (n = 0, 1, 2, \dots).$$

Specializations

If we put $a = q^\alpha$, $b = q^\beta$ and let $q \to 1$, then

$$\mu_n \to \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Note that

$$\frac{(\frac{1}{2})_n}{(2)_n} = \frac{C_n}{2^{2n}}, \qquad \frac{(\frac{1}{2})_n}{(1)_n} = \frac{1}{2^{2n}}\binom{2n}{n}, \qquad \frac{(\frac{3}{2})_n}{(2)_n} = \frac{1}{2^{2n}}\binom{2n+1}{n}.$$

Moments of the little q-Jacobi polynomials

Moments of the little q-Jacobi polynomials

Here we consider the series

$$\mu_n = \langle x^n, 1\rangle = \frac{(aq;q)_n}{(abq^2;q)_n} \quad (n = 0, 1, 2, \dots).$$

Specializations

If we put $a = q^\alpha$, $b = q^\beta$ and let $q \to 1$, then

$$\mu_n \to \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Note that

$$\frac{(\frac{1}{2})_n}{(2)_n} = \frac{C_n}{2^{2n}}, \quad \frac{(\frac{1}{2})_n}{(1)_n} = \frac{1}{2^{2n}}\binom{2n}{n}, \quad \frac{(\frac{3}{2})_n}{(2)_n} = \frac{1}{2^{2n}}\binom{2n+1}{n}.$$

q-Catalan Hankel determinants

Theorem (ITZ'09)

Let n be a positive integer and t non-negative integer. Then

$$\det\left(\mu_{i+j+r-2}\right)_{1\le i,j\le n} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \left\{\frac{(aq;q)_r}{(abq^2;q)_r}\right\}^n \times \prod_{k=1}^{n} \frac{(q,aq^{r+1},bq;q)_{n-k}}{(abq^{n-k+r+1};q)_{n-k}(abq^{r+2};q)_{2(n-k)}}.$$

Theorem (ITZ'09)

Let n be a positive integer and $k_1, k_2, \ldots, k_n$ non-negative integers.

$$\det\left(\mu_{k_i+j-2}\right)_{1\le i,j\le n1} = a^{\binom{n}{2}} q^{\binom{n+1}{3}} \prod_{i=1}^{n} \frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \times \prod_{1\le i<j\le n} (q^{k_i}-q^{k_j}) \prod_{i=1}^{n} (bq;q)_{n-i}.$$

q-Catalan Hankel determinants

Theorem (ITZ'09)

Let n be a positive integer and t non-negative integer. Then

$$\det\left(\mu_{i+j+r-2}\right)_{1\le i,j\le n} = a^{\frac{1}{2}n(n-1)}q^{\frac{1}{6}n(n-1)(2n-1)}\left\{\frac{(aq;q)_r}{(abq^2;q)_r}\right\}^n \times\prod_{k=1}^{n}\frac{(q,aq^{r+1},bq;q)_{n-k}}{(abq^{n-k+r+1};q)_{n-k}(abq^{r+2};q)_{2(n-k)}}.$$

Theorem (ITZ'09)

Let n be a positive integer and k_1, k_2,..., k_n non-negative integers.

$$\det\left(\mu_{k_i+j-2}\right)_{1\le i,j\le n1} = a^{\binom{n}{2}}q^{\binom{n+1}{3}}\prod_{i=1}^{n}\frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \times\prod_{1\le i<j\le n}(q^{k_i}-q^{k_j})\prod_{i=1}^{n}(bq;q)_{n-i}.$$

Methods to prove the above theorems

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

Methods to prove the above theorems

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

Methods to prove the above theorems

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

Methods to prove the above theorems

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

Methods to prove the above theorems

Proof methods

The methods to prove the theorems

- Lattice path method (the Lindström-Gessel-Viennot theorem)
- Orthogonal polynomials and continued fractions (the little q-Jacobi polynomials)
- LU-decompositions (q-Dougall's formula)
- Desnanot-Jacobi adjoint matrix theorem (Dodgson's formula)

An Pfaffian analogue of q-Catalan Hankel determinant

Theorem

Let $n \geq 1$ and $r \geq -1$ be integers. Then we have

$$\operatorname{Pf}\left((q^{i-1} - q^{j-1}) \frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}} \right)_{1 \leq i,j \leq 2n}$$
$$= a^{n(n-1)} q^{n(n-1)(4n+1)/3 + n(n-1)r}$$
$$\times \prod_{k=1}^{n-1} (bq;q)_{2k} \prod_{k=1}^{n} \frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

If we put $a = q^{\alpha}$ and $b = q^{\beta}$ and let $q \to 1$, then we obtain the following corollary.

Theorem

Let $n \geq 1$ and $r \geq -1$ be integers. Then we have

$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\leq i,j\leq 2n}$$
$$= a^{n(n-1)}q^{n(n-1)(4n+1)/3+n(n-1)r}$$
$$\times \prod_{k=1}^{n-1}(bq;q)_{2k}\prod_{k=1}^{n}\frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

If we put $a = q^{\alpha}$ and $b = q^{\beta}$ and let $q \to 1$, then we obtain the following corollary.

A corollary

Corollary

Let $n \geq 1$ and $r \geq -1$ be integers. Then we have

$$\mathrm{Pf}\left((j-i)\frac{(\alpha+1)_{i+j+r-2}}{(\alpha+\beta+2)_{i+j+r-2}}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1}(\beta+1)_{2k}\prod_{k=1}^{n}\frac{(2k-1)!(\alpha+1)_{2k+r-1}}{(\alpha+\beta+2)_{2(k+n)+r-3}},$$

where we use the notation

$$(\alpha)_n = \begin{cases} \prod_{i=1}^{n}(\alpha+i-1) & \text{if } n \geq 0, \\ 1/\prod_{i=1}^{-n}(\alpha+i+n-1) & \text{if } n < 0. \end{cases}$$

(See Krattenthaler'02.)

Corollary

Let $n \geq 1$ and $r \geq -1$ be integers.

1. Let C_n denote the Catalan numbers.

$$\operatorname{Pf}\left((j-i)C_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1} \frac{(4k+1)!}{(2k)!} \prod_{k=1}^{n} \frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-2\}!},$$

2. Let $D_n = \binom{2n}{n}$ denote the central binomial coefficients.

$$\operatorname{Pf}\left((j-i)D_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1} \frac{(4k)!}{(2k)!} \prod_{k=1}^{n} \frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-3\}!}.$$

Corollary

Let $n \geq 1$ and $r \geq -1$ be integers.

1. Let C_n denote the Catalan numbers.

$$\mathrm{Pf}\left((j-i)C_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1} \frac{(4k+1)!}{(2k)!} \prod_{k=1}^{n} \frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-2\}!},$$

2. Let $D_n = \binom{2n}{n}$ denote the central binomial coefficients.

$$\mathrm{Pf}\left((j-i)D_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1} \frac{(4k)!}{(2k)!} \prod_{k=1}^{n} \frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-3\}!}.$$

Corollary

Let $n \geq 1$ and $r \geq -1$ be integers.

1. Let C_n denote the Catalan numbers.

$$\mathrm{Pf}\left((j-i)C_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1}\frac{(4k+1)!}{(2k)!}\prod_{k=1}^{n}\frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-2\}!},$$

2. Let $D_n = \binom{2n}{n}$ denote the *central binomial coefficients*.

$$\mathrm{Pf}\left((j-i)D_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n-1}\frac{(4k)!}{(2k)!}\prod_{k=1}^{n}\frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-3\}!}.$$

More Specializations

Corollary

1. Let $\mu_n = (\alpha + 1)_n$ for $n \geq 0$, which is known as the moment sequence of Laguerre polynomials.

$$\operatorname{Pf}\left((j-i)\mu_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n}(2k-1)!(\alpha+1)_{2k+r-1}.$$

2. Let $\mu_n = (2n+1)!!$ denote the double factorial of $2n+1$ for $n \geq 0$, which is known as the moment sequence of Hermite polynomials.

$$\operatorname{Pf}\left((j-i)\mu_{i+j+r-2}\right)_{1\leq i,j\leq 2n} = \frac{1}{2^n}\prod_{k=1}^{n}(4k-2)!!(4k+2r-1)!!$$

More Specializations

Corollary

1. Let $\mu_n = (\alpha+1)_n$ for $n \geq 0$, which is known as the moment sequence of Laguerre polynomials.

$$\mathrm{Pf}\Big((j-i)\mu_{i+j+r-2}\Big)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n}(2k-1)!(\alpha+1)_{2k+r-1}.$$

2. Let $\mu_n = (2n+1)!!$ denote the double factorial of $2n+1$ for $n \geq 0$, which is known as the moment sequence of Hermite polynomials.

$$\mathrm{Pf}\Big((j-i)\mu_{i+j+r-2}\Big)_{1\leq i,j\leq 2n} = \frac{1}{2^n}\prod_{k=1}^{n}(4k-2)!!(4k+2r-1)!!$$

Corollary

1. Let $\mu_n = (\alpha+1)_n$ for $n \geq 0$, which is known as the moment sequence of Laguerre polynomials.

$$\mathrm{Pf}\Big((j-i)\mu_{i+j+r-2}\Big)_{1\leq i,j\leq 2n} = \prod_{k=1}^{n}(2k-1)!(\alpha+1)_{2k+r-1}.$$

2. Let $\mu_n = (2n+1)!!$ denote the double factorial of $2n+1$ for $n \geq 0$, which is known as the moment sequence of Hermite polynomials.

$$\mathrm{Pf}\Big((j-i)\mu_{i+j+r-2}\Big)_{1\leq i,j\leq 2n} = \frac{1}{2^n}\prod_{k=1}^{n}(4k-2)!!(4k+2r-1)!!$$

Submatrices

Definition

Let $A = (a^i_j)_{i,j\geq 1}$ be a matrix (of finite or infinite row/column length). If $I = \{i_1, \ldots, i_r\}$ (resp. $J = \{j_1, \ldots, j_r\}$) are a set of row (resp. column) indices, then we write $A^I_J = A^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ for the $r \times r$ submatrix obtained from A by choosing the rows indexed by I and columns indexed by J. Let $a^I_J = a^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ denote $\det A^I_J$ if $|I| = |J| > 0$, and 1 if $I = J = \emptyset$. For positive integer n we let $[n] = \{1, \ldots, n\}$.

Example

For $A = (a^i_j)_{i,j\geq 1}$ we use the notation

$$A^{1,2,4}_{2,3,5} = \begin{pmatrix} a^1_2 & a^1_3 & a^1_5 \\ a^2_2 & a^2_3 & a^2_5 \\ a^4_2 & a^4_3 & a^4_5 \end{pmatrix}, \qquad a^{1,2,4}_{2,4,5} = \det A^{1,2,4}_{2,4,5}.$$

Submatrices

Definition

Let $A = (a_j^i)_{i,j\geq 1}$ be a matrix (of finite or infinite row/column length). If $I = \{i_1, \dots, i_r\}$ (resp. $J = \{j_1, \dots, j_r\}$) are a set of row (resp. column) indices, then we write $A_J^I = A_{j_1,\dots,j_r}^{i_1,\dots,i_r}$ for the $r \times r$ submatrix obtained from A by choosing the rows indexed by I and columns indexed by J. Let $a_J^I = a_{j_1,\dots,j_r}^{i_1,\dots,i_r}$ denote $\det A_J^I$ if $|I| = |J| > 0$, and 1 if $I = J = \emptyset$. For positive integer n we let $[n] = \{1, \dots, n\}$.

Example

For $A = (a_j^i)_{i,j\geq 1}$ we use the notation

$$A_{2,3,5}^{1,2,4} = \begin{pmatrix} a_2^1 & a_3^1 & a_5^1 \\ a_2^2 & a_3^2 & a_5^2 \\ a_2^4 & a_3^4 & a_5^4 \end{pmatrix}, \qquad a_{2,4,5}^{1,2,4} = \det A_{2,4,5}^{1,2,4}.$$

LDU-decomposition

Let n be a positive integer, and $A = (a_j^i)_{1\le i,j\le n}$ be an $n\times n$ matrix such that $a_{[i]}^{[i]} \neq 0$ for $1 \le i \le n$. Then A is uniquely written as

$$A = L\,D\,U,$$

where $D = (d_i\delta_j^i)_{1\le i,j\le n}$ is a diagonal matrix, $L = (l_j^i)_{1\le i,j\le n}$ (resp. $U = (u_j^i)_{1\le i,j\le n}$) is a lower (resp. upper) unitriangular matrix. In fact

$$d_i = \frac{a_{[i]}^{[i]}}{a_{[i-1]}^{[i-1]}}, \qquad l_j^i = \frac{a_{[j]}^{[j-1],i}}{a_{[j]}^{[j]}}, \qquad u_j^i = \frac{a_{[i-1],j}^{[i]}}{a_{[i]}^{[i]}}.$$

LDU-decomposition

Let n be a positive integer, and $A = (a^i_j)_{1 \le i,j \le n}$ be an $n \times n$ matrix such that $a^{[i]}_{[i]} \ne 0$ for $1 \le i \le n$. Then A is uniquely written as

$$A = L\,D\,U,$$

where $D = (d_i \delta^i_j)_{1 \le i,j \le n}$ is a diagonal matrix, $L = (l^i_j)_{1 \le i,j \le n}$ (resp. $U = (u^i_j)_{1 \le i,j \le n}$) is a lower (resp. upper) unitriangular matrix. In fact

$$d_i = \frac{a^{[i]}_{[i]}}{a^{[i-1]}_{[i-1]}}, \qquad l^i_j = \frac{a^{[j-1],i}_{[j]}}{a^{[j]}_{[j]}}, \qquad u^i_j = \frac{a^{[i]}_{[i-1],j}}{a^{[i]}_{[i]}}.$$

Example

$n = 4$

$$
\begin{pmatrix}
a_1^1 & a_2^1 & a_3^1 & a_4^1 \\
a_1^2 & a_2^2 & a_3^2 & a_4^2 \\
a_1^3 & a_2^3 & a_3^3 & a_4^3 \\
a_1^4 & a_2^4 & a_3^4 & a_4^4
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{a_1^2}{a_1^1} & 1 & 0 & 0 \\
\frac{a_1^3}{a_1^1} & \frac{a_{12}^{13}}{a_{12}^{12}} & 1 & 0 \\
\frac{a_1^4}{a_1^1} & \frac{a_{12}^{14}}{a_{12}^{12}} & \frac{a_{123}^{124}}{a_{123}^{123}} & 1
\end{pmatrix}
\begin{pmatrix}
a_1^1 & 0 & 0 & 0 \\
0 & \frac{a_{12}^{12}}{a_1^1} & 0 & 0 \\
0 & 0 & \frac{a_{123}^{123}}{a_{12}^{12}} & 0 \\
0 & 0 & 0 & \frac{a_{1234}^{1234}}{a_{123}^{123}}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{a_2^1}{a_1^1} & \frac{a_2^1}{a_1^1} & \frac{a_2^1}{a_1^1} \\
0 & 1 & \frac{a_{13}^{12}}{a_{12}^{12}} & \frac{a_{14}^{12}}{a_{12}^{12}} \\
0 & 0 & 1 & \frac{a_{124}^{123}}{a_{123}^{123}} \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Proof

Remark

The proof of the decomposition is made by the Desnanot-Jacobi adjoint matrix theorem:

$$a^{[k-2]}_{[k-2]} a^{[k]}_{[k]} = a^{[k-2],k-1}_{[k-2],k-1} a^{[k-2],k}_{[k-2],k} - a^{[k-2],k-1}_{[k-2],k} a^{[k-2],k}_{[k-2],k-1}.$$

A strategy to evaulate a determinant

Assume one has a conjecture for $a^{[i]}_{[i]} = a^{12\dots i}_{12\dots i}$. If he can guess a certain formula for

$$a^{[i]}_{[i-1],j} = a^{12\dots i}_{1\dots i-1,j} \text{ and } a^{[j-1],i}_{[j]} = a^{1\dots j-1,i}_{12\dots j}$$

then he should prove

$$\sum_{k=1}^{\min(i,j)} l^i_k d_k u^k_j = a^i_j.$$

Proof

Remark

The proof of the decomposition is made by the Desnanot-Jacobi adjoint matrix theorem:

$$a^{[k-2]}_{[k-2]} a^{[k]}_{[k]} = a^{[k-2],k-1}_{[k-2],k-1} a^{[k-2],k}_{[k-2],k} - a^{[k-2],k-1}_{[k-2],k} a^{[k-2],k}_{[k-2],k-1}.$$

A strategy to evaulate a determinant

Assume one has a conjecture for $a^{[i]}_{[i]} = a^{12\dots i}_{12\dots i}$. If he can guess a certain formula for

$$a^{[i]}_{[i-1],j} = a^{12\dots i}_{1\dots i-1,j} \text{ and } a^{[j-1],i}_{[j]} = a^{1\dots j-1,i}_{12\dots j}$$

then he should prove

$$\sum_{k=1}^{\min(i,j)} l^i_k d_k u^k_j = a^i_j.$$

Skew-symmetric matrices

Definition

We say a matrix $A = (a^i_j)_{i,j\geq 1}$ (resp. $A = (a^i_j)_{1\leq i,j\leq n}$) is *skew-symmetric* if it satisfies $a^j_i = -a^i_j$ for $i, j \geq 1$ (resp. $1 \leq i, j \leq n$).

Example

The 4×4 matrix

$$\begin{pmatrix} 0 & a^1_2 & a^1_3 & a^1_4 \\ -a^1_2 & 0 & a^2_3 & a^2_4 \\ -a^1_3 & -a^2_3 & 0 & a^3_4 \\ -a^1_4 & -a^2_4 & -a^3_4 & 0 \end{pmatrix}$$

is skew-symmetric.

Skew-symmetric matrices

Definition

We say a matrix $A = (a^i_j)_{i,j\geq 1}$ (resp. $A = (a^i_j)_{1\leq i,j\leq n}$) is *skew-symmetric* if it satisfies $a^j_i = -a^i_j$ for $i, j \geq 1$ (resp. $1 \leq i, j \leq n$).

Example

The 4×4 matrix

$$\begin{pmatrix} 0 & a^1_2 & a^1_3 & a^1_4 \\ -a^1_2 & 0 & a^2_3 & a^2_4 \\ -a^1_3 & -a^2_3 & 0 & a^3_4 \\ -a^1_4 & -a^2_4 & -a^3_4 & 0 \end{pmatrix}$$

is skew-symmetric.

Definition

We define 2×2 skew-symmetric matrix J_2 by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $J_{2n} = J_2 \oplus \cdots \oplus J_2$ denote the $2n \times 2n$ matrix whose main diagonal 2×2 blocks are all J_2 and the other blocks are 2×2 zero matrices O_2.

Example

$$J_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Definition

We define 2×2 skew-symmetric matrix J_2 by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $J_{2n} = J_2 \oplus \cdots \oplus J_2$ denote the $2n \times 2n$ matrix whose main diagonal 2×2 blocks are all J_2 and the other blocks are 2×2 zero matrices O_2.

Example

$$J_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Definition

If we are given an $n \times n$ skew-symmetric matrix $A = (a^i_j)_{1 \le i,j \le n}$ of even degree, then $\det A$ is the square of a polynomial of its entries a^i_j. So the *Pfaffian* of A, denoted by $\mathrm{Pf}\, A$, is definded to be the square root of $\det A$, where we take the branch which takes the value $\mathrm{Pf}\, J_{2n} = 1$.

Example

$$\mathrm{Pf}\begin{pmatrix} 0 & a^1_2 & a^1_3 & a^1_4 \\ -a^1_2 & 0 & a^2_3 & a^2_4 \\ -a^1_3 & -a^2_3 & 0 & a^3_4 \\ -a^1_4 & -a^2_4 & -a^3_4 & 0 \end{pmatrix} = a^1_2 a^3_4 - a^1_3 a^2_4 + a^1_4 a^2_3$$

Pfaffian

Definition

If we are given an $n \times n$ skew-symmetric matrix $A = (a^i_j)_{1 \leq i,j \leq n}$ of even degree, then $\det A$ is the square of a polynomial of its entries a^i_j. So the *Pfaffian* of A, denoted by $\mathrm{Pf}\, A$, is definded to be the square root of $\det A$, where we take the branch which takes the value $\mathrm{Pf}\, J_{2n} = 1$.

Example

$$\mathrm{Pf} \begin{pmatrix} 0 & a^1_2 & a^1_3 & a^1_4 \\ -a^1_2 & 0 & a^2_3 & a^2_4 \\ -a^1_3 & -a^2_3 & 0 & a^3_4 \\ -a^1_4 & -a^2_4 & -a^3_4 & 0 \end{pmatrix} = a^1_2 a^3_4 - a^1_3 a^2_4 + a^1_4 a^2_3$$

Definition

For a skew-symmetric matrix A, we usually take $I = J$ so that, hereafter, we write $A_I = A_{i_1,\dots,i_r}$ for A_I^I. Further let $a_I = a_{i_1,\dots,i_r}$ denote $\mathrm{Pf}\,A_I$ if $I \neq \emptyset$, 1 if $I = \emptyset$ when there is no fear of confusion.

Example

$$A_{1356} = \begin{pmatrix} 0 & a_3^1 & a_5^1 & a_6^1 \\ -a_3^1 & 0 & a_5^3 & a_6^3 \\ -a_5^1 & -a_5^3 & 0 & a_6^5 \\ -a_6^1 & -a_6^3 & -a_6^5 & 0 \end{pmatrix}$$

$$a_{1356} = a_3^1 a_6^5 - a_5^1 a_6^3 + a_6^1 a_5^3$$

Subpfaffians

Definition

For a skew-symmetric matrix A, we usually take $I = J$ so that, hereafter, we write $A_I = A_{i_1,\dots,i_r}$ for A_I^I. Further let $a_I = a_{i_1,\dots,i_r}$ denote $\operatorname{Pf} A_I$ if $I \neq \emptyset$, 1 if $I = \emptyset$ when there is no fear of confusion.

Example

$$A_{1356} = \begin{pmatrix} 0 & a_3^1 & a_5^1 & a_6^1 \\ -a_3^1 & 0 & a_5^3 & a_6^3 \\ -a_5^1 & -a_5^3 & 0 & a_6^5 \\ -a_6^1 & -a_6^3 & -a_6^5 & 0 \end{pmatrix}$$

$$a_{1356} = a_3^1 a_6^5 - a_5^1 a_6^3 + a_6^1 a_5^3$$

2×2 block notation

2×2 block notation

We write the matrix $A = (a^i_j)_{1 \le i,j \le 2n}$ by 2×2 blocks as

$$A = \begin{pmatrix} A^1_1 & A^1_2 & \dots & A^1_n \\ A^2_1 & A^2_2 & \dots & A^2_n \\ \vdots & \vdots & \ddots & \vdots \\ A^n_1 & A^n_2 & \dots & A^n_n \end{pmatrix},$$

where A^i_j is the 2×2 block matrix $A^i_j = \begin{pmatrix} a^{2i-1}_{2j-1} & a^{2i-1}_{2j} \\ a^{2i}_{2j-1} & a^{2i}_{2j} \end{pmatrix}$

Pfaffian decomposition

Theorem (Pfaffian decomposition)

Let n be a positive integer, and $A = (a_j^i)_{1 \le i,j \le 2n}$ be a skew-symmetric matrix of size $2n$ such that $a_{[2i]} \neq 0$ for $1 \le i \le n$. then A is uniquely written as

$$A = {}^t V \, T \, V.$$

Here T and V are composed of 2×2 blocks

$$T = \begin{pmatrix} T_1 & O_2 & \dots & O_2 \\ O_2 & T_2 & \dots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \dots & T_n \end{pmatrix}, \qquad V = \begin{pmatrix} J_2 & V_2^1 & \dots & V_n^1 \\ O_2 & J_2 & \dots & V_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \dots & J_2 \end{pmatrix},$$

for $1 \le i \le n$.

Pfaffian decomposition

Theorem (Entries of T and V)

Here we have $T_i = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}$ and $V^i_j = \begin{pmatrix} v^{2i-1}_{2j-1} & v^{2i-1}_{2j} \\ v^{2i}_{2j-1} & v^{2i}_{2j} \end{pmatrix}$, and each t_i and v^k_l is written as

$$t_i = \frac{a_{[2i]}}{a_{[2i-2]}}, \qquad v^k_l = \frac{a_{[2i-2],k,l}}{a_{[2i]}}$$

for $1 \leq i \leq n$ and $1 \leq k, l \leq 2n$.

Remark

The proof of the decomposition is made by the Pfaffian analogue of the Desnanot-Jacobi adjoint matrix theorem:

$$a_{[k-4]}a_{[k]} = a_{[k-4],k-3,k-2}a_{[k-4],k-1,k} - a_{[k-4],k-3,k-1}a_{[k-4],k-2,k} + a_{[k-4],k-3,k}a_{[k-4],k-2,k-1}.$$

Pfaffian decomposition

Theorem (Entries of T and V)

Here we have $T_i = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}$ and $V_j^i = \begin{pmatrix} v_{2j-1}^{2i-1} & v_{2j}^{2i-1} \\ v_{2j-1}^{2i} & v_{2j}^{2i} \end{pmatrix}$, and each t_i and v_l^k is written as

$$t_i = \frac{a_{[2i]}}{a_{[2i-2]}}, \qquad v_l^k = \frac{a_{[2i-2],k,l}}{a_{[2i]}}$$

for $1 \le i \le n$ and $1 \le k, l \le 2n$.

Remark

The proof of the decomposition is made by the Pfaffian analogue of the Desnanot-Jacobi adjoint matrix theorem:

$$a_{[k-4]}a_{[k]} = a_{[k-4],k-3,k-2}a_{[k-4],k-1,k} - a_{[k-4],k-3,k-1}a_{[k-4],k-2,k} + a_{[k-4],k-3,k}a_{[k-4],k-2,k-1}.$$

An example of Pfaffian decomposition

Example

For $A = (a^i_j)_{1\le i,j\le 6}$ we have

$$T = \begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 \\ -a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a_{1234}}{a_{12}} & 0 & 0 \\ 0 & 0 & -\frac{a_{1234}}{a_{12}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a_{123456}}{a_{1234}} \\ 0 & 0 & 0 & 0 & -\frac{a_{123456}}{a_{1234}} & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 1 & \frac{a_{13}}{a_{12}} & \frac{a_{14}}{a_{12}} & \frac{a_{15}}{a_{12}} & \frac{a_{16}}{a_{12}} \\ -1 & 0 & \frac{a_{23}}{a_{12}} & \frac{a_{24}}{a_{12}} & \frac{a_{25}}{a_{12}} & \frac{a_{26}}{a_{12}} \\ 0 & 0 & 0 & 1 & \frac{a_{1235}}{a_{1234}} & \frac{a_{1236}}{a_{1234}} \\ 0 & 0 & -1 & 0 & \frac{a_{1245}}{a_{1234}} & \frac{a_{1246}}{a_{1234}} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

A strategy to evaulate a Pfafian

A strategy to evaulate a Pfafian

Assume one has a conjecture for $a_{[2i]} = a_{1,2,\dots,2i}$. If he can guess a certain formula for

$$a_{[2i-1],j} = a_{1\dots 2i-1,j} \text{ and } a_{[2i-2],2i,j} = a_{12\dots 2i-2,2i,j}$$

then he should prove

$$\sum_{k\geq 1}\left(v_i^{2k-1}t_k v_j^{2k} - v_i^{2k}t_k v_j^{2k-1}\right) = a_j^i$$

Our Pfaffian

Let

$$a_j^i = (q^{i-1} - q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}.$$

A strategy to evaulate a Pfafian

A strategy to evaulate a Pfafian

Assume one has a conjecture for $a_{[2i]} = a_{1,2,\dots,2i}$. If he can guess a certain formula for

$$a_{[2i-1],j} = a_{1\dots 2i-1,j} \text{ and } a_{[2i-2],2i,j} = a_{12\dots 2i-2,2i,j}$$

then he should prove

$$\sum_{k\geq 1}\left(v_i^{2k-1}t_k v_j^{2k} - v_i^{2k}t_k v_j^{2k-1}\right) = a_j^i$$

Our Pfaffian

Let

$$a_j^i = (q^{i-1} - q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}.$$

Guess

Our Goal

We want prove

$$a_{[2i]} = a^{i(i-1)} q^{i(i-1)(4i+1)/3+i(i-1)r} \times \prod_{k=1}^{i-1} (bq;q)_{2k} \prod_{k=1}^{i} \frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+i)+r-3}}.$$

Our Guess

We guess a formula for

$$a_{[2i-1],j} \quad \text{and} \quad a_{[2i-2],2i,j}.$$

Guess

Our Goal

We want prove

$$a_{[2i]} = a^{i(i-1)} q^{i(i-1)(4i+1)/3+i(i-1)r} \times \prod_{k=1}^{i-1} (bq;q)_{2k} \prod_{k=1}^{i} \frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+i)+r-3}}.$$

Our Guess

We guess a formula for

$$a_{[2i-1],j} \quad \text{and} \quad a_{[2i-2],2i,j}.$$

Pfaffian decomposition

Theorem

We obtain the entries of T and V as follows.

$$t_i = a^{2(i-1)} q^{2(i-1)(2i+r-1)} \frac{(q;q)_{2i-1}(aq;q)_{2i+r-1}(bq;q)_{2(i-1)}}{(abq^2,q)_{4i+r-3}(abq^{2i+r-1};q)_{2(i-1)}},$$

$$v_j^i = \begin{cases} \frac{(q^{j-i};q)_i}{(q;q)_i} \cdot \frac{(aq^{i+r+1};q)_{j-i-1}}{(abq^{2i+r+1};q)_{j-i-1}} & \text{if } i \text{ is odd,} \\ q\frac{(q^{j-i};q)_1(q^{j-i+2};q)_{i-2}}{(q;q)_{i-1}} \cdot \frac{(aq^{i+r};q)_{j-i} f(i,j,r)}{(abq^{2i+r-3};q)_1(abq^{2i+r-1};q)_{j-i+1}} & \text{if } i \text{ is even,} \end{cases}$$

where

$$f(i,j,r) = (1-q^{i-1})(1-aq^{i+r-1})(1-abq^{i+j+r-2})/(1-q) + aq^{2i+r-3}(1-b)(1-q^{j-i+1}).$$

Proof

We have to prove

$$\sum_{\substack{k\geq 1\\ k \text{ odd}}} a^{k-1}q^{k(k-1)+1}\cdot\frac{(q^{i-k+1};q)_{k-1}(q^{j-k+1};q)_{k-1}}{(q;q)_k}$$

$$\times\frac{(bq;q)_{k-1}(abq^2;q)_{k-2}(abq^{2k};q)_1}{(aq;q)_k(abq^{i+1};q)_k(abq^{j+1};q)_k}\cdot g_k(i,j;a,b,q)$$

$$=\frac{(q^{i-1}-q^{j-1})(aq;q)_{i+j-2}(abq^2;q)_{i-1}(abq^2;q)_{j-1}}{(aq;q)_{i-1}(aq;q)_{j-1}(abq^2;q)_{i+j-2}},$$

where $g_k(i,j;a,b,q)$ is set to be

$$g_k(i,j;a,b,q)=(1-q^k)(1-aq^k)\Big\{$$

$$(1-q^{i-k})(1-q^{j-k-1})(1-abq^{i+k})(1-abq^{j+k-1})$$

$$-(1-q^{i-k-1})(1-q^{j-k})(1-abq^{k+i-1})(1-abq^{j+k})\Big\}/(1-q)$$

$$+aq^{k-2}(1-b)\left(q^i-q^j\right)\left(1-q^{i-k}\right)\left(1-q^{j-k}\right)\left(1-abq^{2k+1}\right).$$

Proof

We also observe that this identity holds for the sum which runs over even integers, i.e.

$$\sum_{\substack{k\geq 0\\ k\text{ even}}} a^{k-1}q^{k(k-1)+1}\cdot\frac{(q^{i-k+1};q)_{k-1}(q^{j-k+1};q)_{k-1}}{(q;q)_k}$$
$$\times\frac{(bq;q)_{k-1}(abq^2;q)_{k-2}(abq^{2k};q)_1}{(aq;q)_k(abq^{i+1};q)_k(abq^{j+1};q)_k}\cdot g_k(i,j;a,b,q)$$
$$=\frac{(q^{i-1}-q^{j-1})(aq;q)_{i+j-2}(abq^2;q)_{i-1}(abq^2;q)_{j-1}}{(aq;q)_{i-1}(aq;q)_{j-1}(abq^2;q)_{i+j-2}}.$$

Hence, by adding and subtracting these two idetities, those are equivalent to the following identites.

Proof

$$\sum_{k\geq 0} a^{k-1}q^{k(k-1)+1}\cdot\frac{(q^{i-k+1};q)_{k-1}(q^{j-k+1};q)_{k-1}}{(q;q)_k}$$
$$\times\frac{(bq;q)_{k-1}(abq^2;q)_{k-2}(abq^{2k};q)_1}{(aq;q)_k(abq^{i+1};q)_k(abq^{j+1};q)_k}\cdot g_k(i,j;a,b,q)$$
$$=\frac{2(q^{i-1}-q^{j-1})(aq;q)_{i+j-2}(abq^2;q)_{i-1}(abq^2;q)_{j-1}}{(aq;q)_{i-1}(aq;q)_{j-1}(abq^2;q)_{i+j-2}},$$

and

$$\sum_{k\geq 0}(-1)^k a^{k-1}q^{k(k-1)+1}\cdot\frac{(q^{i-k+1};q)_{k-1}(q^{j-k+1};q)_{k-1}}{(q;q)_k}$$
$$\times\frac{(bq;q)_{k-1}(abq^2;q)_{k-2}(abq^{2k};q)_1}{(aq;q)_k(abq^{i+1};q)_k(abq^{j+1};q)_k}\cdot g_k(i,j;a,b,q)=0.$$

Proof

For the first identity, rewrite $g_k(i,j;a,b,q)$ as follows and apply the q-Dougall formula to each term

$$g_k(i,j;a,b,q) = (q^i - q^j)\Big[q^{-2-k}(q + aq^{i+j})(1-q^k)(1-q^{k-1})(1-abq^k)(1-abq^{k}$$

$$+ q^{-3}\Big\{aq(bq - ab - 1 + b)(1 - q^{i+j}) + q(1-a)(q-ab)$$

$$+ aq(1+bq)(1-q^i)(1-q^j) + (q + aq^{i+j})(1-q)(1-abq)\Big\}(1-q^k)(1-ab$$

$$+ aq^{-2+k}(1-b)(1-abq)(1-q^i)(1-q^j)\Big].$$

q-Dougall formula

$${}_6\phi_5\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}.$$

Proof

For the second identity, generalize as follows and use induction.

$$\sum_{k=0}^{m}(-1)^k a^{k-1} q^{k(k-1)+1}(1-abq^{2k})$$
$$\times \frac{(bq;q)_{k-1}(abq^2;q)_{k-2}(cq^{-k+1};q)_k(dq^{-k+1};q)_k \widehat{g}_k(a,b,c,d;q)}{(q;q)_k(aq;q)_k(abcq;q)_k(abdq;q)_k}$$
$$= \frac{a^m c^{m+1} d^{m+1}(d-c)(1-abq^{2m+1})(abq^2;q)_{m-1}(bq;q)_m(c^{-1};q)_{m+1}(d^{-1};q)_{m+}}{(-q)^{m+1}(q;q)_m(aq;q)_m(abcq;q)_m(abdq;q)_m}$$

where

$$\widehat{g}_k(a,b,c,d;q) = (1-q^k)(1-aq^k)\Big\{$$
$$(1-cq^{-k})(1-dq^{-k-1})(1-abcq^k)(1-abdq^{k-1})$$
$$-(1-cq^{-k-1})(1-dq^{-k})(1-abcq^{k-1})(1-abdq^k)\Big\}\Big/(1-q)$$
$$+aq^{k-2}(1-b)(c-d)(1-cq^{-k})(1-dq^{-k})(1-abq^{2k+1}).$$

Conjecture

Al-Salam-Carlitz polynomials

The *Al-Salam-Carlitz polynomials* are defined by

$$U_n^{(a)}(x;q) = (-1)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, ax^{-1} \\ 0 \end{matrix} ; q, qx \right).$$

Then the nth moment have the expression:

$$G_n(a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k.$$

Conjecture

$$\operatorname{Pf}\left((q^{i-1} - q^{j-1}) G_{i+j-2}(a;q) \right)_{1 \le i,j \le 2n}$$

$$= a^{n(n-1)} q^{\frac{1}{3}\lfloor n/2 \rfloor (16\lfloor n/2 \rfloor^2 - 1) + (-1)^n 4 \lfloor n/2 \rfloor^2} \prod_{k=1}^{n} (q;q)_{2k-1} \sum_{k=0}^{n} q^{\lfloor (n-2k)^2/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} a^k$$

Conjecture

Al-Salam-Carlitz polynomials

The *Al-Salam-Carlitz polynomials* are defined by

$$U_n^{(a)}(x;q) = (-1)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, ax^{-1} \\ 0 \end{matrix} ; q, qx\right).$$

Then the nth moment have the expression:

$$G_n(a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k.$$

Conjecture

$$\operatorname{Pf}\Big((q^{i-1} - q^{j-1}) G_{i+j-2}(a;q)\Big)_{1\le i,j\le 2n}$$

$$= a^{n(n-1)} q^{\frac{1}{3}\lfloor n/2\rfloor(16\lfloor n/2\rfloor^2-1)+(-1)^n 4\lfloor n/2\rfloor^2} \prod_{k=1}^{n} (q;q)_{2k-1} \sum_{k=0}^{n} q^{\lfloor (n-2k)^2/2\rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} a^k$$

Several numbers counting paths

Definition (Motzkin, Delannoy, Schröder and Narayana numbers)

Let $M_n = \sum_{k=0}^{n} \binom{n}{2k} C_k$ denote the *Motzkin numbers*, $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ the *central Delannoy numbers*, and $S_n = \sum_{k=0}^{n} \binom{n+k}{2k} C_k$ *Schröder numbers*. The *Narayana numbers* $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, $n = 1, 2, 3 \dots, 1 \le k \le n$, gives the *Narayana polynomials*

$$N_n(a) = \sum_{k=0}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} a^k,$$

which is the moment sequece of a generalized Thebyshev polynomials of the first kind.

Several numbers counting paths

The numbers

The first few terms of Motzkin, Delannoy, Schröder and Narayana numbers are as follows.

$$\{M_n\}_{n\geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, 323, \dots$$
$$\{D_n\}_{n\geq 0} = 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, \dots$$
$$\{S_n\}_{n\geq 0} = 1, 2, 6, 22, 90, 394, 1806, 8558, 41586, \dots$$
$$\{N_n(a)\}_{n\geq 0} = a, a^2 + a, a^3 + 3a^2 + a, a^4 + 6a^3 + 6a^2 + a, \dots$$

Conjectures

Conjecture

Let $n \geq 1$ be an integers. Then the following identities would hold:

$$\mathrm{Pf}\Big((j-i)M_{i+j-3}\Big)_{1\leq i,j\leq 2n} = \prod_{k=0}^{n-1}(4k+1),$$

$$\mathrm{Pf}\Big((j-i)D_{i+j-3}\Big)_{1\leq i,j\leq 2n} = 2^{n^2-1}(2n-1)\prod_{k=1}^{n-1}(4k-1),$$

$$\mathrm{Pf}\Big((j-i)S_{i+j-2}\Big)_{1\leq i,j\leq 2n} = 2^{n^2}\prod_{k=0}^{n-1}(4k+1),$$

$$\mathrm{Pf}\Big((j-i)N_{i+j-2}(a)\Big)_{1\leq i,j\leq 2n} = a^{n^2}\prod_{k=0}^{n-1}(4k+1).$$

Sloane's database

Remark

Note that

$$\mathrm{Pf}\left((j-i)C_{i+j-2}\right)_{1\le i,j\le 2n} = \prod_{k=0}^{n-1}(4k+1).$$

The first few terms are 1, 1, 5, 45, 585, 9945, 208845, 5221125, 151412625, $\dots$. While these numbers are known as the numbers of increasing quaternary trees on n vertices (On-line Encyclopedia for Integer Sequences), we don't know the direct connection between the identity and combinatorial enumeration.

The end

Thank you!