Whitney algebras and Letterplace superalgebras

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Contents

I'll give an outline of the theory of Whitney algebras of a matroid, with the notions of geometric product and the excahnge relations, I'll give the letterplace superalgebra coding of these algebras, and I'll show how this coding allows to get the exchange relations directly from the superstraightening laws. I'll not speak about the Lax Hopf algebra structure of Whitney algebras. All this is part of a work in progress with A. Brini, H. Crapo, W. Schmitt.

For the theory of Whitney algebras, I refer to

H. Crapo, W. Schmitt; *J. Combin. Theory Ser. A 91 (2000), no. 1-2, 215–263.*

For the theory of letterplace superalgebras and letterplace coding of the basic algebras, I refer to

A. Brini; *Sém. Lothar. Combin. 55 (2005/07), Art. B55g*

1. **Tensor powers of exterior algebras**

- **Tensor powers of exterior algebras**

  Let $\mathbb{K}$ be a field, V a vector space of dimension d over $\mathbb{K}$.

  Let $\Lambda(V)$ be the exterior algebra of V. For $v_1, \ldots, v_p$ in V, we have $v_1 \cdots v_p \neq 0$ in $\Lambda(V)$ iff $v_1, \ldots, v_p$ are linearly independent; in this case, the product $v = v_1, \ldots, v_p$ is called an *extensor of step* p, and we write $s(v) = p$. The extensors of step p span a linear subspace $\Lambda_p(V)$ of $\Lambda(V)$, and these spaces give rise to a $\mathbb{Z}-$grading $\Lambda(V) = \oplus_{i=0}^d \Lambda_i(V)$ of the algebra $\Lambda(V)$.

  The space tensor square $\Lambda(V)^{\otimes 2} = \Lambda(V) \otimes \Lambda(V)$ has a $(\mathbb{Z} \times \mathbb{Z})-$grading $\Lambda(V)^{\otimes 2} = \oplus_{i,j=0}^d \Lambda_i(V) \otimes \Lambda_j(V)$, and a structure of algebra, in which the product is given by $(u \otimes v) \times (u' \otimes v') = (-1)^{s(v)s(u')} uu' \otimes vv'$.

  $\Lambda(V)$ has a structure of bialgebra, in which the coproduct

  $$\delta : \Lambda(V) \to \Lambda(V)^{\otimes 2}, \qquad \delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)},$$

  is induced by setting $\delta(x) = x \otimes 1 + 1 \otimes x$, for all $x \in V$. For each pair (i,j), the coproduct induces a linear morphism

  $$\delta_{(i,j)} : \Lambda(V) \to \Lambda_i(V) \otimes \Lambda_j(V), \qquad \delta_{(i,j)}(u) = \sum_{(u)_{(i,j)}} u_{(1)} \otimes u_{(2)}.$$

  This expression is called the *coproduct slice of type* (i,j) *of* u.

- **Linear versus additive relations**

  Let $v_1, v_2, \ldots, v_p$ be vectors in V, and let $v = v_1 v_2 \cdots v_p$ be their product in $\Lambda(V)$. If $v_1, v_2, \ldots, v_p$ are linearly dependent, then we have $v = 0$, and

  $$\delta_{(i,j)}(v) = \sum_{(v)_{(i,j)}} v_{(1)} \otimes v_{(2)} = 0,$$

  i.e. "all the coproduct slices" of v are zero.

  These relations give informations on the linear relations between the given vectors. For example, if the given set of vectors is a minimal dependent set, then we have the relation

  $$\delta_{(p-1,1)}(v) = \sum \pm v_1 \cdots \widehat{v_i} \cdots v_p \otimes v_i = 0;$$

  all the extensors in the first tensor fold are scalar multiples of one of them, ... and we can get the coefficients of "the" linear relation between the given vectors. Notice that the only scalars that are explicitly mentioned in this relation are ± 1.

  Along these lines it can be given also an analogous form for Cramer's rule.

- **Geometric Products**

  For each $h = 0, 1, 2, \ldots$ we have a linear mapping

  $$\diamond^h : \Lambda(V)^{\otimes 2} \to \Lambda(V)^{\otimes 2},$$

$$\diamond^h(u \otimes v) = \sum_{(u)_{(*,h)}} u_{(1)} v \otimes u_{(2)}.$$

We call this mapping the *geometric product of order h*. If u, v are extensors, associated to subspaces $[u]$ and $[v]$ of V, and $d([u] \cap [v]) = h$, then

$$\diamond^h(u \otimes v) = w \otimes z,$$

where w and z are extensors, and

$$[w] = [u] + [v], \qquad [z] = [u] \cap [v].$$

Under the assumption $d([u] \cap [v]) = h$, we have also

$$\diamond^h(u \otimes v) = (-1)^{(d[u]-h)(d[v]-h)} \diamond^h (v \otimes u).$$

These facts can be proved easily by factorizing u and v over an extensor representing $[u] \cap [v]$.
Geometric products are deeply linked with Grassmann's regressive product, (Ausdehnungslehre, 1844).

All the topics of this discussion can be generalized to arbitrary tensor powers $\Lambda(V)^{\otimes n}$, $n = 1, 2, \ldots$

2. **Whitney algebras of a matroid**

- **Whitney algebras of a matroid**
  Let $M = M(S)$ be a matroid on a finite set S.
  Let $F = \oplus_{x \in S} \mathbb{Z}x$ be the free $\mathbb{Z}-$module over the set S, and let $\Lambda(F)$ be the free exterior $\mathbb{Z}-$algebra on F. The quotient

$$W^m(M) = \frac{\Lambda(F)^{\otimes m}}{I(M)}$$

  of the $\mathbb{Z}-$algebra $\Lambda(F)^{\otimes m}$ $m-$th tensor power of $\Lambda(F)$ modulo the two-sided ideal $I(M)$ generated by the slices

$$\delta_{(p,\ldots,q)}(v) = \sum_{(v)_{(p,\ldots,q)}} v_{(1)} \otimes \cdots \otimes v_{(m)}$$

  of words $v = x_1 \cdots x_p$ associated to sets $\{x_1, \ldots, x_p\} \subseteq S$ which are dependent in M, is called *the $m-$th Whitney algebra of M*. The image of a tensor product $x \otimes y \otimes \cdots \otimes z$ in the quotient will be denoted by $x \circ y \circ \cdots \circ z$.
  In Whitney algebras there holds an identity that can be regarded as a universal form of Cramer's rule.
- **Representations**
  Let $M = M(S)$ be a matroid on a finite set S, and V a vector space over some field $\mathbb{K}$. A *representation* of M in V is a mapping

$$g : S \to V$$

  such that a set $A \subseteq S$ is independent in M iff the set $g(A)$ is linearly independent in V and g is one-to-one on A.

Proposition 1. *Let $M = M(S)$ be a representable matroid, and let $g : S \to V$ be a representation of M on a vector space V; then for each $n = 1, 2, \ldots$ there is exactly one ring morphism*

$$\hat{g}^n : W^n(M) \to \Lambda^{\otimes n}(V)$$

such that $\hat{g}^n(1 \circ \cdots \circ x \circ 1 \circ \cdots \circ 1) = 1 \otimes \cdots \otimes g(x) \otimes 1 \otimes \cdots \otimes 1$ for all x in S.

Theorem 1. *A matroid $M = M(S)$ is representable over some vector space V iff for every words $w_1, \ldots, w_n$ associated to independent subsets of S,*

$$w_1 \circ \cdots \circ w_n \neq 0 \qquad in \quad W^n(M).$$

- **Geometric Products**

  Let $M = M(S)$ be a matroid on a finite set S, and let $h = 01, 2, \ldots$ The *geometric product of order h on M* is the linear mapping

$$\diamond^2 : W^2(M) \to W^2(M)$$

  given by

$$\diamond^h(u \circ v) = \sum_{(u)_{(*,h)}} u_{(1)} v \circ u_{(2)}.$$

  This definition, given by appealing to representatives, is consistent.

  Proposition 2. *Let $M = M(S)$ be a representable matroid, and let $g : S \to V$ be a representation of M in some vector space V, and let $\hat{g}^2 : W^2(M) \to \Lambda(V)^{\otimes 2}$ be the corresponding ring morphism. Then*

$$\hat{g}^2\left(\diamond^h(u \circ v)\right) = \diamond^h\left(g(u) \otimes g(v)\right),$$

  for every $h = 0, 1, 2, \ldots$.

- **Geometric Products; Exchange relations**

  Theorem 2. *Let $M = M(S)$ be a matroid on a set S; let $A, B \subseteq S$ be flats in M, let u, v be words representing bases of A, B resp., and let h be the nonnegative integer defined by $\rho(A) + \rho(B) = \rho(A \vee B) + h$. Then*

$$\diamond^h(u \circ v) = (-1)^{(\rho(A)-h)(\rho(B)-h)} \diamond^h (v \circ u),$$

  This theorem is one of the main results of the theory of Whitney algebras. The proof of the analogous result in the tensor square of an exterior algebra is far from applicable in this context; the original proof of this theorem was based on a hard technical proposition, the "Zipper Lemma."

3. **Letterplace superalgebras**

- **Letterplace superalgebras**

  Let L be a $\mathbb{Z}_2-$graded set, i.e. a set endowed with a distinguished disjoint union decomposition $L = L_0 \dot{\cup} L_1$, $0, 1 \in \mathbb{Z}_2$. Let L^d be the set obtained from

L by adjoining, for each $x \in L_0$, the sequence of symbols $x^{(p)}$, $p = 1, 2, \ldots$, called its *divided powers*. Any word w on this extended set L^d has a length $l(w)$ (to which each divided power contributes through its exponent), and a $\mathbb{Z}_2-$grade $|w|$.
The *supersymmetric algebra*

$$Super(L)$$

on the $\mathbb{Z}_2-$graded set L is the unitary associative $\mathbb{Z}-$algebra generated by the elements of the extended set L^d modulo the relations

$$x^2 = 0, \quad for\ x \in L_1;$$

$$x^{(p)}x^{(q)} = \binom{p+q}{p} x^{(p+q)} \quad for\ x \in L_0;$$

$$uv = (-1)^{|u||v|}vu, \quad for\ u, v\ words\ on\ L^d.$$

This algebra has a natural $\mathbb{Z} -$ *grading*, a natural $\mathbb{Z}_2-$grading, and these two gradings are compatible.
The supersymmetric algebra $Super(L)$ is a bialgebra, in which the coproduct

$$\delta : Super(L) \to Super(L)^{\otimes 2}, \qquad \delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

is induced by setting

$$\delta(1) = 1 \otimes 1,$$

$$\delta(x) = x \otimes 1 + 1 \otimes x, \qquad for\ x \in L_1$$

$$\delta(x^{(r)}) = \sum_{p+q=r} x^{(p)} \otimes x^{(q)}, \qquad for\ x \in L_0.$$

Let $L = L_0 \dot{\cup} L_1$ and $P = P_0 \dot{\cup} P_1$ be two $\mathbb{Z}_2-$graded sets, whose elements are called *letters* and *places*, resp. The symbols $(x|y)$, for x in L and y in P are called *letterplaces*, they form a set $(L|P)$ which has a natural $\mathbb{Z}_2-$grading, given by $|(x|y)| = |x| + |y|$.
The *letterplace superalgebra*

$$Super(L|P)$$

on the $\mathbb{Z}_2-$graded sets L, P is the supersymmetric $\mathbb{Z}-$algebra over the $\mathbb{Z}_2-$graded set $(L|P)$.

- **Biproducts, Bitableaux, Supertraightening Laws**
  To any two monomials u and v of the same length in $Super(L)$ and $Super(P)$, there corresponds an element

  $$(u|v)$$

  in $Super(L|P)$, called the *biproduct* of u and v. Biproducts satisfy all Laplace expansions

  $$(u_1u_2|v) = \sum_{(v)} (-1)^{|u_2||v_{(1)}|}(u_1|v_{(1)})(u_2|v_{(2)});$$

$$(u|v_1v_2) = \sum_{(u)} (-1)^{|u_{(2)}||v_1|}(u_{(1)}|v_1)(u_{(2)}|v_2).$$

Biproducts are far-reaching generalizations of determinants and permanents. By means of biproducts one can define bitableaux. The main result on bitableaux is the following

Theorem 3. *(Superstraightening Law) For any three monomials u, v, w in $Super(L)$, and any two monomials x, y in $Super(P)$, we have*

$$\sum_{(v)} \left(\begin{array}{c|c} uv_{(1)} & x \\ v_{(2)}w & y \end{array} \right) = (-1)^{|u||v|} \sum_{(u),(y)} (-1)^{l(u_{(2)})} \left(\begin{array}{c|c} vu_{(1)} & xy_{(1)} \\ u_{(2)}w & y_{(2)} \end{array} \right).$$

The main consequence of this therorem is that the superstandard bitableaux form a $\mathbb{Z}-$linear basis of $Super(L|P)$.

4. **Letterplace superalgebra coding of Whitney algebras**

- **Letterplace superalgebra coding of Whitney algebras**

  Let $M = M(S)$ be a matroid on a finite set S, and let $F = \oplus_{x\in S}\mathbb{Z}x$ be the free $\mathbb{Z}-$module on the set S.

  There is an isomorphism $\Lambda(F) \cong Super(L)$, between the exterior $\mathbb{Z}-$algebra on F and the supersymmetric $\mathbb{Z}-$algebra on the $\mathbb{Z}_2-$graded set $L = L_1 = S$. For any positive integer n, there is an isomorhism

  $$\Lambda(F)^{\otimes n} \cong Super(L|P),$$

  between the $n-$th tensor power $\mathbb{Z}-$algebra of $\Lambda(F)$ and the letterplace $\mathbb{Z}-$superalgebra over the letter set $L = L_1 = S$ and the place set $P = P_0 = \{1, \ldots, n\}$, induced by setting

  $$1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \mapsto (x|i),$$

  where x ranges in S and i is the tensor fold in which it occurs.

  This isomorphism maps the slices $\delta_{(p,\ldots,q)}(u) = \sum_{(v)_{(p,\ldots,q)}} v_{(1)} \otimes \cdots \otimes v_{(n)}$ of type $(p, \ldots, q)$ in $\Lambda(F)^{\otimes n}$ to the biproducts $(v|1^{(p)} \cdots n^{(q)})$ in $Super(L|P)$, and induces an isomorphism from the quotient

  $$W^n(M) = \frac{\Lambda(F)^{\otimes m}}{I(M)}$$

  which defines the $n-$th Whitney algebra of M to the quotient

  $$\frac{Super(L|P)}{J(M)}$$

  of the letterplace $\mathbb{Z}-$algebra $Super(L|P)$ modulo the two-sided ideal $J(M)$ generated by the biproducts

  $$(v|1^{(p)} \cdots n^{(q)})$$

  of words $v = x_1 \cdots x_p$ associated to sets $\{x_1, \ldots, x_p\} \subseteq S$ which are dependent in M.

- **Geometric products; exchange relations**

  Under the given isomorphism, the geometric product

  $\diamond^h(u \circ v) = \sum_{(u)_{(*,h)}} u_{(1)} v \circ u_{(2)}$,

  in the Whitney algebra $W^2(M)$ is mapped to the bitableau

  $\sum_{(u)} \left(\begin{array}{c|c} u_{(1)}v & 1^{(r)} \\ u_{(2)} & 2^{(h)} \end{array} \right)$,

  in the quotient.

  In the latterplace setting, the excahnge theorem reads: let $A, B \subseteq S$ be flats in M, let u, v be words representing bases of A, B resp., and let h be the nonnegative integer defined by $\rho(A) + \rho(B) = \rho(A \vee B) + h$. Then

$$\sum_{(u)} \left(\begin{array}{c|c} u_{(1)}v & 1^{(r)} \\ u_{(2)} & 2^{(k)} \end{array} \right) = (-1)^{(|u|-h)(|v|-h)} \sum_{(v)} \left(\begin{array}{c|c} v_{(1)}u & 1^{(r)} \\ v_{(2)} & 2^{(k)} \end{array} \right).$$

  One can then guess that it is a rather direct consequence of the Supertraightening Law.

5. **Further topics**

- **Polarizations**

  The letterplace superalgebra $Super(L|P)$ has a natural structure of bimodule on the general linear Lie superalgebras $pl(L)$ and $pl(P)$; the representations are afforded by the action of left superpolarizations $\mathcal{D}_{xy}$, $x, y \in L$ and right superpolarizations ${}_{uv}\mathcal{D}$, $u, v \in P$.

  Biproducts and bitableaux can be defined as the result of the action of superpolarization monomials on letterplace monomials, by means of Capelli's method of virtual variables.

  In the letterplace coding of a Whitney algebra, these superpolarizations are indeed classical polarizations. Geometric products are coded by divided powers of right polarizations on places. Thus, the properties of the geometric products are consequences of the Lie bracket relations between elementary matrices.