Parking and Trees

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Séminaire Lotharingien de Combinatoire 65

Parking functions - definition 1

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f :	1	8	5	2	7	4	4	8	1

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$$f: \;\; 1 \;\; 8 \;\; 5 \;\; 2 \;\; 7 \;\; 4 \;\; 4 \;\; 8 \;\; 1$$

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3. $f \in \mathbf{PF}_n \Longrightarrow P(f)([n]) = [n]$

$$\begin{matrix} f\text{:}(& \cdots & k & \cdots &) \\ P(f)\text{:}(& \underbrace{\cdots \quad n+1}_{k,k+1,\ldots,n+1} & & \cdots &) \end{matrix} \Longrightarrow \left| P(f)^{-1}([k-1]) \right| < k-1$$

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Parking functions and trees

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f :	1	8	5	2	7	4	4	8	1
σ :	1								

σ^{-1} :	1								

$\longrightarrow$ $\longrightarrow$

Parking functions and trees

	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
σ :	1	8							
σ^{-1} :	1							2	

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σ :	1	8	5						

σ^{-1} :	1				3			2	

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σ :	1	8	5	2					
σ^{-1} :	1	4			3			2	

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σ :	1	8	5	2	7				
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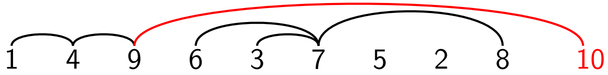

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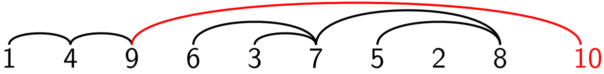

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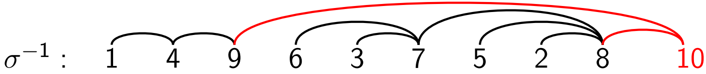

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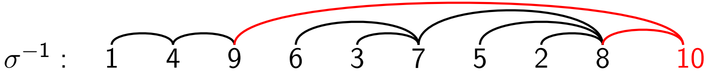
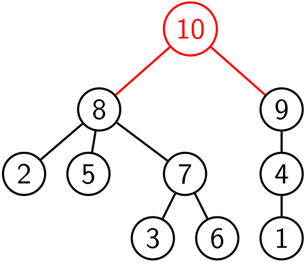


$\phi : S_n \to \mathbf{DT}_{n+1}$



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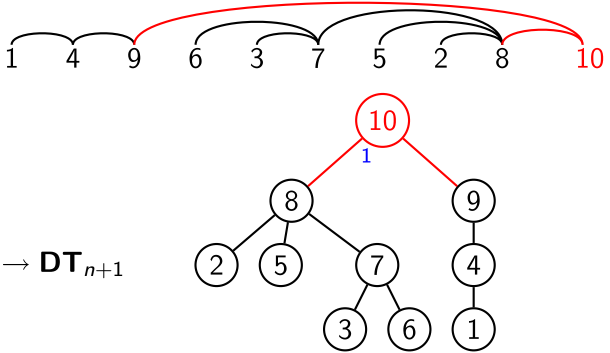

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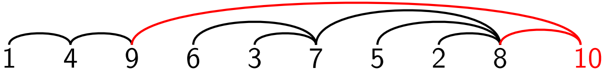

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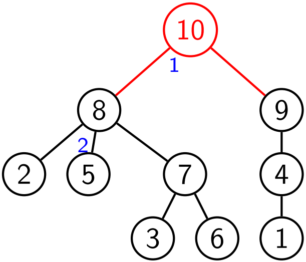

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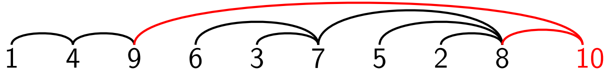

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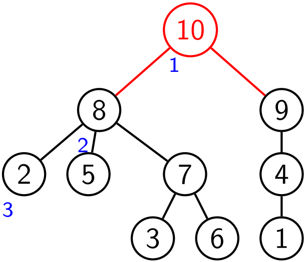

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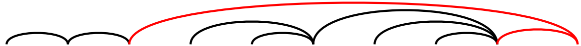

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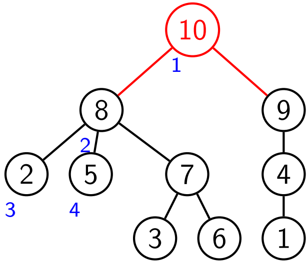

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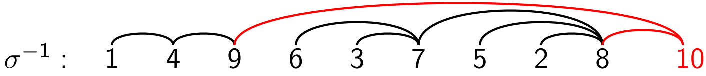

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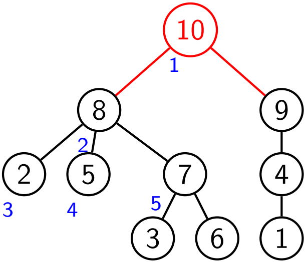

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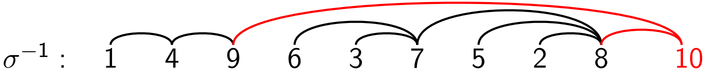

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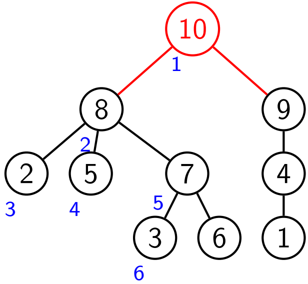

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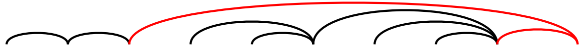

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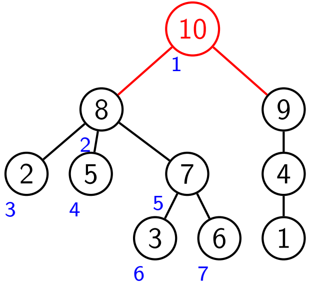

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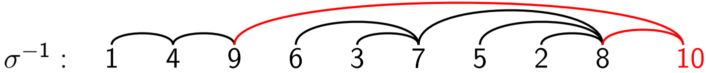

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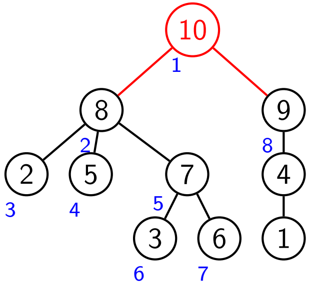
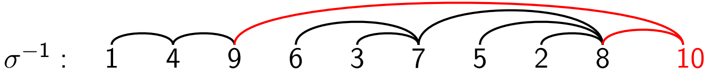
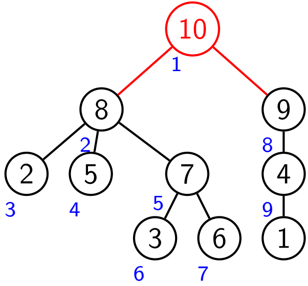


Parking functions and trees

	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
σ :	1	8	5	2	7	4	6	9	3



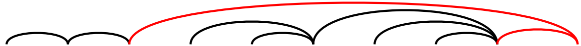
$$\phi : S_n \rightarrow \mathbf{DT}_{n+1}$$



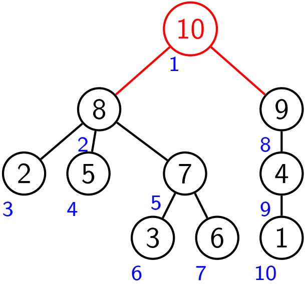
Parking functions and trees

	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
σ :	1	8	5	2	7	4	6	9	3

σ^{-1} : 1 4 9 6 3 7 5 2 8 10

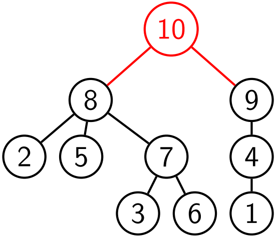


$\phi : S_n \to \mathbf{DT}_{n+1}$



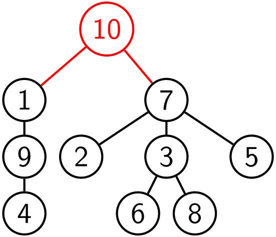
Labeled trees and parking functions

1 8 5 2 7 4 6 9 3 $\longmapsto$

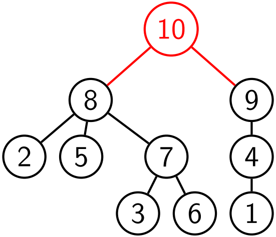


Labeled trees and parking functions

f: 1 8 5 2 7 4 4 8 1 $\mapsto$



$\sigma = P(f)$: 1 8 5 2 7 4 6 9 3 $\mapsto$



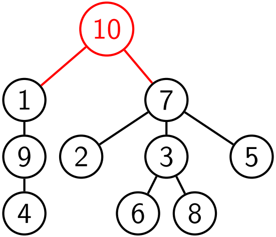
Labeled trees and parking functions

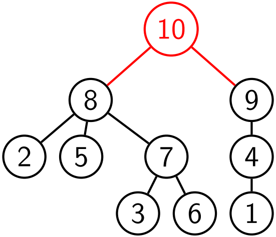
f: 1 8 5 2 7 4 4 8 1 $\mapsto$

$\sigma = P(f)$: 1 8 5 2 7 4 6 9 3 $\mapsto$

. . . .

pr(f): **0 0 0 0 0 0 2 1 2** $\rightarrow_{\sum}$ **5**





Labeled trees and parking functions

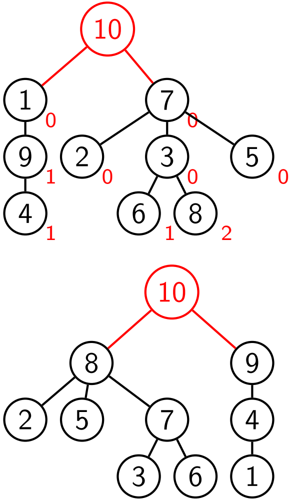
f: 1 8 5 2 7 4 4 8 1 $\mapsto$

Inversions: 91 41 83 87 63

$\sigma = P(f)$: 1 8 5 2 7 4 6 9 3 $\mapsto$

. . . .

pr(f): **0 0 0 0 0 0 2 1 2** $\rightarrow_{\sum}$ **5**



Theorem (AGO, Michel Las Vergnas)

There exists a bijection

$$\psi : \mathbf{PF}_n \to \mathbf{T}_{n+1} \quad \text{s.t.}$$

- $\#\mathrm{Inv}\big(\psi(f)\big) = \#\mathrm{Probes}(f)$;

---

a

b

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---

[a] Kreweras'80

[b]

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- $\psi|_{S_n} = \phi : S_n \to \mathbf{DT}_{n+1}$;

[a] Kreweras'80

[b]

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[a] Kreweras'80
[b] Heesung Shin, SLC 61, Curia

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- $\mathbf{dp}_{\psi(f)} = \mathbf{dp}_f$.

---

[a] Kreweras'80

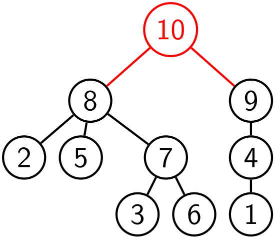
[b] Heesung Shin, SLC 61, Curia

Probes and Inversions

σ : 1 8 5 2 7 4 6 9 3

Probes and Inversions

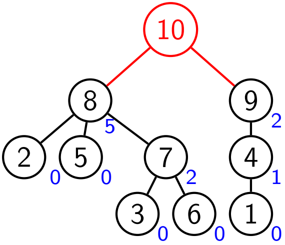
σ : 1 8 5 2 7 4 6 9 3



Probes and Inversions

σ : 1 8 5 2 7 4 6 9 3

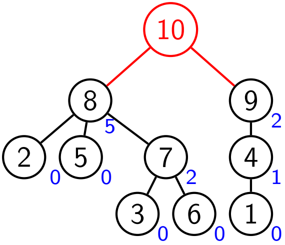
dp$_T$:



Probes and Inversions

	1	2	3	4	5	6	7	8	9
σ :	1	8	5	2	7	4	6	9	3

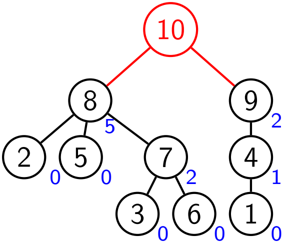
$\mathbf{dp}_T$:



Probes and Inversions

	1	2	3	4	5	6	7	8	9
σ :	1	8	5	2	7	4	6	9	3
$\mathbf{dp}_\sigma$:	**0**	**0**	**0**	**1**	**0**	**0**	**2**	**5**	**2**

$\mathbf{dp}_T$:

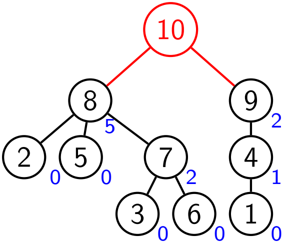


Probes and Inversions

$$\mathbf{dp}_\sigma(i) := \sigma(i) - \min\left\{k \in \sigma([i]) \colon \{k, k+1, \ldots, \sigma(i)\} \subset \sigma([i])\right\}$$

	1	2	3	4	5	6	7	8	9
σ :	1	8	5	2	7	4	6	9	3
$\mathbf{dp}_\sigma$:	**0**	**0**	**0**	**1**	**0**	**0**	**2**	**5**	**2**

$\mathbf{dp}_T$:



Probes and Inversions

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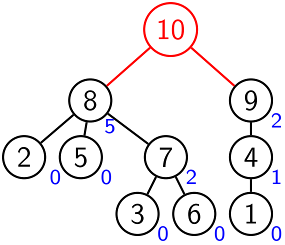
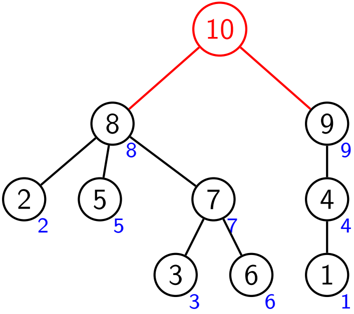
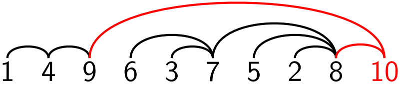
	1	2	3	4	5	6	7	8	9
σ :	1	8	5	2	7	4	6	9	3
$\mathbf{dp}_\sigma$:	**0**	**0**	**0**	**1**	**0**	**0**	**2**	**5**	**2**
	1	**8**	**5**	**1**	**7**	**4**	**4**	**4**	**1**

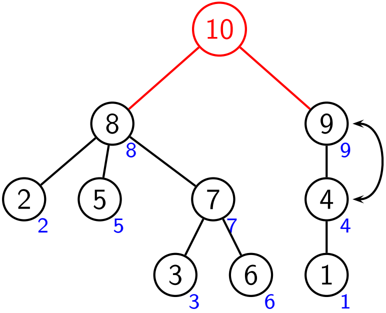
$\mathbf{dp}_T$:

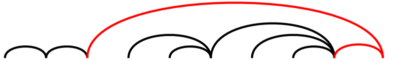




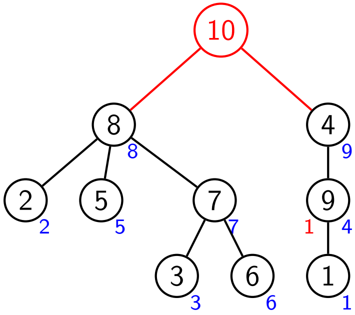


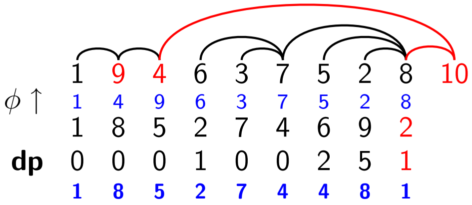
$\phi\uparrow$									
	1	8	5	2	7	4	6	9	3
dp	0	0	0	1	0	0	2	5	2
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**

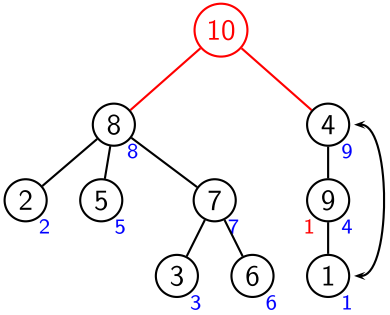


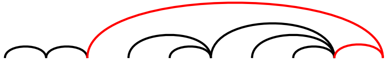


	1	4	9	6	3	7	5	2	8	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	3	
dp	0	0	0	1	0	0	2	5	2	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	

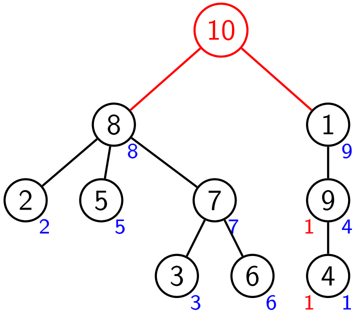

10
8
4
8
9
2
5
7
9
2
5
7
1
4
3
6
1
3
6
1

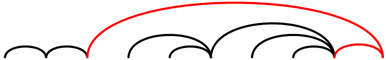

1 9 4 6 3 7 5 2 8 10
$\phi \uparrow$
1 4 9 6 3 7 5 2 8
1 8 5 2 7 4 6 9 2
dp 0 0 0 1 0 0 2 5 1
1 8 5 2 7 4 4 8 1



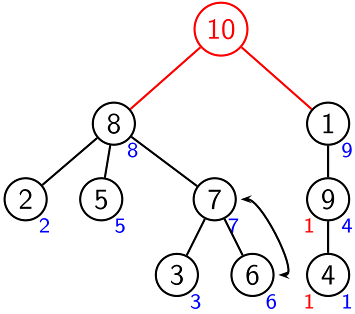


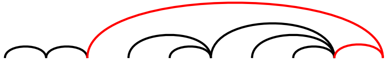
	1	9	4	6	3	7	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	2	
dp	0	0	0	1	0	0	2	5	1	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



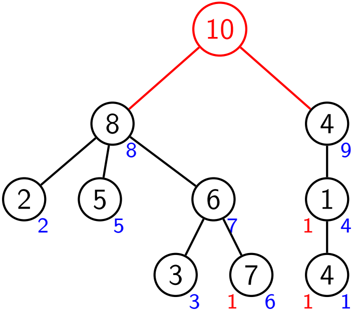


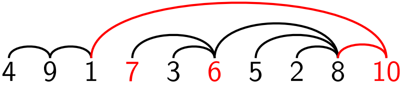
	4	9	1	6	3	7	5	2	8	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	1	
dp	0	0	0	1	0	0	2	5	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



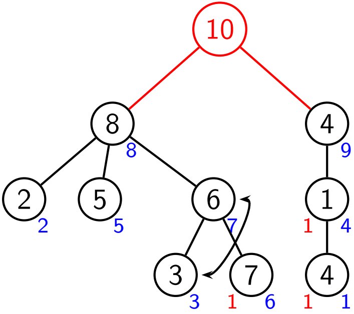


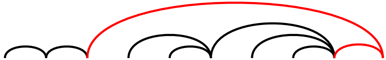
	4	9	1	6	3	7	5	2	8	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	6	9	1	
dp	0	0	0	1	0	0	2	5	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



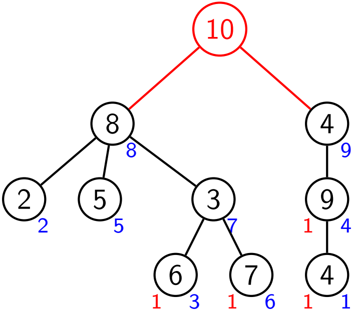


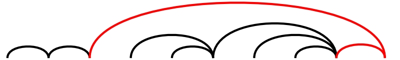
$\phi\uparrow$	1	4	9	6	3	7	5	2	8
	1	8	5	2	7	4	5	9	1
dp	0	0	0	1	0	0	1	5	0
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**



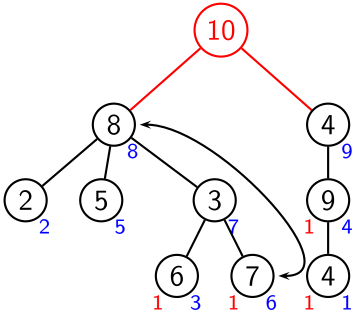


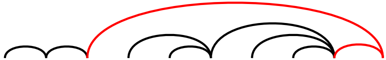
	4	9	1	7	3	6	5	2	8	10
$\phi \uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	5	9	1	
dp	0	0	0	1	0	0	1	5	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



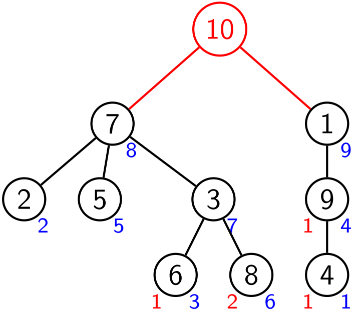


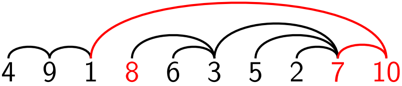
	4	9	1	7	6	3	5	2	8	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	4	9	1	
dp	0	0	0	1	0	0	0	5	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



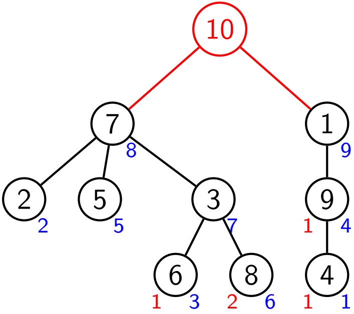


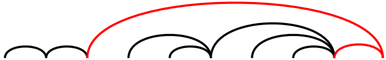
	4	9	1	7	6	3	5	2	8	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	4	9	1	
dp	0	0	0	1	0	0	0	5	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	



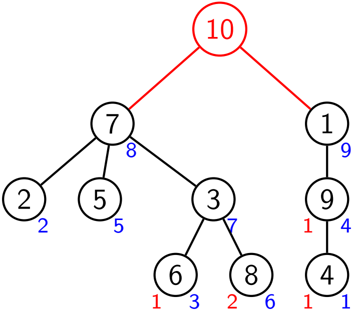


$\phi \uparrow$	1	4	9	6	3	7	5	2	8
	1	8	5	2	7	4	4	8	1
dp	0	0	0	1	0	0	0	4	0
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**



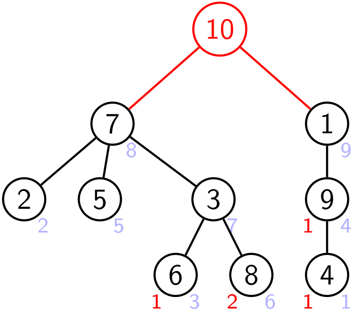


	4	9	1	8	6	3	5	2	7	10
$\phi\uparrow$	1	4	9	6	3	7	5	2	8	
	1	8	5	2	7	4	4	8	1	
dp	0	0	0	1	0	0	0	4	0	
	1	**8**	**5**	**2**	**7**	**4**	**4**	**8**	**1**	

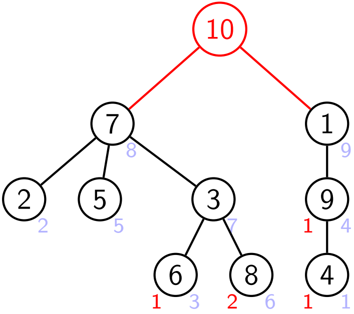


4	9	1	8	6	3	5	2	7
1	4	9	6	3	7	5	2	8

Trees vs. Parking Functions



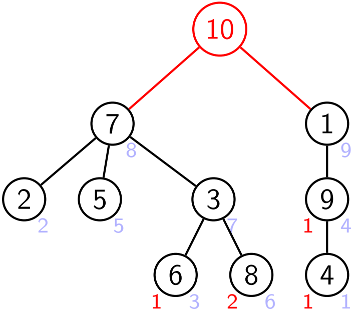
$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8



91 41 83 87 63

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

Trees vs. Parking Functions

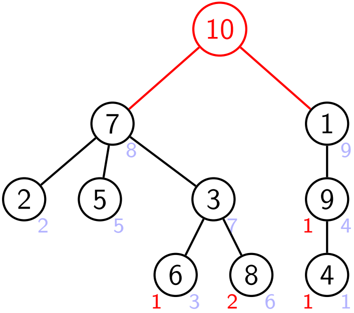


91 41 83 87 63

$$\pi^{-1}\downarrow \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \qquad\qquad 1$$

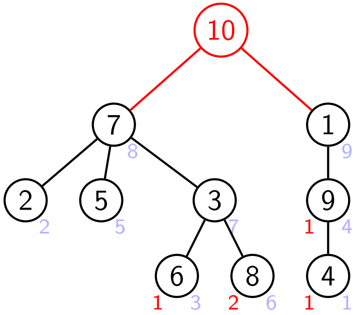
Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \quad 9\ 1$$

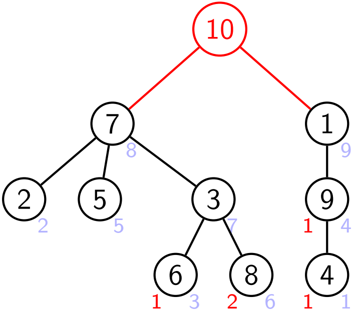


91 41 83 87 63

$\pi^{-1}\downarrow$

4	9	1	8	6	3	5	2	7
1	4	9	6	3	7	5	2	8

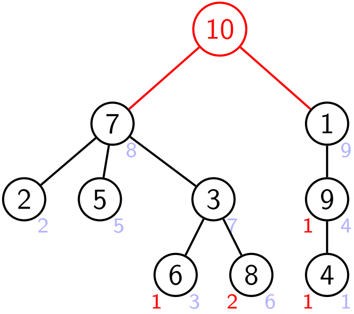
$\beta =$ 4 9 1



91 41 83 87 63

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \quad 7\ 4\ 9\ 1$$

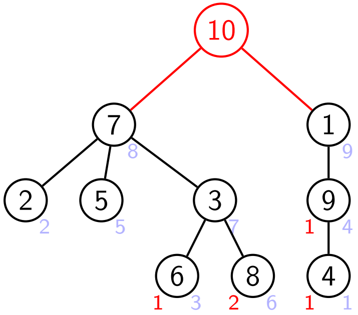


91 41 83 87 63

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \quad 2\ 7\ 4\ 9\ 1$$

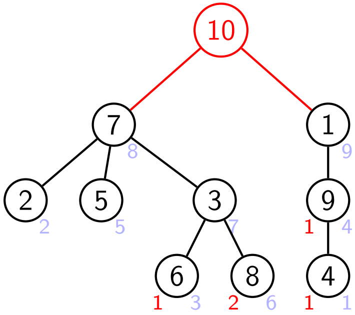
Trees vs. Parking Functions



91 41 83 87 63

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

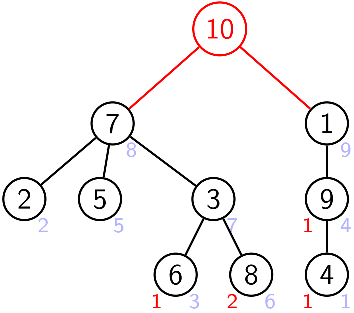
$$\beta = \quad 3\ 2\ 7\ 4\ 9\ 1$$



91 41 83 87 63

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

$$\beta = \quad 6\ 3\ 2\ 7\ 4\ 9\ 1$$



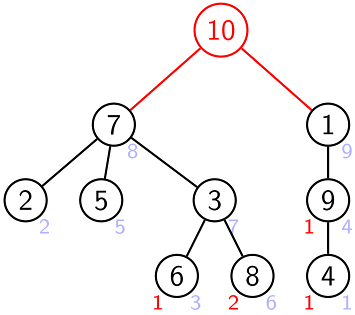
91 41 83 87 63

$\pi^{-1}\downarrow$

4	9	1	8	6	3	5	2	7
1	4	9	6	3	7	5	2	8

$\beta =$ 8 6 3 2 7 4 9 1

Trees vs. Parking Functions

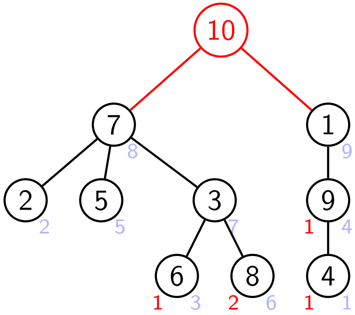


91 41 83 87 63

$\pi^{-1}\downarrow$

4	9	1	8	6	3	5	2	7
1	4	9	6	3	7	5	2	8

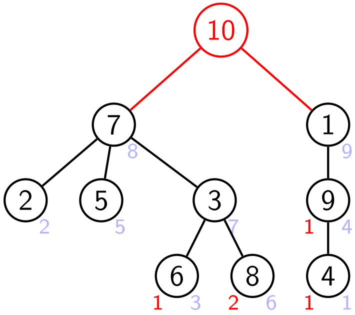
$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

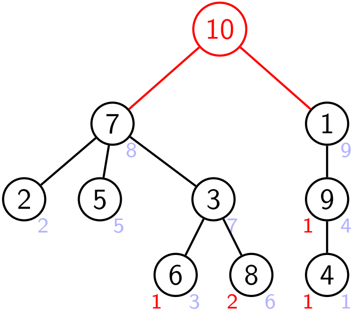
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$

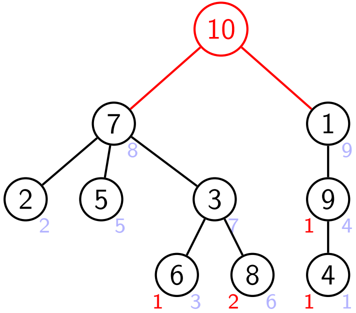


91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)				

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$

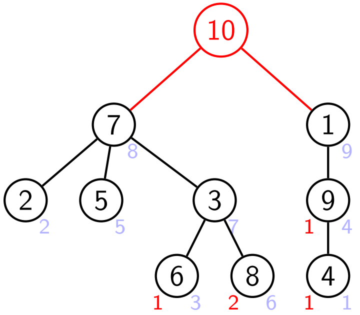
Trees vs. Parking Functions



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)			

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

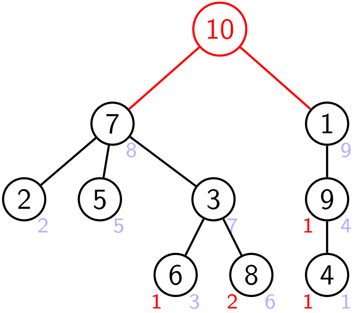
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)	(87)		

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

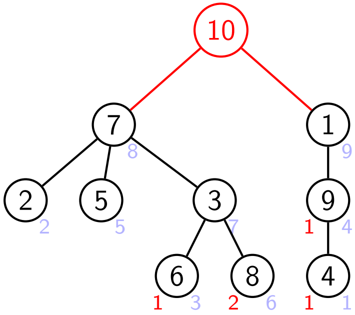
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)	(87)		

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

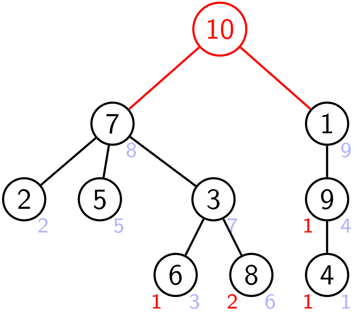
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)	(87)	(63)	

$$\pi^{-1}\downarrow \quad \begin{matrix} 4 & 9 & 1 & 8 & 6 & 3 & 5 & 2 & 7 \\ 1 & 4 & 9 & 6 & 3 & 7 & 5 & 2 & 8 \end{matrix}$$

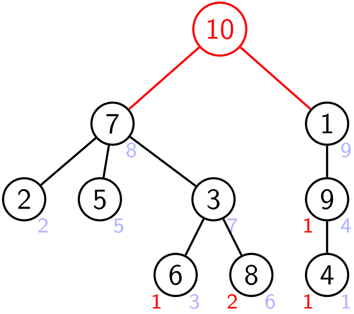
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)	(87)	(63)	(83)

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

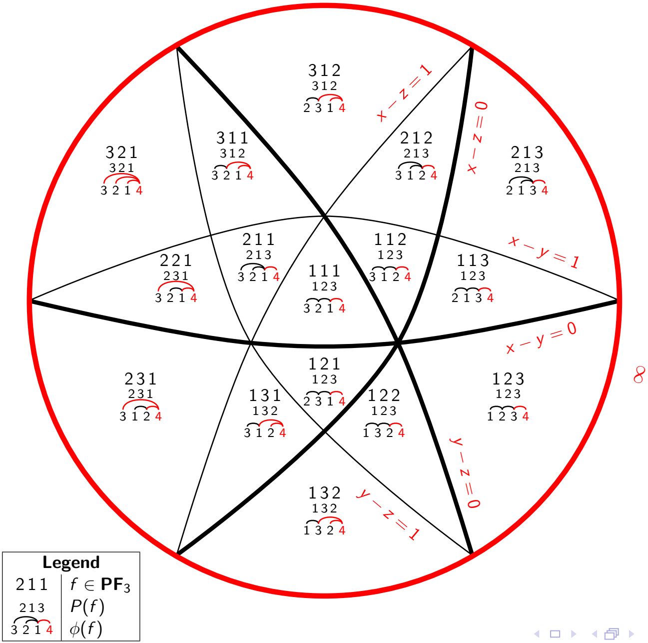
$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$



91	41	83	87	63
$\beta(8)\beta(9)$	$\beta(7)\beta(9)$	$\beta(2)\beta(4)$	$\beta(2)\beta(6)$	$\beta(3)\beta(4)$
(41)	(91)	(87)	(63)	(83)

$\pi^{-1}\downarrow$	4	9	1	8	6	3	5	2	7
	1	4	9	6	3	7	5	2	8

$$\beta = 5\ 8\ 6\ 3\ 2\ 7\ 4\ 9\ 1$$


x − z = 1
x − z = 0
x − y = 1
x − y = 0
y − z = 0
y − z = 1
8
312
312
2 3 1 4
311
312
3 2 1 4
321
321
3 2 1 4
212
213
3 1 2 4
213
213
2 1 3 4
211
213
3 2 1 4
221
231
3 2 1 4
111
123
3 2 1 4
112
123
3 1 2 4
113
123
2 1 3 4
231
231
3 1 2 4
131
132
3 1 2 4
121
123
2 3 1 4
122
123
1 3 2 4
123
123
1 2 3 4
132
132
1 3 2 4
Legend
211 | f ∈ PF3
213 | P(f)
3 2 1 4 | φ(f)

Parking functions - the number

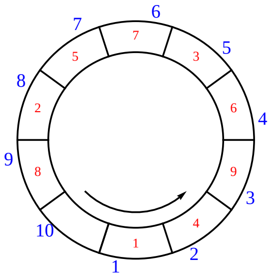
	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
$\sigma = P(f)$:	1	8	5	2	7	4	6	9	3
σ^{-1} :	1	4	9	6	3	7	5	2	8

$\longrightarrow$ $\longrightarrow$

Parking functions - the number

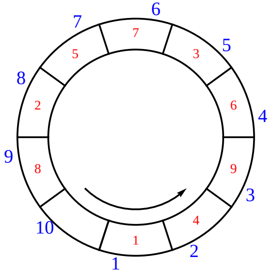
	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
$\sigma = P(f)$:	1	8	5	2	7	4	6	9	3
σ^{-1} :	1	4	9	6	3	7	5	2	8

$\longrightarrow$ $\longrightarrow$



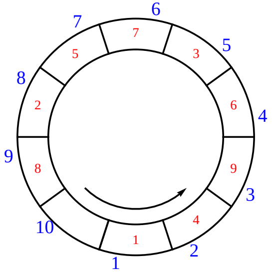
Parking functions - the number

	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	4	4	8	1
$\sigma = P(f)$:	1	8	5	2	7	4	6	9	3



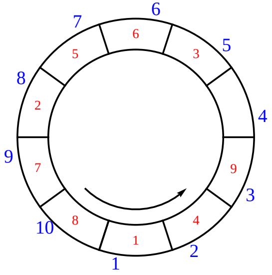
Parking functions - the number

	1	2	3	4	5	6	7	8	9
						5	5		
f :	1	8	5	2	7	~~4~~	~~4~~	8	1
$\sigma = P(f)$:	1	8	5	2	7	~~4~~	~~6~~	~~9~~	3
						6	9	10	



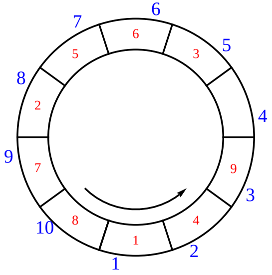
Parking functions - the number

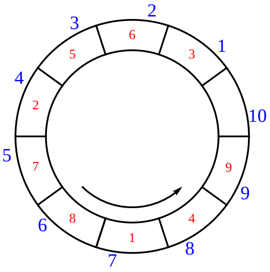
	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	5	5	8	1
$\sigma = P(f)$:	1	8	5	2	7	6	9	10	3



Parking functions - the number

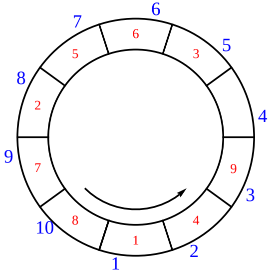
	1	2	3	4	5	6	7	8	9
f :	1	8	5	2	7	5	5	8	1
$\sigma = P(f)$:	1	8	5	2	7	6	9	10	3

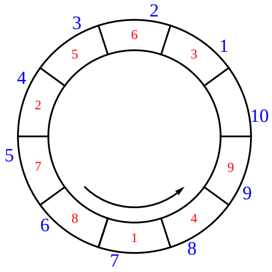




Parking functions - the number

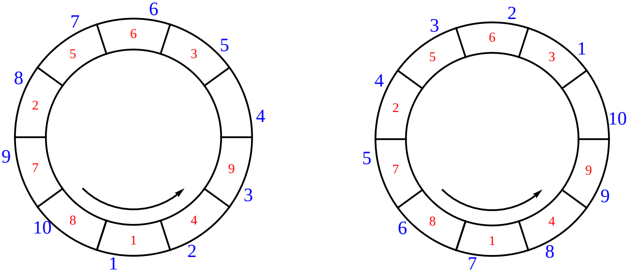
	1	2	3	4	5	6	7	8	9
$g \in \mathbf{PF}_9$	7	4	1	8	3	2	2	5	8
f :	1	8	5	2	7	5	5	8	1
$\sigma = P(f)$:	1	8	5	2	7	6	9	10	3





Parking functions - the number

	1	2	3	4	5	6	7	8	9
$g \in \mathbf{PF}_9$	7	4	1	8	3	2	2	5	8
f :	1	8	5	2	7	5	5	8	1
$\sigma = P(f)$:	1	8	5	2	7	6	9	10	3



$$|\mathbf{PF}_n| = (n+1)^n/(n+1) = (n+1)^{n-1} = |\mathbf{T}_{n+1}|$$