Refined Gelfand models for B_n and D_n

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Starting point: Gelfand models

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$$(M, \rho) \cong \bigoplus_{\phi \in Irr(G)} (V_\phi, \phi)$$

$$Irr(G) = \{\text{irreducible representations of } G\}.$$

Gelfand models in recent literature

- Inglis-Richardson-Saxl, for symmetric groups;
- Kodiyalam-Verma, for symmetric groups;
- Aguado-Araujo-Bigeon, for Weyl groups;
- Baddeley, for wreath products;
- Adin-Postnikov-Roichman, for the groups $G(r, n)$...

Plan of the talk

- Caselli, for *involutory reflection groups*, a family of complex reflection groups which is bigger than $\{G(r,n)\}$ and contains all infinite families of irreducible finite Coxeter groups.

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We will:

- Introduce the Gelfand model due to F.Caselli, for the particular cases of B_n and D_n;
- provide a refinement for such model in these two cases.

The group B_n

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We denote by $|g|$ the permutation associated to g:

$$|g| := (2, 4, 3, 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$g = (2, -4, 3, 1) \in B_4$.

- split g into two double-rowed vectors according to the sign:

$$g_0 = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \end{pmatrix} \qquad g_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

RS corresponence for B_n

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- split g into two double-rowed vectors according to the sign:

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- perform RS to the two double-rowed vectors:

$$g_0 \xrightarrow{RS} (P_0, Q_0) = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \right)$$

$$g_1 \xrightarrow{RS} (P_1, Q_1) = \left(\boxed{4} , \boxed{2} \right)$$

RS corresponence for B_n

- glue the images of g_0 and g_1 together:

$$g \xrightarrow{RS} (P_0, P_1; Q_0, Q_1) = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 4 \\ \hline \end{array}; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 2 \\ \hline \end{array} \right)$$

A crucial remark

Let M be a model for B_n. It turns out that

$$\dim(M) = \#\{g \in B_n : g^2 = 1\}.$$

We observe that:

g is an involution if and only if $g \xrightarrow{RS} (P_0, P_1; P_0, P_1)$.

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$$\{\text{ involutions of } B_n\} = \{\text{ symmetric matrices of } B_n\} =: \mathrm{Sym}(B_n)$$

Basis of the model

Thus, when constructing a model for B_n, it is natural to look for a model structure on a vector space spanned by the elements

$$\{g \in B_n : g \xrightarrow{RS} (P_0, P_1; P_0, P_1)\}.$$

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$\phi_v(g)$ being a scalar.

Irreducible representations for B_n

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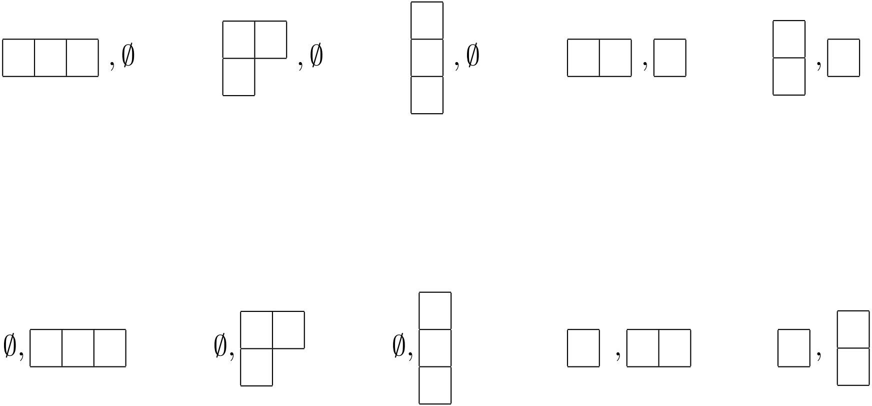
Example: B_3

The irreducible representations of B_3 are:



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Recall that the representation that makes M a model has the form

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Definition

Two elements of B_n are S_n-conjugate if they are conjugate via an element of S_n.

Thus M naturally splits into submodules $M(c)$, where each c is a S_n-conjugacy class of involutions of B_n.

Which of the irreducible representations of B_n are afforded by each of this natural submodules?

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And so it is!

Decomposition of M

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$$Sh(c) = \bigcup_{v\in c} Sh(v).$$

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...$M(c)$ affords the irreducible representations of B_n parametrized by those shapes.

Two involutions v and w of B_n are S_n-conjugate if and only if

$$v \xrightarrow{RS} (P_0, P_1; P_0, P_1) \qquad w \xrightarrow{RS} (Q_0, Q_1; Q_0, Q_1)$$

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with:

- P_0 and Q_0 have the same number of boxes;
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$$Sh(v) = Sh(w) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad v \text{ and } w \text{ are } S_n - \text{conjugate}$$

Example in B_3:

$$Sh(v) = \left(\yng(2,1), \; \emptyset \right) \qquad Sh(w) = \left(\yng(1,1,1), \; \emptyset \right)$$

v and w are S_3 conjugate.

Example

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$$M(c) \cong \left(\yng(2,1), \yng(2,1,1)\right) \oplus \left(\yng(1,1,1), \yng(3,1)\right) \oplus \left(\yng(2,1), \yng(3,1)\right) \oplus \left(\yng(1,1,1), \yng(2,1,1)\right).$$

The proof

Key point: the submodule spanned by the set

$$Sym_0(B_n) := \left\{ \begin{array}{c} \text{symmetric elements of } B_n \text{ which are} \\ \text{products of signed cycles of length 2 only} \end{array} \right\}$$
$$= \left\{ \begin{array}{c} \text{symmetric elements of } B_n \text{ whose} \\ \text{diagonal has zero entries only} \end{array} \right\}$$

is actually made up of the irreducible representations parametrized by the elements of $Sh(Sym_0(B_n))$.

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$$Sh(Sym_0(B_n)) = \{(\lambda, \mu) : \lambda, \mu \text{ have no columns of odd length}\}.$$

Strategy: a hint from S_n

Theorem (Inglis, Richardson, Saxl, 1990)

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Then

$$\pi_k \simeq \bigoplus_{\substack{\lambda \vdash 2k \\ \lambda \text{ with even parts only}}} V_\lambda$$

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Apply this to

$$\Pi_m = \bigoplus_{v\in Sym_0(B_{2m})} \mathbb{C}\, C_v!$$

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$h = (2, -4, -3, 1) \in B_4$.

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in D_4$$

Another generalization for RS correspondence

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$$\bar{g} \in \frac{B_n}{\pm Id} \xrightarrow{RS_2} (\{P_0, P_1\}; \{Q_0, Q_1\})$$

$$\uparrow \qquad \uparrow$$

UNORDERED PAIRS!!!!

The model for D_n

Model for B_n spanned by

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Model for D_n: instead of looking at D_n...

... we look at the quotient $\dfrac{B_n}{\pm Id}$

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the same *unordered* pair

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Generators for the model of D_n

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$$= \left\{ \mathrm{Sym}\left(\frac{B_n}{\pm Id}\right) := \{\bar{g} \in \tfrac{B_n}{\pm Id} : g \xrightarrow{RS} (P_0, P_1; P_0, P_1) \text{ for (any) } g \text{ lift of } \bar{g}\}; \right.$$

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$$= \begin{cases} \mathrm{Sym}\left(\frac{B_n}{\pm Id}\right) := \{\bar{g} \in \frac{B_n}{\pm Id} : g \xrightarrow{RS} (P_0, P_1; P_0, P_1) \text{ for (any) } g \text{ lift of } \bar{g}\}; \\ \mathrm{Asym}\left(\frac{B_n}{\pm Id}\right) := \{\bar{g} \in \frac{B_n}{\pm Id} : g \xrightarrow{RS} (P_0, P_1; P_1, P_0) \text{ for (any) } g \text{ lift of } \bar{g}\}. \end{cases}$$

Example: antisymmetric elements

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- Notice that a pair $(P_0, P_1; P_1, P_0)$ can be the RS image of a $g \in B_n$ only if P_0 and P_1 have the same shape λ.

A model for D_n: the module (Caselli, 2009)

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$$M = \bigoplus_{v \in \mathrm{Sym}} \mathbb{C}\, C_v \oplus \bigoplus_{v \in \mathrm{Asym}} \mathbb{C}\, C_v$$

A model for D_n: the representation (Caselli, 2010)

The morphism $\rho : D_n \to GL(M)$ has the form

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$\psi_v(g)$ being a scalar.

The irreducible representations of D_n

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$$\{\text{ordered pairs of Ferrers diagrams } (\lambda, \mu) \text{ such that } |\lambda| + |\mu| = n\}.$$

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Restrict them to D_n:

$$\begin{cases} \lambda \neq \mu : \quad (\lambda, \mu) \downarrow_{D_n} = (\mu, \lambda) \downarrow_{D_n} \text{ doesn't split} \\ \end{cases}$$

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Where can we find these representations in the model M?

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- $\{\lambda, \lambda\}^+$, with $\lambda \vdash \frac{n}{2}$ (SPLIT REP);
- $\{\lambda, \lambda\}^-$, with $\lambda \vdash \frac{n}{2}$ (SPLIT REP)

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Possible shapes via RS_2 of the generators of M:

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A very welcome coincidence

Irreducible representations of D_n:

- $\{\lambda, \mu\}$, with $\lambda \neq \mu$, $|\lambda| + |\mu| = n$ (UNSPLIT REP);
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HOW NICE!

A natural decomposition for M

M naturally splits *first of all* into the two fat submodules

$$\bigoplus_{v \in \mathrm{Sym}} \mathbb{C}\, C_v \qquad \bigoplus_{v \in \mathrm{Asym}} \mathbb{C}\, C_v$$

Refinement for M

Again, this decomposition is well-behaved w.r.t. the RS_2 correspondence!

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Theorem (C., F., 2010)

The split representations of D_n can be labelled in such a way that

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$$\Downarrow$$

$$\bigoplus_{v \in \mathrm{Sym}} \mathbb{C}\, C_v \simeq \bigoplus_{\lambda \neq \mu} \{\lambda, \mu\} \oplus \bigoplus_{\lambda \vdash \frac{n}{2}} \{\lambda, \lambda\}^+.$$

Example

$v = (-6, 4, 3, 2, -5, -1) \in B_6.$

Let $\bar{c}$ be the S_6-conjugacy class of $\bar{v}$. Then

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$$M(\bar{c}) \cong \left(\yng(2,1), \yng(1,1,1)\right) \oplus \left(\yng(2,1), \yng(2,1)\right)^{+} \oplus \left(\yng(1,1,1), \yng(1,1,1)\right)^{+}.$$

A further refinement

The submodule

$$\bigoplus_{v \in \mathrm{Sym}} \mathbb{C}\, C_v$$

admits a finer refinement which is analogous to the case of B_n and is also well-behaved with respect to RS_2.

Further generalizations

The whole argument can be generalized to a much wider class of groups.

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Definition

Let $G < GL(n, \mathbb{C})$ and let M be a Gelfand model for G. G is involutory if

$$\dim(M) = \#\{g \in G : g\bar{g} = 1\},$$

where $\bar{g}$ denotes the complex conjugate of g.

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Theorem (Caselli, 2009)

A group $G(r, p, n)$ is involutory if and only if $\mathrm{GCD}(p, n) = 1, 2$.

Thank you!