Genocchi numbers and alternative tableaux

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Séminaire lotharingien de combinatoire
30 Mars 2010

Introduction

The Genocchi numbers are : $G_2 = 1,\ G_4 = 1,\ G_6 = 3,\ G_8 = 17,\ \ldots$
and $\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x \cdot \tan\left(\frac{x}{2}\right)$.

Consider the recurrence $F_1 = 1$, and :

$$F_n(x,y,z) = (x+y)(x+z)F_{n-1}(x+1,y,z) - x^2 F_{n-1}(x,y,z).$$

Proposition (Dumont, Foata)

$F_n(x,y,z)$ is symmetric in x, y, z, with non-negative coefficients, and $F_n(1,1,1) = G_{2n+2}$.

- Gandhi polynomials $F_n(x, 1, 1)$:
  - [Carlitz 1972] [Riordan and Stein 1973].
- Combinatorial interpretations of F_n :
  - [Dumont and Foata 1976] [Viennot 1981] [Han 1993]
- Explicit formula for $F_n(x, y, z)$:
  - [Carlitz 1980] [Han 1993] [Zeng 1995]
- J-fraction for $\sum t^n F_n$:
  - [Dumont, Randrianarivony, Zeng 1995] [Gessel and Zeng]

A more general sequence is defined by $\Gamma_1 = 1$ and

$$\begin{aligned}\Gamma_n(x,y,z,\bar{x},\bar{y},\bar{z}) =& (x+\bar{z})(y+\bar{x})\Gamma_{n-1}(x+1,y,z,\bar{x}+1,\bar{y},\bar{z}) \\ &+ \big(x(\bar{y}-y)-\bar{x}(\bar{z}-z)-x\bar{x}\big)\Gamma_{n-1}(x,y,z,\bar{x},\bar{y},\bar{z}).\end{aligned}$$

[Dumont, Randrianarivony, Zeng]

- $F_n(x,y,z) = \Gamma_n(x,y,z,x,y,z)$
- there is also a J-fraction for $\sum_n \Gamma_n t^n$
- If $(u,v,w) \mapsto (x,y,z)$ has signature 1 (resp. -1), we have :

$$\Gamma(u,v,w,\bar{u},\bar{v},\bar{w}) = \begin{cases} \Gamma(x,y,z,\bar{x},\bar{y},\bar{z}) \\ \text{resp. } \Gamma(\bar{x},\bar{y},\bar{z},x,y,z) \end{cases}$$

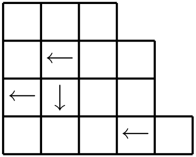
Outline

- Alternative tableaux
  - Definition
  - Combinatorial interpretation of F_n and Γ_n
- «Matrix Ansatz» for alternative tableaux
  - General case
  - Case of F_n and Γ_n
- J-fraction for $\sum t^n F_n$ and $\sum t^n \Gamma_n$
  - Weighted Motzkin paths
  - Case of F_n and Γ_n

Let λ be a Young diagram. An *alternative tableau* is a (partial) filling of λ with arrows $\leftarrow$ and $\downarrow$ such that :

- any cell below a $\downarrow$ (resp. to the left of a $\leftarrow$) are empty.

Example

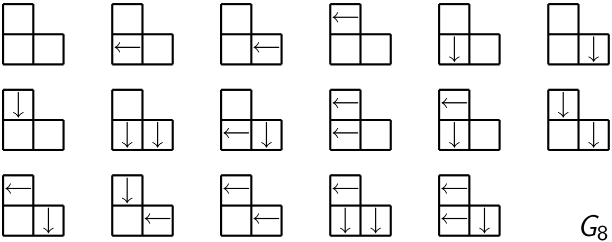


Permutation tableaux [Steigrímsson, Corteel, Williams], alternative tableaux [Viennot, Nadeau], links with combinatorics of permutations and PASEP [Corteel-Williams, Viennot]

It is known that G_{2n+4} is the number of alternative tableaux of *staircase* shape with n rows and n columns.

(G_{2n} counts some permutations called Dumont permutations, which correspond to staircase alternative tableaux via bijections of Steingrímsson and Williams, Corteel and Nadeau, Viennot).

Example



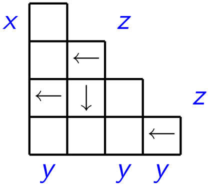
$G_8 = 17$

Example

We define three statistics :

- $\alpha(T) =$ number of rows without $\leftarrow$,
- $\beta(T) =$ number of columns without $\downarrow$,
- $\gamma(T) =$ number of corners with $\leftarrow$ or $\downarrow$,

and $w(T) = x^{\alpha(T)} y^{\beta(T)} z^{\gamma(T)}$



$w(T) = xy^3z^2$

Theorem

Let $Stc(n-1)$ be the set of staircase alternative tableaux with $n-1$ rows and $n-1$ columns. Then :

$$F_n(x, y, z) = \sum_{T \in \mathrm{Stc}(n-1)} w(T)$$

Proposition

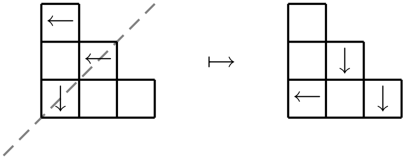
F_n is symmetric in x, y and z.

Proof

F_n is symmetric in y and z by the recurrence :

$$F_n(x,y,z) = (x+y)(x+z)F_{n-1}(x+1,y,z) - x^2 F_{n-1}(x,y,z).$$

F_n is symmetric in x and y from the combinatorial interpretation (we can conjugate alternative tableaux).



Proof of the theorem

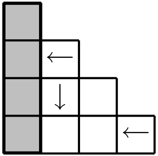
We suppose it is true for F_{n-1}. We have then :

Lemma

$$\sum_{\substack{T \in \mathrm{Stc}(n-1),\ \textit{with no} \downarrow \\ \textit{in the 1st column}}} w(T) = y(x+z)F_{n-1}(x+1, y, z)$$

Proof

We build $T \in \mathrm{Stc}(n-1)$ from $T' \in \mathrm{Stc}(n-2)$ by adding a column.



The upper cell can be empty or contain $\leftarrow$, whence a factor $(x+z)$.
For every row of T' without $\leftarrow$, we add either an empty cell, or a cell containing $\leftarrow$, whence the substitution x to $x+1$.

Lemma

$$\sum_{\substack{T \in \mathrm{Stc}(n-1) \\ \text{with no } \downarrow \text{ in the 1st column} \\ \text{at least a } \leftarrow \text{ in the 1st column}}} w(T) = y(x+z)F_{n-1}(x+1,y,z) - yxF_{n-1}(x,y,z)$$

Lemma

$$\sum_{\substack{T \in \mathrm{Stc}(n-1) \\ \text{with a } \downarrow \text{ in the 1st column}}} w(T) = x(x+z)F_{n-1}(x+1,y,z) - x^2F_{n-1}(x,y,z)$$

We add the first and third lemmas and obtain :

$$\begin{aligned}\sum_{T\in \mathrm{Stc}(n-1)} w(T) &= (x+y)(x+z)F_{n-1}(x+1,y,z) \\ &\qquad -x^2F_{n-1}(x,y,z) \\ &= F_n(x,y,z)\end{aligned}$$

This proves the theorem by recurrence.

The case of Γ_n

Recall that

$$\Gamma_n(x,y,z,\bar{x},\bar{y},\bar{z}) = (x+\bar{z})(y+\bar{x})\Gamma_{n-1}(x+1,y,z,\bar{x}+1,\bar{y},\bar{z}) + \big(x(\bar{y}-y)-\bar{x}(\bar{z}-z)-x\bar{x}\big)\Gamma_{n-1}(x,y,z,\bar{x},\bar{y},\bar{z}).$$

Theorem

We have

$$\Gamma_n = \sum_{T\in \mathrm{Stc}(n-1)} x^{\#\text{ of empty rows}} \times \bar{x}^{\#\text{ of non-empty rows with no }\leftarrow} \times y^{\#\text{ of non-empty columns with no }\downarrow} \times \bar{y}^{\#\text{ of empty columns}} \times z^{\#\text{ of corners containing a }\downarrow} \times \bar{z}^{\#\text{ of corners containing a }\leftarrow}$$

Sketch of proof

Let $\Gamma_n^+ = \Gamma_n(x+1, y, z, \bar{x}+1, \bar{y}, \bar{z})$. We need to distinguish six cases.

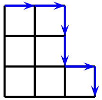
The upper left corner contains :

The leftmost column is :

	$\downarrow$	$\leftarrow$	*nothing*
empty	$\times$	$\times$	*Case 4* $x\bar{y}\Gamma_{n-1}$
non-empty with no $\downarrow$	$\times$	*Case 2* $y\bar{z}\Gamma^+_{n-1}$	*Case 5* $xy(\Gamma^+_{n-1} - \Gamma_{n-1})$
with a $\downarrow$	*Case 1* $\bar{x}z\Gamma_{n-1}$	*Case 3* $\bar{x}\bar{z}(\Gamma^+_{n-1} - \Gamma_{n-1})$	*Case 6* $x\bar{x}(\Gamma^+_{n-1} - \Gamma_{n-1})$

II.
Enumération of alternative alternative tableaux by «Matrix Ansatz»

Let λ be a Young diagram. It corresponds to a word m in D and E.
($\rightarrow$ becomes D, $\downarrow$ becomes E)



$m = DDEEDE$

Proposition (Corteel-Williams)

Given D, E, $\langle W|$ and $|V\rangle$ such that :

$$DE - ED = D + E, \qquad \langle W|E = x\langle W|, \qquad D|V\rangle = y|V\rangle,$$

we have :

$$\langle W|m|V\rangle = \sum x^{\alpha(T)} y^{\beta(T)}$$

where we sum over alternative tableaux T of shape λ.

$$DE - ED = D + E, \qquad \langle W|E = x\langle W|, \qquad D|V\rangle = y|V\rangle,$$

Explanation : using $DE = ED + D + E$, we can rewrite m in the form :

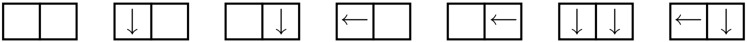
$$m = \sum_{i,j \geq 0} c_{i,j} E^i D^j$$

with $c_{i,j} \geq 0$. Then we have $\langle W|m|V\rangle = \sum_{i,j \geq 0} c_{i,j} x^i y^j$.

Example

$$\begin{aligned} DDE &= DED + DE + DD = DED + ED + D + E + DD \\ &= EDD + 2ED + 2DD + E + D, \end{aligned}$$

$$\langle W|m|V\rangle = xy^2 + 2xy + 2y^2 + x + y.$$



Proof

Reccurrence over $|\lambda|$.

The initial case is when $m = E^i D^j$ (diagram λ with i empty rows and j empty columns). Then :

$$\langle W|E^i D^j|V\rangle = x^i y^j$$

because $\langle W|E = x\langle W|$ and $F|V\rangle = y|V\rangle$.

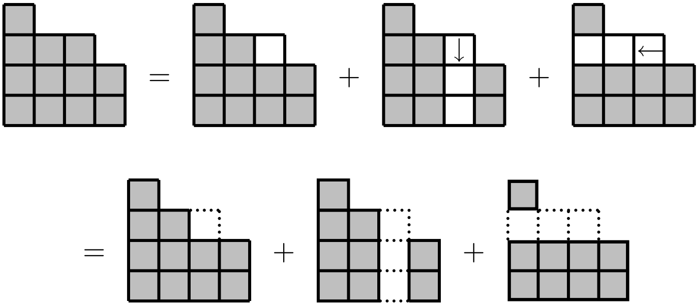
$\lambda =$ 

Proof

Recurrence on operators :

$$DED(DE)DEE = DED(ED)DEE + DED(E)DEE + DED(D)DEE$$

Recurrence on tableaux :

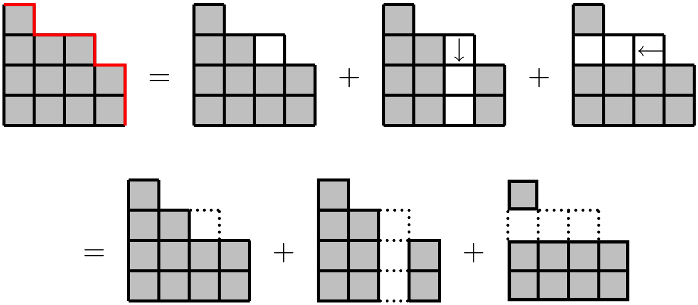


Proof

Recurrence on operators :

$$\color{red}{DED}(\color{red}{DE})\color{red}{DEE} = DED(ED)DEE + DED(E)DEE + DED(D)DEE$$

Recurrence on tableaux :

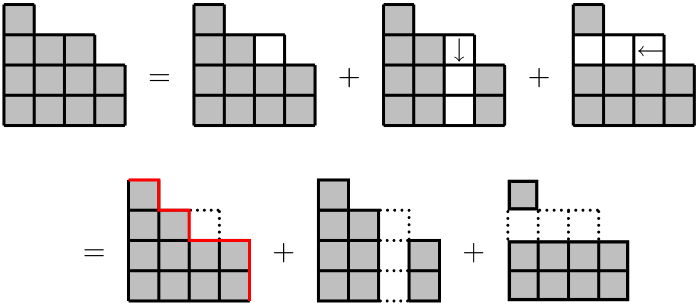


Proof

Recurrence on operators :

$$DED(DE)DEE = DED(ED)DEE + DED(E)DEE + DED(D)DEE$$

Recurrence on tableaux :

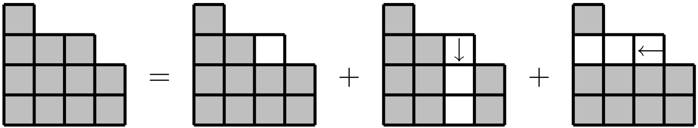


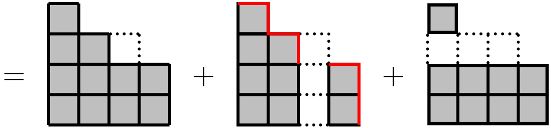
Proof

Recurrence on operators :

$$DED(DE)DEE = DED(ED)DEE + DED(E)DEE + DED(D)DEE$$

Recurrence on tableaux :





Proof

Recurrence on operators :

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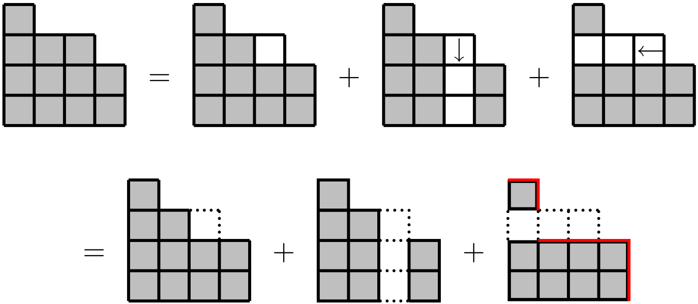
Recurrence on tableaux :

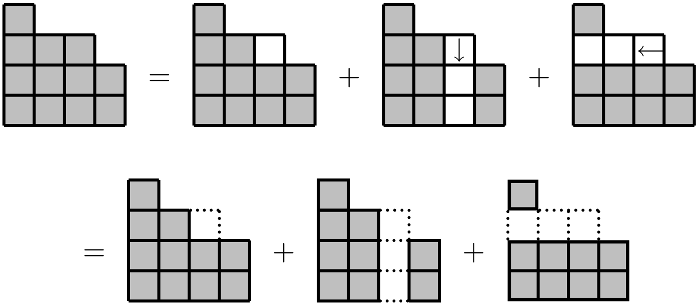


Proof

Recurrence on operators :

$$DED(DE)DEE = DED(ED)DEE + DED(E)DEE + DED(D)DEE$$

Recurrence on tableaux :



The recurrence relation are identical.

The staircase Young diagram corresponds to the word $(DE)^{n-1}$, so :

Proposition

$$F_n(x,y,1) = \sum_{T \in \mathrm{Stc}(n-1)} w(T)|_{z=1} \; = \; \langle W|(DE)^{n-1}|V\rangle.$$

To count the corners containing $\leftarrow$ or $\downarrow$, we write $DE = ED + D + E$ and mark D and E with a z :

Proposition

$$F_n(x,y,z) = \sum_{T \in \mathrm{Stc}(n-1)} w(T) \; = \; \langle W|(ED + zD + zE)^{n-1}|V\rangle.$$

Proposition

We have $\Gamma_n = \langle W|M^n|V\rangle$ *where*

$$M = ED + (\bar{z} + x - \bar{x})D + (z + \bar{y} - y)E + (\bar{y} - y)(x - \bar{x})I.$$

where $DE - ED = D + E, \qquad \langle W|E = \bar{x}\langle W|, \qquad D|V\rangle = y|V\rangle.$

Sketch of proof

We need to distinguish two kinds of empty rows with weights either $\bar{x}$ *or* $x - \bar{x}$*, instead of one kind with weight* x*. (Respectively, empty columns with weights* y *or* $\bar{y} - y$ *instead of* $\bar{y}$*.)*
Then we distinguish six kinds of corners :

- *containing* $\downarrow$*, this corresponds to the matrix* zE*,*
- *containing* $\leftarrow$*, this corresponds to the matrix* $\bar{z}D$*,*
- *four kinds of empty corners corresponding to matrices* ED*,* $(x - \bar{x})D$*,* $(\bar{y} - y)E$*, and* $(x - \bar{x})(\bar{y} - y)I$

III.
J-fraction pour $\sum t^n F_n$

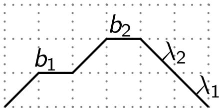
Lemma (Flajolet)

$$[t^n]\frac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\ddots}}}=\sum_{\substack{\text{Motzkin path}\\ P\text{ of length } n}} w(P),$$

Example

where the weight $w(P)$ is the product :

- *b_i for each steo $\rightarrow$ at height i,*
- *λ_i for each step $\searrow$ starting at height i.*



Theorem (Dumont, Randrianarivony, Zeng)

$$\sum_{n=1}^{\infty} F_n t^{n-1} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\ddots}}}$$

with

$$b_i = (x+i)(y+i) + (x+i)(z+i) + (y+i)(z+i) - i(i+1),$$
$$\lambda_i = i(x+y+i-1)(x+z+i-1)(y+z+i-1).$$

Theorem (Dumont, Randrianarivony, Zeng)

$$\sum_{n=1}^{\infty} \Gamma_n t^{n-1} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\ddots}}}$$

with

$$b_i = (x+i)(\bar{y}+i) + (\bar{x}+i)(z+i) + (y+i)(\bar{z}+i) - i(i+1),$$
$$\lambda_i = i(\bar{x}+y+i-1)(x+\bar{z}+i-1)(\bar{y}+z+i-1).$$

Lemma

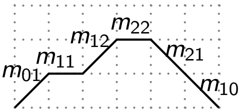
Let $M = (m_{ij})_{i,j \in \mathbb{N}}$ *be a tridiagonal matrix,* $|V\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, *and* $\langle W| = (1, 0, \dots)$. *Then*

$$\langle W|M^n|V\rangle = \sum_{\substack{\text{Motzkin path} \\ P \text{ of length } n}} w(P),$$

where the weight $w(P)$ *is the product of :*

- m_{ii} *for each step* $\rightarrow$ *at height* i,
- $m_{i,i+1}$ *for each step* $\nearrow$ *starting at height* i,
- $m_{i,i-1}$ *for each step* $\searrow$ *starting at height* i.

Example



Proof

$\langle W|M^n|V\rangle$ is the coefficient $(M^n)_{0,0}$ of M^n. So :

$$\langle W|M^n|V\rangle = \sum_{i_1,\dots,i_{n-1}\geq 0} m_{0i_1} m_{i_1 i_2} \dots m_{i_{n-2}i_{n-1}} m_{i_{n-1}0}.$$

M being tridiagonal, we can assume $|i_j - i_{j+1}| \leq 1$, so that $0, i_1, i_2, \dots$ defines a Motzkin path with weight $m_{0i_1} m_{i_1 i_2} \dots m_{i_{n-2}i_{n-1}} m_{i_{n-1}0}$.

Since $F_n(x,y,z) = \langle W|(ED + zD + zE)^{n-1}|V\rangle$, if $M = ED + zD + zE$ is tridiagonal we can prove the continued fraction expansion, with :

$$b_i = m_{i,i}, \qquad \lambda_i = m_{i-1,i} m_{i,i-1}.$$

Problem : find D and E such that

$$DE - ED = D + E, \quad \langle W|E = x\langle W|, \quad D|V\rangle = y|V\rangle,$$

and $DE + zD + zE$ is tridiagonal.

cf. Derrida & al, [Exact solution of a 1D asymmetric exclusion model using a matrix formulation]

$$D = \begin{pmatrix} y & a_0 & & & (0) \\ & y+1 & a_1 & & \\ & & y+2 & a_2 & \\ & & & y+3 & \ddots \\ (0) & & & & \ddots \end{pmatrix}, E = \begin{pmatrix} x & & & & (0) \\ a_0 & x+1 & & & \\ & a_1 & x+2 & & \\ & & a_2 & x+3 & \\ (0) & & & \ddots & \ddots \end{pmatrix},$$

with $a_i = \sqrt{(i+1)(x+y+i)}$, satisfy :

$$DE - ED = D + E, \quad \langle W|E = x\langle W|, \quad D|V\rangle = y|V\rangle,$$

With $M = ED + zE + zD$, coefficients of M are :

$$\begin{aligned}
m_{i,i} &= e_{i,i-1}d_{i-1,i} + e_{i,i}d_{i,i} + ze_{i,i} + zd_{i,i} \\
&= a_{i-1}^2 + (x+i)(y+i) + z(x+i) + z(y+i) \\
&= i(x+y+i-1) + (x+i)(y+i) + z(x+y+2i) \\
&= (x+i)(y+i) + (x+i)(z+i) + (y+i)(z+i) - i(i+1) \\
m_{i-1,i} &= e_{i-1,i-1}d_{i-1,i} + zd_{i-1,i} \\
&= (x+i-1)a_{i-1} + za_{i-1} = a_{i-1}(x+z+i-1), \\
m_{i,i-1} &= e_{i,i-1}d_{i-1,i-1} + ze_{i,i-1} \\
&= a_{i-1}(y+i-1) + za_{i-1} = a_{i-1}(y+z+i-1), \\
m_{i-1,i}m_{i,i-1} &= i(x+y+i-1)(x+z+i-1)(y+z+i-1),
\end{aligned}$$

Thus we have $m_{i,i} = b_i$ and $m_{i,i-1}m_{i-1,i} = \lambda_i$, with b_i and λ_i as announced.
Hence $\sum_n F_n t^n = 1/(1 - b_0 t - \lambda_1 t^2/(1 - b_1 t - \lambda_2 t^2/...))$

With $M = ED + (\bar{z} + x - \bar{x})D + (z + \bar{y} - y)E + (\bar{y} - y)(x - \bar{x})I$, coefficients of M are :

$$\begin{aligned}
m_{i,i} &= e_{i,i}d_{i,i} + e_{i,i-1}d_{i-1,i} + (\bar{z} + x - \bar{x})d_{i,i} + (z + \bar{y} - y)e_{i,i} + (\bar{y} - y)(x - \bar{x}) \\
&= (\bar{x} + i)(y + i) + (y + \bar{x} + i - 1)i + (\bar{z} + x - \bar{x})(y + i) + (z + \bar{y} - y)(\bar{x} + i) + (\bar{y} - y)(x - \bar{x}) \\
&= x\bar{y} + y\bar{z} + z\bar{x} + i\big(\bar{x} + \bar{y} + \bar{z} + x + y + z\big) + i(2i - 1) \\
&= (x + i)(\bar{y} + i) + (\bar{x} + i)(z + i) + (y + i)(\bar{z} + i) - i(i + 1) \\
m_{i,i+1} &= e_{i,i}d_{i,i+1} + (\bar{z} + x - \bar{x})d_{i,i+1} = (\bar{x} + i)a_i + (z + \bar{y} - y)a_i = (x + \bar{z} + i)a_i, \\
m_{i+1,i} &= e_{i+1,i}d_{i,i} + (z + \bar{y} - y)e_{i+1,i} = a_i(y + i) + (z + \bar{y} - y)a_i \\
&= a_i(z + \bar{y} + i).
\end{aligned}$$

Thus we have $m_{i,i} = b_i$ and $m_{i,i-1}m_{i-1,i} = \lambda_i$, with b_i and λ_i as announced.
Hence $\sum_n \Gamma_n t^n = 1/(1 - b_0 t - \lambda_1 t^2/(1 - b_1 t - \lambda_2 t^2/...))$

Conclusion

- F_n is the nth moment of some Hahn polynomials. More generally, the Matrix Ansatz method links various classes of tableaux with J-fractions, or moments of orthogonal polynomials :
  - rook placements and q-Hermite, q-Charlier,
  - 0-1 tableaux [Leroux] q-Charlier polynomials,
  - alternative tableaux and q-Laguerre, Al-Salam-Chihara polynomials
- Some classical sequence of orthogonal polynomials give other generalizations of F_n whose combinatorial properties are not yet fully known :
  - moments q-Hahn polynomials,
  - moments of Wilson polynomials