Fusion process and representations of affine Hecke algebras

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joint work with V. Guizzi and M. Nazarov

Upshot

We want to provide a detailed account of a combinatorial construction (after Cherednik) which exhibits the irreducible (finite dimensional) modules of the affine Hecke algebra of GL_n as cyclic modules over the Hecke algebra of GL_n.

The main theorem we are going to discuss has been (roughly) stated by Cherednik without proof.

Affine Hecke algebras: Bernstein presentation

The affine Hecke algebra $\widetilde{\mathcal{H}}_n$ (of GL_n) is the $\mathbb{C}(q)$-algebra with the following presentation

Affine Hecke algebras: Bernstein presentation

Generators for $\widetilde{\mathcal{H}}_n$

$$T_1, \dots, T_{n-1}$$
$$X_1^{\pm 1}, \dots, X_n^{\pm 1}$$

Relations $\widetilde{\mathcal{H}}_n$

$$\begin{aligned}
&T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, && 1 \le k \le n-2,\\
&T_k T_l = T_l T_k, && 1 < |k-l|,\\
&(T_k - q)(T_k + 1) = 0, && 1 \le k \le n-1,\\
&X_k X_l = X_l X_k, && 1 \le k, l \le n,\\
&X_l T_k = T_k X_l, && l \ne k, k+1,\\
&T_k X_k T_k = q\, X_{k+1}, && 1 \le k \le n-1.
\end{aligned}$$

The **“finite” Hecke algebra** $\mathcal{H}_n$ is the subalgebra of $\widetilde{\mathcal{H}}_n$ generated by the T_i; it has the following presentation, which shows it as a q deformation of $\mathbb{C}[S_n]$.

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\end{aligned}$$

Remarks

Recall the defining relations or $\widetilde{\mathcal{H}}_n$:

$$T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \qquad T_k T_l = T_l T_k,$$
$$(T_k - q)(T_k + 1) = 0, \qquad X_k X_l = X_l X_k,$$
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The $X_i^{\pm 1}$ generate a (maximal) commutative subalgebra $\mathcal{A}_n$ of $\widetilde{\mathcal{H}}_n$.

$$\widetilde{\mathcal{H}}_n = \mathcal{H}_n \otimes \mathcal{A}_n \quad \text{as vector spaces}$$

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Corollary

Any irreducible $\widetilde{\mathcal{H}}_n$-module is finite dimensional

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Under this hypothesis, irreducible representations of $\widetilde{\mathcal{H}}_n$ are parametrized by *multisegments*:

$$M = \sum_{i \leq j} m_{ij}[i, j]$$

with $\sum_{i \leq j} m_{ij}(j - i + 1) = n$.

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Zelevinsky classification sounds like: for a certain ordering of M, a suitable induced module built up starting form M has a unique irreducible quotient. The irreducible $\widetilde{\mathcal{H}}_n$-modules obtained in this way are non-equivalent for different multisegments M and form a complete set with all eigenvalues of $X_1, \dots, X_n$ in $q^{\mathbb{Z}}$.

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- For each Cherednik diagram λ we produce a pair $(E_\lambda, \chi_\lambda)$ where $E_\lambda \in \mathcal{H}_n$ and $\chi_\lambda : \mathcal{A}_n \to \mathbb{C}(q)$ is a character of the algebra $\mathcal{A}_n$, such that:

(i) E_λ is an eigenvector for $\mathcal{A}_n$ inside $\mathrm{Ind}\,_{\mathcal{A}_n}^{\widetilde{\mathcal{H}}_n}\chi_\lambda$,

$$\mathrm{Ind}\,_{\mathcal{A}_n}^{\widetilde{\mathcal{H}}_n}\chi_\lambda = \widetilde{\mathcal{H}}_n \otimes_{\mathcal{A}_n} \mathbb{C}_{\chi_\lambda} = (\mathcal{H}_n \otimes_{\mathbb{C}(q)} \mathcal{A}_n) \otimes_{\mathcal{A}_n} \mathbb{C}_{\chi_\lambda} = \mathcal{H}_n$$

(ii) the space $V_\lambda = \mathcal{H}_n \cdot E$ which is a $\widetilde{\mathcal{H}}_n$-module by (i), is irreducible.

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- The modules V_λ are a complete set of inequivalent irreducible representations of $\widetilde{\mathcal{H}}_n$.
- Proving the above step is non trivial. However, it can be done by resorting to instances in which the results are known: indeed we use a version of a Drinfeld functor which reduces our problem to a statement about quantum affine algebras, known by the work of Akasaka-Kashiwara.

Main goal

The combinatorial definition of E_λ and χ_λ will be the main theme of this lecture.

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- Our approach provides at least one vector in each irreducible! Indeed, when the action of $\mathcal{A}_n$ is semisimple, one can write down combinatorially a basis for V_λ. These are precisely the cases in which λ is a skew-shape.
- Ours seems to be the first instance of a combinatorial treatment of non-semisimple representations.

Cherednik's diagrams

Given $(a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathbb{Z}^r$ s.t. $a_i \leq b_i \ \forall\, i$, we'll consider subsets

$$B = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq r,\ a_i \leq j \leq b_i\}.$$

represented, with matrix-style coordinates, in the plane $\mathbb{Z}^2$ by diagrams having boxes centered in

$$(1, a_1), \ldots, (1, b_1), (2, a_2), \ldots, (2, b_2), \ldots \ldots (r, a_r), \ldots, (r, b_r)$$

Definition

The set B is a Cherednik diagram if for each $i = 1, \ldots, r-1$ either $b_{i+1} \leq b_i$, or $b_{i+1} = b_i + 1$ and $a_{i+1} \leq a_i + 1$.

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$$(1,a_1),\ldots,(1,b_1),(2,a_2),\ldots,(2,b_2),\ldots\ldots(r,a_r),\ldots,(r,b_r)$$

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Let $\mathcal{C}_n$ denote the collection of Cherednik diagrams with n boxes.

Remarks and examples

The conditions can rephrased as follows. Think of λ as the juxtaposition of its rows $\lambda_1, \dots, \lambda_r$, where $\lambda_i = [a_i, b_i]$.

1. For any i, if the final box of λ_i lies on the diagonal $x + y = h$, then final box of λ_{i+1} lies on the diagonal $x + y = k$, then $h \geq k$.
2. if $h = k$, the the length of λ_{i+1} is greater or equal than the length of λ_i.

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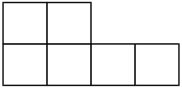
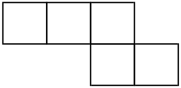
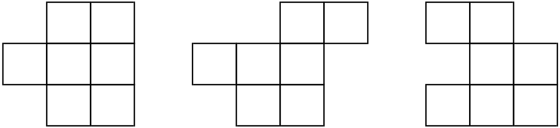
From the above remarks, it is clear that $\mathcal{C}_n$ contains all ordinary and skew diagrams with n boxes.

$\mathcal{C}_5, \mathcal{C}_6$ contains neither $\{[1,2],[1,4]\}$ nor $\{[1,3],[3,4]\}$





$\mathcal{C}_7$ contains $\{[1,2],[0,2],[1,2]\}$, $\{[1,2],[-1,1],[0,1]\}$, $\{[1,2],[2,3],[0,3]\}$



Fusion procedure

The construction of E_λ will be performed via a suitable version of the *fusion procedure*.
This method was initially used to reproduce the Young symmetrizers in the group ring of $\mathcal{S}_n$.
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Roughly speaking, it consists in obtaining E_λ as a limiting value of suitable rational functions, by using a mixture of combinatorics and noncommutative ring theory.

The basic functions

For any $x \in \mathbb{C}(q)$ put

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$$\begin{aligned}\varphi_k(x_1, \ldots, x_n) &= T_k + \langle x_{k+1}/x_k \rangle \\ &= T_k + \frac{1-q}{1 - x_k x_{k+1}^{-1}}\end{aligned}$$

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with values in the algebra $\mathcal{H}_n$.
Using any reduced decomposition $w = s_{k_1} \ldots s_{k_m} \in S_n$ define the rational function

$$\varphi_w = \varphi_{k_1} ({}^{s_{k_1}} \varphi_{k_2}) ({}^{s_{k_1} s_{k_2}} \varphi_{k_3}) \ldots ({}^{s_{k_1} \ldots s_{k_{m-1}}} \varphi_{k_m}).$$

Example

For instance, if $w = s_1 s_2 s_3$ we have

$$\begin{aligned}\varphi_w &= (T_1 + \langle x_2/x_1\rangle)(T_2 + {}^{s_1}\langle x_3/x_2\rangle)(T_3 + {}^{s_1 s_2}\langle x_4/x_3\rangle) \\ &= (T_1 + \langle x_2/x_1\rangle)\left(T_2 + {}^{(12)}\langle x_3/x_2\rangle\right)\left(T_3 + {}^{(123)}\langle x_4/x_3\rangle\right) \\ &= (T_1 + \langle x_2/x_1\rangle)(T_2 + \langle x_3/x_1\rangle)(T_3 + \langle x_4/x_1\rangle).\end{aligned}$$

Properties

Proposition

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The above proposition is better understood if we write φ_w as

$$\begin{aligned}\varphi_w &= (\varphi_{i_1})({}^{s_{i_1}}\varphi_{i_2})({}^{s_{i_1}s_{i_2}}\varphi_{i_3})\dots({}^{s_{i_1}\dots s_{i_{m-1}}}\varphi_{i_m}) \\ &= (T_{i_1}+\langle\beta_1\rangle)(T_{i_2}+\langle\beta_2\rangle)(T_{i_3}+\langle\beta_3\rangle)\dots(T_{i_m}+\langle\beta_m\rangle)\end{aligned}$$

where $\beta_k = s_{i_1}\dots s_{i_{k-1}}(\alpha_{i_k})$ for each $k = 1,\dots,m$ and $\alpha_i = x_i x_{i+1}^{-1}$ is a simple root written multiplicatively. Therefore, if we set

$$\varphi_{\beta_k}^{i_k} = T_{i_k} + \langle\beta_k\rangle$$

then we get

$$\varphi_w = \varphi_{\beta_1}^{i_1}\varphi_{\beta_2}^{i_2}\dots\varphi_{\beta_m}^{i_m}$$

Properties

$$\varphi_w = \varphi^{i_1}_{\beta_1}\varphi^{i_2}_{\beta_2}\ \dots\ \varphi^{i_m}_{\beta_m}$$

Therefore, when we change the reduced expression for w, the upper indices change according to the braid relations, whereas the lower roots change according to the "moves" that regulate the convex orders on $N(w) = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in -\Delta^+\}$.
In this case these moves are

$$\begin{aligned} \alpha, \alpha+\beta, \beta \quad &\leftrightarrow \quad \beta, \alpha+\beta, \alpha \\ \alpha, \beta \quad &\leftrightarrow \quad \beta, \alpha \qquad\qquad \text{if } \alpha \perp \beta \end{aligned}$$

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- Set

$$\delta_\lambda(x_1, \ldots, x_n) = \prod \left(1 - \frac{x_l}{x_k}\right),$$

  the product being taken over all pairs (k, l) such that $k < l$ while in Λ the numbers k, l occur in the leftmost boxes of two parallel rows of λ.

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- Let $w_0 \in S_n$ be the element of maximal length.

Main Theorem: statement

$$\mathcal{F}_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{C}(q)^n \mid x_k\, q^{c_l} = q^{c_k}\, x_l$$
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Theorem

For any $\lambda \in \mathcal{C}_n$ the restriction of the rational function $\delta_\lambda\, \varphi_{w_0}$,

$$(\delta_\lambda\, \varphi_{w_0})(x_1, \dots, x_n) = \prod \left(1 - \frac{x_l}{x_k}\right) \overrightarrow{\prod} (\, T_{i_s} + \langle \beta_s \rangle)$$

to the subspace $\mathcal{F}_\lambda$ is regular and non-zero at the point $(x_1, \dots, x_n) = (q^{c_1}, \dots, q^{c_n})$.
Its value E_λ is a cyclic generator for V_λ as a submodule of the $\widetilde{\mathcal{H}}_n$-module obtained by inducing the character of $\mathcal{A}_n$

$$\chi_\lambda(X_k) = q^{c_k}$$

Comments on the proof

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Recall that if $\beta = x_i x_j^{-1}$, we have

$$\varphi_\beta^k = T_k + \frac{1-q}{1 - x_i x_j^{-1}}.$$

This function has a simple pole at our point $(q^{c_1}, \ldots, q^{c_n})$ precisely when $c_i = c_j$. The root β (which we identify combinatorialy with the pair (i, j), $i < j$) will be called **singular** if $c_i = c_j$.

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The basic tool to deal with singular roots is the following

Lemma

Suppose that the pair (i,j) is singular. Set $\underline{x} = (q^{c_1}, \dots, q^{c_n})$.
(a) *If $i, i+1$ belong to the same row of Λ then*

$$(\varphi_{i,i+1}^k \varphi_{i,j}^{k+1} \varphi_{i+1,j}^k)|_{\mathcal{F}_\lambda}(\underline{x}) = (1+T_k)(T_{k+1}T_k - q\,T_{k+1} - q).$$

(b) *If $j-1, j$ belong to the same row of Λ then*

$$(\varphi_{i,j-1}^k \varphi_{i,j}^{k+1} \varphi_{j-1,j}^k)|_{\mathcal{F}_\lambda}(\underline{x}) = (T_k T_{k+1} - q\,T_{k+1} - q)(1+T_k).$$

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Indeed, a singular root is always decomposable, i.e. of the form $\alpha + \beta$. We will introduce an ordering such that for any singular $\alpha + \beta$, we have either

$$\alpha, \alpha + \beta \ldots\ldots \beta \quad \text{or} \quad \alpha \ldots\ldots \alpha + \beta, \beta$$

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Using the local moves on the ordering, we will achieve

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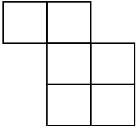
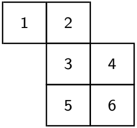
Using the local moves on the ordering, we will achieve

$$\alpha, \alpha+\beta, \gamma_1, \ldots \gamma_k, \beta \rightarrow \alpha, \alpha+\beta, \beta, \gamma_1', \ldots \gamma_k'$$

so that we can use the lemma.
Caveat: we should perform this procedure for each singular pair, so we should check some kind of compatibility conditions.
Moreover, not all singularities are removed in this way: here the correction factor δ_λ plays its role.

Final example

$\lambda =$ 

$\Lambda =$ 

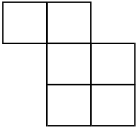
$\mathcal{F}_\lambda = \{x_2 = qx_1, x_4 = qx_3, x_6 = qx_5\}$

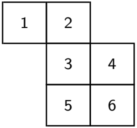
Final example

$$\lambda = \begin{array}{|c|c|c} \cline{1-2} & & \\ \hline & & \multicolumn{1}{c|}{} \\ \cline{2-3} & & \multicolumn{1}{c|}{} \\ \cline{2-3} \end{array} \qquad \Lambda = \begin{array}{|c|c|c} \cline{1-2} 1 & 2 & \\ \hline & 3 & \multicolumn{1}{c|}{4} \\ \cline{2-3} & 5 & \multicolumn{1}{c|}{6} \\ \cline{2-3} \end{array} \qquad \mathcal{F}_\lambda = \{x_2 = qx_1, x_4 = qx_3, x_6 = qx_5\}$$

$$\begin{aligned}
\Delta^+_{11} &: \quad (1,2), \\
\Delta^+_{12} &: \quad (\mathbf{1},\mathbf{3}),\ (1,4),\ \underline{(2,3),\ (\mathbf{2},\mathbf{4})}, \\
\Delta^+_{22} &: \quad (3,4), \\
\Delta^+_{13} &: \quad (1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)}, \\
\Delta^+_{23} &: \quad (3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6)}, \\
\Delta^+_{33} &: \quad (5,6).
\end{aligned}$$

Final example

$\lambda =$ 

$\Lambda =$ 

$\mathcal{F}_\lambda = \{x_2 = qx_1, x_4 = qx_3, x_6 = qx_5\}$

$$\Delta^+_{11}: \quad (1,2),$$
$$\Delta^+_{12}: \quad (\mathbf{1},\mathbf{3}),\ (1,4),\ \underline{(2,3),\ (\mathbf{2},\mathbf{4})},$$
$$\Delta^+_{22}: \quad (3,4),$$
$$\Delta^+_{13}: \quad (1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},$$
$$\Delta^+_{23}: \quad (3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6)},$$
$$\Delta^+_{33}: \quad (5,6).$$

$$w_0 = s_1 s_2 s_3 s_1 s_2 s_1 s_4 s_3 s_5 s_4 s_2 s_1 s_3 s_2 s_1$$

Remarks

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

In the above display (i,j) is a shortcut for $\varphi_{i,j}$. We shall start desingularizing from the rightmost singular pair in the "red" piece. So we move $(3,4)$ next to $(3,6),(4,6)$, which are already adjacent thanks to the special choice of the order. Then we apply the Lemma evaluating the three factors restricted to $\mathcal{F}_\lambda$. Here the main technical point is that $\varphi_{3,4}$, upon restriction to $\mathcal{F}_\lambda$, is a constant function, hence it is unaffected by evaluation! Therefore we can take back $\varphi_{3,4}$ to its original position.
The next display should suggest that a careful choice of the order on pairs to desingularize allows us to perform the previous procedure for any singular pair (up to those which are canceled by the correction factor δ_λ).

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (3,4), (2,5), (1,6), (2,6), (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (2,5), (3,4), (1,6), (2,6), (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (2,5), (1,6), (3,4), (2,6), (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (2,5), (1,6), (2,6), (3,4), (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), (3,4), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6),} (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6)}, (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6)}, (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(2,3), (1,3), (2,4), (1,4), (3,4), (2,5), (1,5), (1,2), \underline{(1,6), (2,6)}, (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6),} (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6),} (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(2,3), (1,3), (2,4), (1,4), (3,4), (2,5), (1,5), (1,2), \boxed{(1,2,6),} (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6)}, (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \boxed{(1,2,6),} (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The moves

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6)},(5,6)$

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6),}(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),(1,6),(2,6),(4,5),(3,5),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\underline{(1,6),(2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),\underline{(2,3),(2,4)},(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

The moves

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6)},(5,6)$

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6),}(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),(1,6),(2,6),(4,5),(3,5),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\underline{(1,6),(2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),\underline{(2,3),(2,4)},(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

The moves

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6)},(5,6)$

$(1,2),(\mathbf{1},\mathbf{3}),\ (\mathbf{1},\mathbf{4}),\ \underline{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{4})},(\mathbf{3},\mathbf{4}),(1,5),\ (2,5),\ \underline{(\mathbf{1},\mathbf{6}),\ (2,6)},(3,5),\ (4,5),\ \underline{(\mathbf{3},\mathbf{6}),\ (4,6),}(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),(1,6),(2,6),(4,5),(3,5),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\underline{(1,6),(2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

$(1,2),(1,3),(1,4),\underline{(2,3),(2,4)},(3,4),(1,5),(2,5),\boxed{(1,2,6),}(4,5),(3,5),(3,4),\boxed{(3,4,6)},(5,6)$

The moves

(1, 2), (**1**, **3**), (**1**, **4**), (**2**, **3**), (**2**, **4**), (**3**, **4**), (1, 5), (2, 5), (**1**, **6**), (2, 6), (3, 5), (4, 5), (**3**, **6**), (4, 6), (5, 6)

(1, 2), (**1**, **3**), (**1**, **4**), (**2**, **3**), (**2**, **4**), (**3**, **4**), (1, 5), (2, 5), (**1**, **6**), (2, 6), (3, 5), (4, 5), (**3**, **6**), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), [(3, 4, 6)], (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), [(3, 4, 6)], (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), [(1, 2, 6),] (4, 5), (3, 5), (3, 4), [(3, 4, 6)], (5, 6)

(1, 2), (1, 3), (1, 4), [(2, 3, 4),] (3, 4), (1, 5), (2, 5), [(1, 2, 6),] (4, 5), (3, 5), (3, 4), [(3, 4, 6)], (5, 6)

The moves

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (\mathbf{1},\mathbf{3}), (\mathbf{1},\mathbf{4}), \underline{(\mathbf{2},\mathbf{3}), (\mathbf{2},\mathbf{4})}, (\mathbf{3},\mathbf{4}), (1,5), (2,5), \underline{(\mathbf{1},\mathbf{6}), (2,6)}, (3,5), (4,5), \underline{(\mathbf{3},\mathbf{6}), (4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), (1,6), (2,6), (4,5), (3,5), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \underline{(1,6), (2,6)}, (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,5), (2,5), \boxed{(1,2,6),}\ (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(1,2), (1,3), (1,4), \boxed{(2,3,4),}\ (3,4), (1,5), (2,5), \boxed{(1,2,6),}\ (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

$(1,2), \boxed{\boxed{(1,3)}}, (1,4), \boxed{(2,3,4),}\ (3,4), (1,5), (2,5), \boxed{(1,2,6),}\ (4,5), (3,5), (3,4), \boxed{(3,4,6)}, (5,6)$

The final outcome

$$E_\lambda = (\partial_\lambda \varphi_0)|_{\mathcal{F}_\lambda}(1, q, 1, q, q^{-1}, 1) =$$
$$(T_1+1)(1-q)(T_3+1)(T_1 T_2 - q\, T_2 - q)(T_1+1)(T_4-q)$$
$$(T_3 - q^2(q+1)^{-1})(T_5 T_4 - q\, T_5 - q)(T_2 - q)$$
$$(T_1 - q^2(q+1)^{-1})(T_3 T_2 - q\, T_3 - q)(T_1+1).$$

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$$E_\lambda = (\partial_\lambda \varphi_0)|_{\mathcal{F}_\lambda}(1, q, 1, q, q^{-1}, 1) =$$
$$(T_1 + 1)(1 - q)(T_3 + 1)(T_1 T_2 - q\, T_2 - q)(T_1 + 1)(T_4 - q)$$
$$(T_3 - q^2(q + 1)^{-1})(T_5 T_4 - q\, T_5 - q)(T_2 - q)$$
$$(T_1 - q^2(q + 1)^{-1})(T_3 T_2 - q\, T_3 - q)(T_1 + 1).$$

E_λ is nonzero! Indeed:

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$$E_\lambda = (\partial_\lambda \varphi_0)|_{\mathcal{F}_\lambda}(1, q, 1, q, q^{-1}, 1) =$$
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$$(T_1 - q^2(q+1)^{-1})(T_3 T_2 - q\, T_3 - q)(T_1 + 1).$$

E_λ is nonzero! Indeed:

$$E_\lambda = q\,(q^2 - 1)\; T_1 T_3 T_4 T_3 T_5 T_4 T_2 T_1 T_3 T_2 T_1 + \sum_{\ell(w) < 11} a_w\, T_w$$