Properties of an asymmetric annihilation process

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SLC 64 – Lyon, March 2010

The annihilation process

Proof of the eigenvalue conjecture (outline)

Linear algebra approach for a generalized partition function

Denominators of weighted shifted tableaux

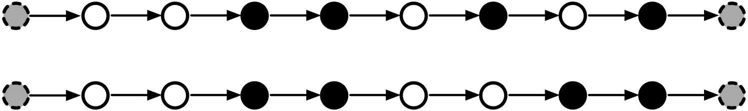
For a detailed discussion of the physical model see:

Arvind Ayyer, Kirone Mallick (C.A.E. Saclay)
Exact results for an asymmetric annihilation process with open boundaries
J. Phys. A: Math. Gen. 343 045033 2010, 22pp.

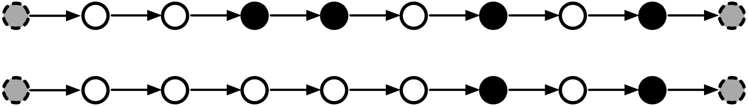
In this paper the model is introduced and various properties are obtained: transition matrices, transfer matrices, partition functions, distributions related to particular states etc. The paper ends with a conjecture for the eigenvalues of the transition matrices. A proof of (a generalized version of) this conjecture is outlined in these slides. Furthermore, transfer matrices and partition functions for a generalized model are given. Finally, an interesting property of shifted standard tableaux that is related to the partition functions is sketched.

The model (1)

- right shift (1)

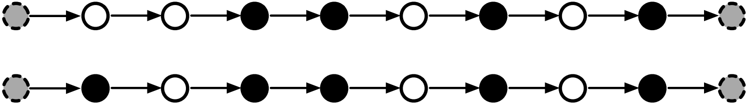


- annihilation (λ)

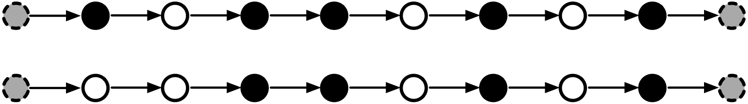


The model (2)

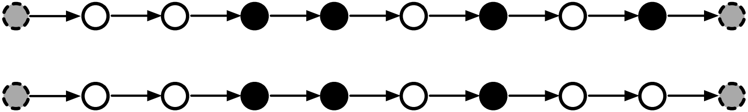
- left creation (α)



- left annihilation ($\alpha\lambda$)



- right annihilation (β)



- Example of a transition matrix ($L = 2$)

	00	01	10	11
00	$\star$	β	$\alpha\lambda$	λ
01		$\star$	λ	$\alpha\lambda$
10	α		$\star$	β
11		α		$\star$

Note: The diagonal elements $\star$ must be chosen so that column sums vanish

- Example of a transition matrix ($L = 3$)

	000	001	010	011	100	101	110	111
000	$\star$	β		λ	$\alpha\lambda$		λ	
001		$\star$	1			$\alpha\lambda$		λ
010			$\star$	β	1		$\alpha\lambda$	
011				$\star$		1		$\alpha\lambda$
100	α				$\star$	β		λ
101		α				$\star$	1	
110			α				$\star$	β
111				α				$\star$

Note: The diagonal elements $\star$ must be chosen so that column sums vanish

- inductive structure of the transition matrices ($\lambda = 1$)
  Let
  $$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
  and $\mathbf{1}_L$ the $(2^L \times 2^L)$-unit matrix
  Define
  $$M_1 = \begin{bmatrix} -\alpha & \alpha + \beta \\ \alpha & -\alpha - \beta \end{bmatrix}$$
  and then inductively the $(2^L \times 2^L)$-matrices
  $$M_L = \begin{bmatrix} M_{L-1} - \alpha(\sigma \otimes \mathbf{1}_{L-2}) & \alpha \mathbf{1}_{L-1} + (\sigma \otimes \mathbf{1}_{L-2}) \\ \alpha \mathbf{1}_{L-1} & M_{L-1} - \alpha(\sigma \otimes \mathbf{1}_{L-2}) - \mathbf{1}_{L-1} \end{bmatrix}$$

- The conjecture by Ayyer and Mallick

  The characteristic polynomial $P_L(x)$ of M_L is given by

$$P_L(x) = A_L(x)\, A_L(x + 2\alpha + \beta)\, B_L(x + \beta)\, B_L(x + 2\alpha)$$

  where

$$A_L(x) = \prod_{k\geq 0} (x + 2k)^{\binom{L-1}{2k}}$$

$$B_L(x) = \prod_{k\geq 0} (x + 2k + 1)^{\binom{L-1}{2k+1}}$$

1. Proof of the eigenvalue conjecture (outline)

- some notation
  - $\mathbb{B}^L$: vector space of bitvectors of length L (over $\mathbb{F}_2$)
  - $V_L = \{ |\, \mathbf{b} \,\rangle \, ; \, \mathbf{b} \in \mathbb{B}^L \}$
  - $\mathcal{V}_L$: vector space with basis V_L (over an extension of $\mathbb{Q}$)
  - $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - For $\mathbf{b} = b_1 b_2 \dots b_L \in \mathbb{B}^L$

$$\sigma^{\mathbf{b}} = \sigma^{b_1 b_2 \dots b_L} = \sigma^{b_1} \otimes \sigma^{b_2} \otimes \cdots \otimes \sigma^{b_L}.$$

    defines a linear transformation $\sigma^{\mathbf{b}}$ of $\mathcal{V}_L = \mathcal{V}_1^{\otimes L}$ (w.r.t. the basis V_L)

- For $1 \leq j \leq L$ define involutive mappings

$$\phi_j : \mathbb{B}^L \to \mathbb{B}^L :$$
$$b_1 \dots b_{j-1} b_j \, b_{j+1} \dots b_L \mapsto \phi_j \mathbf{b} = b_1 \dots b_{j-1} \, \overline{b_j} \, b_{j+1} \dots b_L$$

  by complementing the j-th component,

- and involutions

$$\psi_j : \mathbb{B}^L \to \mathbb{B}^L : \mathbf{b} \mapsto \phi_j \phi_{j+1} \mathbf{b}$$

  by complementing components indexed j and $j+1$.
  ψ_L is the same as ϕ_L.

- The $\mathcal{A}$-transformation
  - $\boldsymbol{\alpha} = (\alpha_{\mathbf{b}})_{\mathbf{b}\in\mathbb{B}^L}$ variables
  - the transformation $\mathcal{A}_L(\boldsymbol{\alpha})$ of $\mathcal{V}_L$ is defined by its matrix (w.r.t. the basis V_L)

$$A_L(\boldsymbol{\alpha}) = \sum_{\mathbf{b}\in\mathbb{B}^L} \alpha_{\mathbf{b}}\,\sigma^{\mathbf{b}}$$

  - direct definition (matrix elements)

$$\langle\, \mathbf{b} \,|\, A_L \,|\, \mathbf{c} \,\rangle = \alpha_{\mathbf{b}\oplus\mathbf{c}},\ \ (\mathbf{b}, \mathbf{c} \in \mathbb{B}^L),$$

    where $\oplus$ denotes $\mathbb{F}_2$-addition (exor) of bit vectors.

- Example ($L = 2$)

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{10} & \alpha_{11} \\ \alpha_{01} & \alpha_{00} & \alpha_{11} & \alpha_{10} \\ \alpha_{10} & \alpha_{11} & \alpha_{00} & \alpha_{01} \\ \alpha_{11} & \alpha_{10} & \alpha_{01} & \alpha_{00} \end{bmatrix}$$

- The $\mathcal{B}$-transformation
  - For $1 \leq j \leq L$ define projection operators $\mathcal{P}_{L,j}$ on $\mathcal{V}_L$ by

$$\mathcal{P}_{L,j} = \sum_{\mathbf{b} \in \mathbb{B}^L} |\, \mathbf{b} \,\rangle\langle\, \mathbf{b} \,| - |\, \psi_j^{b_j}(\mathbf{b}) \,\rangle\langle\, \mathbf{b} \,|$$

  - For variables $\mathbf{b} = (\beta_1, \beta_2, \ldots, \beta_L)$ let

$$\mathcal{B}_L(\boldsymbol{\beta}) = \sum_{1 \leq j \leq L} \beta_j \, \mathcal{P}_{L,j}$$

  - $B_L(\boldsymbol{\beta})$: the matrix representing $\mathcal{B}_L(\boldsymbol{\beta})$ in V_L

- In the sum for $\mathcal{P}_{L,j}$ only summands for which $b_j = 1$, i.e., for which $\psi_j(\mathbf{b}) < \mathbf{b}$, occur.
  Indeed: this condition allows only for two situations to contribute:

$$b_j\, b_{j+1} = 10 \;\mapsto\; \overline{b_j}\,\overline{b_{j+1}} = 01 \qquad \text{(right shift)}$$
$$b_j\, b_{j+1} = 11 \;\mapsto\; \overline{b_j}\,\overline{b_{j+1}} = 00 \qquad \text{(annihilation)}$$

  These operators ψ_j encode the transitions of our model.
- By definition, $B_L(\beta)$ is an upper triangular matrix.

- Example for $L = 3$ (with $\boldsymbol{\beta} = (\beta, \gamma, \delta)$ in place of $(\beta_1, \beta_2, \beta_3)$):

$$B_3(\beta, \gamma, \delta) =$$

$$\begin{bmatrix} 0 & -\delta & 0 & -\gamma & 0 & 0 & -\beta & 0 \\ 0 & \delta & -\gamma & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & \gamma & -\delta & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta+\gamma & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & -\delta & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & \delta+\beta & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma+\beta & -\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta+\gamma+\beta \end{bmatrix}.$$

- another linear transformation (over $\mathbb{F}_2$)

  $\Delta : \mathbb{B}^L \to \mathbb{B}^L$:

$$\mathbf{b} = [b_1 b_2 \dots b_L] \mapsto \mathbf{b}^{\Delta} = [b_1 b_2 \dots b_L] \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & & & & \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

- As an example ($L = 3$):

$\mathbf{b}$	000	001	010	011	100	101	110	111
$\mathbf{b}^{\Delta}$	000	100	110	010	111	011	001	101

- Main result: Consider the transformation

$$\mathcal{M}_L(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{A}_L(\boldsymbol{\alpha}) - \mathcal{B}_L(\boldsymbol{\beta}).$$

  with its matrix (w.r.t. V_L) $M_L(\boldsymbol{\alpha}, \boldsymbol{\beta}) = A_L(\boldsymbol{\alpha}) - B_L(\boldsymbol{\beta})$

  Then

$$\det M_L(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \det \left[A_L(\boldsymbol{\alpha}) - B_L(\boldsymbol{\beta})\right] = \prod_{\mathbf{b} \in \mathbb{B}^L} \left(\lambda_{\mathbf{b}^\Delta} - \boldsymbol{\beta}^{\texttt{rev}} \cdot \mathbf{b}\right)$$

  where

  - the $\lambda_{\mathbf{b}}$ are the eigenvalues of $\mathcal{A}_L(\boldsymbol{\alpha})$

$$\lambda_{\mathbf{b}} = \sum_{c \in \mathbb{B}^L} (-1)^{\mathbf{b} \cdot \mathbf{c}} \, \alpha_{\mathbf{c}} \qquad (\mathbf{b} \in \mathbb{B}^L)$$

  - $\boldsymbol{\beta}^{\texttt{rev}} = (b_L, b_{L-1}, \ldots, b_2, b_1)$ is the reverse of $\boldsymbol{\beta} = (b_1, b_2, \ldots, b_{L-2}, b_L)$

- Illustration of the Theorem for $L = 3$:

$\mathbf{b}$	$\mathbf{b}^{\Delta}$	$\lambda_{b^{\Delta}}$	$(\delta, \gamma, \beta) \cdot \mathbf{b}$
000	000	$[++++++++] \cdot \boldsymbol{\alpha}$	0
001	100	$[++++----] \cdot \boldsymbol{\alpha}$	β
010	110	$[++----++] \cdot \boldsymbol{\alpha}$	γ
011	010	$[++--++--] \cdot \boldsymbol{\alpha}$	$\beta + \gamma$
100	111	$[+--+-++-] \cdot \boldsymbol{\alpha}$	δ
101	011	$[+--++--+] \cdot \boldsymbol{\alpha}$	$\beta + \delta$
110	001	$[+-+-+-+-] \cdot \boldsymbol{\alpha}$	$\gamma + \delta$
111	101	$[+-+--+-+] \cdot \boldsymbol{\alpha}$	$\beta + \gamma + \delta$

So, as an example, the line for $\mathbf{b} = 101$ contributes the factor

$$\alpha_{000} - \alpha_{001} - \alpha_{010} + \alpha_{011} + \alpha_{100} - \alpha_{101} - \alpha_{110} + \alpha_{111} - \beta - \delta$$

to the product.

- Idea of proof:
  - Use the Hadamard transform on $\mathcal{V}_L$, given by

$$H^{\otimes L} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{\otimes L}$$

    to diagonalize $\mathcal{A}_L(\boldsymbol{\alpha})$. The basis

$$W_L = \left\{ H^{\otimes L} |\, \mathbf{b} \,\rangle \,;\, \mathbf{b} \in \mathbb{B}^L \right\} = \left\{ \,|\, w^{\mathbf{b}} \,\rangle \,;\, \mathbf{b} \in \mathbb{B}^L \,\right\}$$

    is the adapted ON-basis of $\mathcal{V}_L$
  - $\mathcal{B}_L$ is not even triangular in the basis W_L, but becomes lower triangular w.r.t. the basis

$$\widetilde{W}_L = \{ \,|\, w^{\mathbf{b}^\Delta} \,\rangle \,;\, \mathbf{b} \in \mathbb{B}^L \,\}$$

    namely

$$\widetilde{H}_L \cdot B_L(\boldsymbol{\beta}) \cdot \widetilde{H}_L = B_L^{\mathtt{t}}(\boldsymbol{\beta}^{\mathtt{rev}})$$

    where $\widetilde{H}_L$ is the matrix of the Hadamard transformation combined with the Δ-transformation.

- Comments
  - The original conjecture by AYYER-MALLICK about the eigenvalues of the *asymmetric annihilation process* is a particular case of the above factorized determinant
  - In the original problem the eigenvalues had high multiplicities and the eigenspaces were maximally degenerate (experimental observation) – in the extended "symbolic" model all eigenvalues are simple
  - For the extended model a partition function can be computed

$$Z(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{\mathbf{0} \neq \mathbf{b} \in \mathbb{B}^L} (\lambda^*_{b^\Delta} + \boldsymbol{\beta}^{\mathtt{rev}} \cdot \mathbf{b})$$

where

$$\lambda^*_{\mathbf{b}} = 2 \sum_{\mathbf{c}\,:\,\mathbf{b}\cdot\mathbf{c}=1} \alpha_{\mathbf{c}} \qquad (\mathbf{b} \in \mathbb{B}^L)$$

which has degree $2^L - 1$ – instead of $\binom{L+1}{2}$ in the original model.

2. Linear algebra approach for a generalized partition function

- Some notation
  - $\mathbb{I}_k$ denotes the $2^k \times 2^k$-identity matrix
  - $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - if N is a $2^n \times 2^n$ matrix with $n > 0$, then

$$N^\sigma = N \cdot (\sigma \otimes \mathbb{I}_{n-1})$$

  - for matrices $M = [m_{i,j}]_{\substack{1 \le i \le k \\ 1 \le j \le \ell}}$ and N (as before)

$$M \otimes_\sigma N = \left[m_{i,j} \cdot N^{\sigma^{i+j}} \right]_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} = \begin{bmatrix} m_{1,1}N & m_{1,2}N^\sigma & m_{1,3}N & \dots \\ m_{2,1}N^\sigma & m_{2,2}N & m_{2,3}N^\sigma & \dots \\ m_{3,1}N & m_{3,2}N^\sigma & m_{3,3}N & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  - a useful homomorphism property

$$(M \otimes_\sigma N)^\sigma = M^\sigma \otimes_\sigma N^\sigma$$

- generalized transition matrices

$$M_1 = \begin{bmatrix} z & a+b \\ a & z-b \end{bmatrix}$$

$$M_2 = \begin{bmatrix} z & b & a & c \\ 0 & z-b & c & a \\ a & 0 & z-c & b \\ 0 & a & 0 & z-b-c \end{bmatrix}$$

$$M_3 =$$

$$\begin{bmatrix} z & b & 0 & c & a & 0 & d & 0 \\ 0 & z-b & c & 0 & 0 & a & 0 & d \\ 0 & 0 & z-c & b & d & 0 & a & 0 \\ 0 & 0 & 0 & z-b-c & 0 & d & 0 & a \\ a & 0 & 0 & 0 & z-d & b & 0 & c \\ 0 & a & 0 & 0 & 0 & z-b-d & c & 0 \\ 0 & 0 & a & 0 & 0 & 0 & z-c-d & b \\ 0 & 0 & 0 & a & 0 & 0 & 0 & z-b-c-d \end{bmatrix}$$

- writing the transition matrices using the matrices $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\tau = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$:
- $$M_1 = z \cdot \mathbb{I}_1 + a\,\sigma + b\,\tau$$
- $$M_2 = z \cdot \mathbb{I}_2 + a \cdot \sigma \otimes \mathbb{I}_1 + b \cdot \mathbb{I}_1 \otimes \tau + c \cdot \tau \otimes_\sigma \mathbb{I}_1$$
- $$M_3 = z \cdot \mathbb{I}_3 + a \cdot \sigma \otimes \mathbb{I}_2 + b \cdot \mathbb{I}_2 \otimes \tau + c \cdot \mathbb{I}_1 \otimes \tau \otimes_\sigma \mathbb{I}_1 + d \cdot \mathbb{I}_0 \otimes \tau \otimes_\sigma \mathbb{I}_2$$

- the general picture

$$M_k = z \cdot \mathbb{I}_k + a_0 \cdot \sigma \otimes \mathbb{I}_{k-1} + \sum_{j=0}^{k-1} a_{j+1} \cdot \mathbb{I}_{k-1-j} \otimes \tau \otimes_\sigma \mathbb{I}_j$$

- wanted: transfer matrices $T_{k+1,k}$ of format $2^{k+1} \times 2^k$ such that

$$M_{k+1} \cdot T_{k+1,k} = T_{k+1,k} \cdot M_k$$

- $k = 1$

$$T_{2,1} = \begin{bmatrix} ab + bc + c^2 & ab + ac + bc \\ ac & c^2 \\ ab + ac & ab \\ 0 & ac \end{bmatrix}$$

$$= \begin{bmatrix} c \\ a \end{bmatrix} \otimes \underbrace{\begin{bmatrix} b + c & b \\ 0 & c \end{bmatrix}}_{\beta(b,c)} + \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} b & b + c \\ c & 0 \end{bmatrix}}_{\beta^{\sigma}(b,c)}$$

$$= \begin{bmatrix} c \\ a \end{bmatrix} \otimes \beta(b, c) + \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \beta^{\sigma}(b, c)$$

- $k = 2$

$$T_{3,2} =$$

$$\begin{bmatrix} d^2c+d^2b+d^3+dcb+acb & d^2b+cda+dcb+acb & dcb+cda+abd+ad^2+acb & d^2c+abd+dcb+acb \\ cda & (d+c)\,d^2 & d^2c & (d+c)\,ad \\ (b+d)\,ad & abd & (b+d)\,d^2 & d^2b \\ 0 & ad^2 & 0 & d^3 \\ a\left(bd+d^2+cb+cd\right) & ba\,(d+c) & acb & ac\,(b+d) \\ 0 & (d+c)\,ad & cda & 0 \\ 0 & 0 & (b+d)\,ad & abd \\ 0 & 0 & 0 & ad^2 \end{bmatrix}$$

$$= \begin{bmatrix} cd+d^2 & ac+ad \\ ad & d^2 \\ ac+ad & 0 \\ 0 & ad \end{bmatrix} \otimes \beta(b,d) + \begin{bmatrix} ac & cd \\ 0 & 0 \\ 0 & ac \\ 0 & 0 \end{bmatrix} \otimes \beta^{\sigma}(b,d)$$

that is

$$T_{3,2} = \begin{bmatrix} d \\ a \end{bmatrix} \otimes B + \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes B^{\sigma}$$

where

$$B = \begin{bmatrix} (c+d)\beta(b,d) & c\,\beta^{\sigma}(b,d) \\ 0 & d\,\beta(b,d) \end{bmatrix} = \beta(c,d) \otimes_{\sigma} \beta(b,d)$$

$$B^{\sigma} = \begin{bmatrix} c\,\beta^{\sigma}(b,d) & (c+d)\beta(b,d) \\ d\,\beta(b,d) & 0 \end{bmatrix} = \beta^{\sigma}(c,d) \otimes_{\sigma} \beta^{\sigma}(b,d)$$

so we get

$$T_{3,2} = \begin{bmatrix} d \\ a \end{bmatrix} \otimes \underbrace{\beta(c,d) \otimes_{\sigma} \beta(b,d)}_{\beta(b,c,d)} + \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \underbrace{\beta^{\sigma}(c,d) \otimes_{\sigma} \beta^{\sigma}(b,d)}_{\beta^{\sigma}(b,c,d)}$$

- $k = 3$

$$T_{4,3} = \begin{bmatrix} e \\ a \end{bmatrix} \otimes \underbrace{\beta(d, e) \otimes_\sigma \beta(c, e) \otimes_\sigma \beta(b, e)}_{\beta(b,c,d,e)}$$

$$+ \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \underbrace{\beta^\sigma(d, e) \otimes_\sigma \beta^\sigma(c, e) \otimes_\sigma \beta^\sigma(b, e)}_{\beta^\sigma(b,c,d,e)}$$

- the general picture: the transfer matrices are

$$T_{k+1,k} = \begin{bmatrix} a_{k+1} \\ a_0 \end{bmatrix} \otimes \beta(a_1, a_2, \ldots, a_{k+1}) + \begin{bmatrix} a_0 \\ 0 \end{bmatrix} \otimes \beta^{\sigma}(a_1, a_2, \ldots, a_{k+1})$$

where

$$\beta(a_1, a_2, \ldots, a_{k+1}) = \beta(a_k, a_{k+1}) \otimes_{\sigma} \beta(a_{k-1}, a_{k+1}) \otimes_{\sigma} \cdots \otimes_{\sigma} \beta(a_1, a_{k+1})$$

- the partition functions
  - the stationary vectors (= right eigenvectors for eigenvalue $a_0 + a_1$)

$$|v_1\rangle = \begin{bmatrix} a_0 + a_1 \\ a_0 \end{bmatrix} \qquad |v_{k+1}\rangle = T_{k+1,k} \,|\, v_k\rangle$$

  - the partition functions

$$z_k = \langle 1^{2^k} \,|\, v_k\rangle$$

    so that

$$z_0 = 2a_0 + a_1 \qquad z_{k+1} = \langle 1^{2^{k+1}} |\, T_{k+1,k} \,|\, v_k\rangle$$

- note

$$\langle 11 \,|\, \beta(b,c) = \langle 11 \,|\begin{bmatrix} b+c & b \\ 0 & c \end{bmatrix} = (b+c)\langle 11|$$

$$\langle 11 \,|\, \beta^{\sigma}(b,c) = \langle 11 \,|\begin{bmatrix} b & b+c \\ c & 0 \end{bmatrix} = (b+c)\langle 11|$$

- so that (as an example)

$$\begin{aligned} z_3 &= \langle 1^8 \,|\, v_3\rangle = \langle 1^8 \,|\, T_{3,2} \,|\, v_2\rangle \\ &= \langle 11|^{\otimes 3}\left(\begin{bmatrix} d \\ a \end{bmatrix} \otimes \beta(c,d) \otimes_{\sigma} \beta(b,d)\right) \,|\, v_2\rangle \\ &\qquad + \langle 11|^{\otimes 3}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \beta^{\sigma}(c,d) \otimes_{\sigma} \beta^{\sigma}(b,d)\right) \,|\, v_2\rangle \\ &= \big((a+d)(c+d)(b+d) + a(c+d)(b+d)\big) \cdot \langle 1^4 \,|\, v_2\rangle \\ &= (2a+d)(b+d)(c+d) \cdot z_2 \end{aligned}$$

- and the general result is

$$z_k = \prod_{1 \leq j \leq k} (2a_0 + a_j) \cdot \prod_{1 \leq i < j \leq k} (a_i + a_j)$$

3. Denominators of weighted shifted tableaux

- some notation
- *shifted shape* (alias *strict partition*) : a sequence $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of integers with $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0$
  - *length* of $\boldsymbol{\lambda}$: $|\boldsymbol{\lambda}| = \ell$ (= number of parts)
  - *size* of $\boldsymbol{\lambda}$: $\|\boldsymbol{\lambda}\| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ (sum of parts)
- A weight function on shifted shapes $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

$$w(\boldsymbol{\lambda}) = x_{\lambda_1} + x_{\lambda_2} + \cdots + x_{\lambda_\ell}$$

$x_1, x_2, \ldots$ are variables

- *Shifted standard Young tableaux* (sSYT) of shape $\boldsymbol{\lambda}$ are defined as usual
- A sSYT t is as a sequence of nested shifted shapes

$$t : \emptyset \subset \boldsymbol{\lambda}^{(1)} \subset \boldsymbol{\lambda}^{(2)} \subset \ldots \subset \boldsymbol{\lambda}^{(\|\boldsymbol{\lambda}\|)} = \boldsymbol{\lambda}$$

  where $\|\boldsymbol{\lambda}^{(j)}\| = j \quad (1 \leq j \leq \|\boldsymbol{\lambda}\|)$
- $sSYT(\boldsymbol{\lambda})$: the set of all sSYTs t of shape $\boldsymbol{\lambda}$
- multiplicative extension of the weight function

$$w(t) = \prod_{1 \leq j \leq \|\boldsymbol{\lambda}\|} w(\boldsymbol{\lambda}^{(j)}).$$

  $w(t)$ is a polynomial in $x_1, x_2, \ldots, x_{\lambda_1}$ of degree $\|\boldsymbol{\lambda}\|$

- now define

$$S(\boldsymbol{\lambda}) := \sum_{t \in sSYT(\boldsymbol{\lambda})} \frac{1}{w(t)} = \frac{num(\boldsymbol{\lambda})}{den(\boldsymbol{\lambda})}$$

where

  - $num(\boldsymbol{\lambda})$ is a polynomial in $x_1, x_2, \ldots, x_{\lambda_1}$ of degree $\deg(den(\boldsymbol{\lambda})) - \|\boldsymbol{\lambda}\|$,
  - the fraction is reduced, i.e. $num(\boldsymbol{\lambda})$ and $den(\boldsymbol{\lambda})$ have no common factor

- Claim:
  - if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$, then

$$den(\boldsymbol{\lambda}) = \prod_{\boldsymbol{\mu} \subseteq (\lambda_1, \lambda_2)} w(\boldsymbol{\mu}),$$

    where the $\boldsymbol{\mu}$'s appearing in the product have length two or one. So $den(\boldsymbol{\lambda})$ depends only on the two largest parts of $\boldsymbol{\lambda}$.
  - if $\boldsymbol{\lambda}$ has length one, i.e. $\boldsymbol{\lambda} = (\lambda_1)$, this reduces to

$$den(\boldsymbol{\lambda}) = \prod_{1 \leq i \leq \lambda_1} x_i.$$

- Illustration

$$t = \begin{array}{ccccc} & & 8 & & \\ & 3 & 6 & & \\ 1 & 2 & 4 & 5 & 7 \end{array}$$

$$= (12457 \mid 36 \mid 8)$$

$$= \langle (1), (2), (2,1), (3,1), (4,1), (4,2), (5,2), (5,2,1) \rangle$$

$$w(t) =$$

$$x_1 \cdot x_2 \cdot (x_2 + x_1) \cdot (x_3 + x_1) \cdot (x_4 + x_1) \cdot (x_4 + x_2) \cdot (x_5 + x_2) \cdot (x_5 + x_2 + x_1)$$

- the general rule

$$S(\boldsymbol{\lambda}) = \frac{1}{w(\boldsymbol{\lambda})} \sum_{\boldsymbol{\mu} \lhd \boldsymbol{\lambda}} S(\boldsymbol{\mu})$$

where $\lhd$ denotes covering w.r.t. inclusion

Illustration for $\boldsymbol{\lambda} = (3,2,1)$:

- $(3,2) \lhd (3,2,1)$
- $\boldsymbol{\lambda} = (3,2)$, $sSYT(3,2) = \{(123 \mid 45), (124 \mid 35)\}$

$$S(3,2) = \frac{1}{w(3,2)}\, S(3,1)$$
$$= \frac{x_1 + x_2 + x_3}{x_1\, x_2\, x_3\, (x_1 + x_2)\, (x_1 + x_3)\, (x_2 + x_3)}$$

- $\boldsymbol{\lambda} = (3,2,1)$, $sSYT(3,2,1) = \{(123 \mid 45 \mid 6), (124 \mid 35 \mid 6)\}$

$$S(3,2,1) = \frac{1}{w(3,2,1)} S(3,2) =$$
$$= \frac{1}{x_1\, x_2\, x_3\, (x_1 + x_2)\, (x_1 + x_3)\, (x_2 + x_3)}$$

Illustration for $\boldsymbol{\lambda} = (4,2,1)$:

- $(3,2,1) \lhd (4,2,1)$, $(4,2) \lhd (4,2,1)$
- $\boldsymbol{\lambda} = (3,2,1)$

$$S(3,2,1) = \frac{1}{x_1\, x_2\, x_3\, (x_1+x_2)\,(x_1+x_3)\,(x_2+x_3)}$$

- $\boldsymbol{\lambda} = (4,2)$

$$sSYT(4,2) = \{(1234 \mid 56), (1235 \mid 46), (1245 \mid 36), (1236 \mid 45), (1246 \mid 35)\}$$

- $$S(4,2) = \frac{1}{w(4,2)}\big(S(4,1) + S(3,2)\big)$$

$$= \frac{x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_1x_2^2 + x_1x_3^2 + x_1x_4^2 + x_2^2x_3 + x_2^2x_4 + \cdots + x_3x_4^2}{x_1x_2x_3x_4\,(x_1+x_2)\,(x_1+x_3)\,(x_1+x_4)\,(x_2+x_3)\,(x_2+x_4)}$$
$$+ \frac{2\,x_1x_2x_3 + 2\,x_1x_2x_4 + 2\,x_1x_3x_4 + 2\,x_2x_3x_4}{x_1x_2x_3x_4\,(x_1+x_2)\,(x_1+x_3)\,(x_1+x_4)\,(x_2+x_3)\,(x_2+x_4)}$$
$$= \frac{m_{(2,1)}(x_1,x_2,x_3,x_4) + 2\,m_{(1,1,1)}(x_1,x_2,x_3,x_4)}{x_1x_2x_3x_4\,(x_1+x_2)\,(x_1+x_3)\,(x_1+x_4)\,(x_2+x_3)\,(x_2+x_4)}$$

- $\boldsymbol{\lambda} = (4, 2, 1)$

$$sSYT(4,2,1) = \{(1234 \mid 56 \mid 7), (1235 \mid 46 \mid 7), (1245 \mid 36 \mid 7), (1236 \mid 45 \mid 7), (1246 \mid 35 \mid 7), (1237 \mid 45 \mid 6), (1247 \mid 35 \mid 6)\}$$

$$\begin{aligned} S(4,2,1) &= \frac{1}{w(4,2,1)}\big(S(4,2) + S(3,2,1)\big) \\ &= \frac{x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3^2 + x_3x_4 + x_4^2}{x_1x_2x_3x_4\,(x_1+x_2)\,(x_1+x_3)\,(x_1+x_4)\,(x_2+x_3)\,(x_2+x_4)} \\ &= \frac{e_2(x_1,x_2,x_3,x_4) + x_3^2 + x_4^2}{x_1x_2x_3x_4\,(x_1+x_2)\,(x_1+x_3)\,(x_1+x_4)\,(x_2+x_3)\,(x_2+x_4)} \end{aligned}$$