RESOLUTIONS OF THREE-ROWED SKEW- AND ALMOST SKEW-SHAPES IN CHARACTERISTIC ZERO

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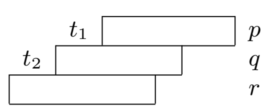
1. Introduction

In [2], a connection was made between the characteristic-free resolution of the three-rowed partition, $(2, 2, 2)$, and its characteristic-zero resolution described by A. Lascoux. The method used there was to take the known general resolution, modify the boundary map exploiting the fact that we can divide when we're over the rationals, and then reduce the large general resolution to the much slimmer one of Lascoux ([4]).

A similar program was carried out by [3] in his thesis, for the partition $(3, 3, 3)$, but it was clear that the method had reached its limit there, since the explicit description of the boundary maps in the characteristic-free case is not yet available. [1]

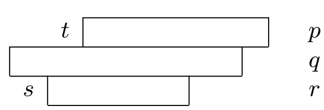
Since it was still very tempting to have an explicit description of the boundary maps in the characteristic-zero situation, we decided to borrow another tool from the characteristic-free construction, namely the mapping cone, and see if we couldn't arrive at the Lascoux resolutions in general, that is not just for partitions, but for all the three-rowed shapes that come up in the representation category: the skew-shapes and almost skew-shapes ([1]).

The combinatorial complexity of a three-rowed skew- or almost skew-shape is to a large extent tied up with the number of triple overlaps that occur in the shape. If the shape is a skew-shape:



then the number of triple overlaps is $r - t_1 - t_2$. To avoid drawing the diagram each time it's needed, we will often denote this shape as $(p, q, r; t_1, t_2)$.

If the shape is an almost skew-shape:



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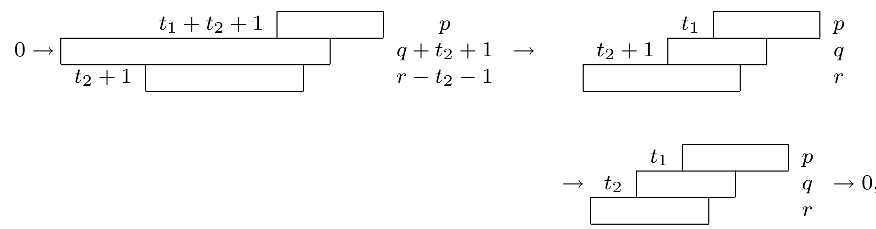
[1]We do intend to return to this approach after we make explicit the boundary maps of the characteristic-free resolutions, but this will no doubt take some time.

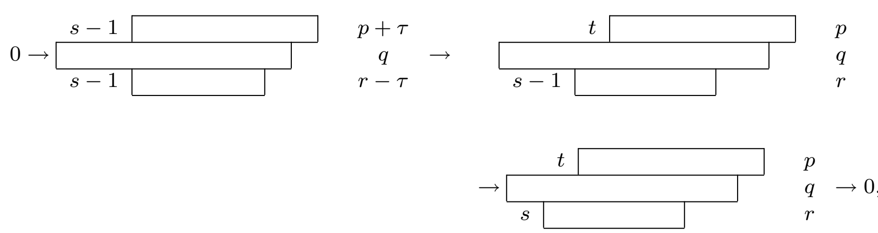
the number of triple overlaps is $r - t + s$. In this case, too, we will often denote the shape by $(p, q, r; t, s)$.

There is a theorem ([1]) that tells us that we have short exact sequences:



(**S**)

and, starting with an almost skew-shape:



(**A**)

where $\tau = t - s + 1$. Here we have resorted to "pictures" rather than the more concise notation since the pictures give a clearer indication of the changes taking place on the shapes as we move from right to left.

In each of these exact sequences, the number of triple overlaps in the middle and leftmost terms is one less than in the initial (or rightmost) term. By using induction on the number of triple overlaps in the term whose resolution we are seeking, we may assume the resolutions known for the terms with fewer triple overlaps. By producing a map between the known resolutions, we can take the mapping cone of this map, and it is well-known that this mapping cone will give us a resolution of the rightmost term. However, this resolution has more terms than are anticipated in the Lascoux resolution (in the case of skew-shapes; Lascoux didn't consider almost skew-shapes), and we would like to excise them. The first time this occurs in the course of our construction, we will prove a lemma which will provide a prescription for doing this excision.

In the next section, we will give a quick review of some of the notation we will be using, and also easily dispatch the resolutions of the two-rowed skew-shapes (there are no almost skew-shapes with two rows). This case, while very special, is essential in getting the induction process going.

In Section 3, we will get the description of the resolution for skew-shapes having at most one triple overlap, and then tackle the case of an almost skew-shape with one triple overlap (there are no almost skew-shapes with fewer than one triple overlap). These cases are rather special; for the skew-shapes, the Jacobi-Trudi determinant tells us we should expect fewer terms than in the general case, and that is what happens.

In the subsequent section, we get the description of the resolutions in general, again working first on the skew-shapes, and then the almost skew. From the exact sequence (**A**) above, it is clear that the approach to almost skew-shapes has to consider the case when $s = 1$ (where the terms that occur to the left of it will be skew-shapes), and the case when $s > 1$.

2. Notation and the two-rowed case

As usual, we will let D_a stand for the a-fold divided power of some underlying free module F, and we consider the n-fold tensor product, $D_{a_1} \otimes \cdots \otimes D_{a_n}$.

Notation: We denote by $\partial_{ji}^{(t)}$ the composition of the maps

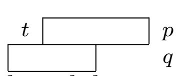
$$D_{a_1} \otimes \cdots \otimes D_{a_i} \otimes \cdots \otimes D_{a_j} \cdots \otimes D_{a_n} \xrightarrow{\Delta_t} D_{a_1} \otimes \cdots \otimes D_{a_i - t} \otimes D_t \otimes \cdots \otimes D_{a_j} \cdots \otimes D_{a_n}$$
$$\xrightarrow{\mu} D_{a_1} \otimes \cdots \otimes D_{a_i - t} \otimes \cdots \otimes D_{a_j + t} \cdots \otimes D_{a_n},$$

where Δ_t denotes the indicated diagonalization $D_{a_i} \to D_{a_i - t} \otimes D_t$, and μ stands for the multiplication within the divided power algebra, $D_t \otimes D_{a_j} \to D_{a_j + t}$. (We have obviously moved D_t past several places to juxtapose it with D_{a_j}, but there are no sign changes to deal with when we do this since the divided power algebra is commutative.)

Definition 2.1. *The map $\partial_{ji}^{(t)}$ is called the t^{th} divided power of the place polarization, ∂_{ji}.*

The way we have written this definition, it looks as though i must be less than j. It is clear from the definition though that the relative size of the indices is of no consequence; it simply moves from the argument in place i to the argument in place j. However, in all that we do in this article, it will be the case that $i < j$. For a fuller discussion of letter-place algebras and place polarizations, we refer the reader to [1].

The main result we want to prove in this section is the following theorem.

Theorem 2.1. *Let* (**M**)= 

be a two-rowed skew-shape, and let $K(p, q; t)$ denote the Weyl module corresponding to this shape. Then the following sequence:

$$0 \to D_{p+t+1} \otimes D_{q-t-1} \xrightarrow{\partial_{21}^{(t+1)}} D_p \otimes D_q \to K(p, q; t) \to 0$$

is exact.

Proof. We could go through a painstaking induction, using mapping cone arguments, to show that this is so. However, we'll jump to a "letter-place" proof of this easy fact, just to illustrate the use of that technique.

It is well-known that, in characteristic zero, the sequence without the leftmost zero is exact. Therefore it will suffice to define a map, s, from $D_p \otimes D_q$ to $D_{p+t+1} \otimes D_{q-t-1}$ such that $s\partial_{21}^{(t+1)} = \mathrm{id}$.

Now a basis element of $D_{p+t+1} \otimes D_{q-t-1}$ can be written, in letter-place double tableau notation, as

$$\left(\begin{array}{c|cc} w_1 & 1^{(p+t+1)} & 2^{(l)} \\ w_2 & 2^{(q-t-1-l)} & \end{array} \right),$$

and its image under $\partial_{21}^{(t+1)}$ is

$$\binom{t+1+l}{l} \left(\begin{array}{c|cc} w_1 & 1^{(p)} & 2^{(t+1+l)} \\ w_2 & 2^{(q-t-1-l)} & \end{array} \right).$$

To split this map, that is, to define our map, s, we have to take a typical basis element of $D_p \otimes D_q$ and say where it goes under s.

A typical such basis element is of the form

$$\left(\begin{array}{c|cc} w_1 & 1^{(p)} & 2^{(k)} \\ w_2 & 2^{(q-k)} & \end{array} \right).$$

Let's agree to define s on such a basis element by sending it to 0 if $k < t+1$, and sending it to $\binom{k}{k-t-1}^{-1} \left(\begin{array}{c|cc} w_1 & 1^{(p+t+1)} & 2^{(k-t-1)} \\ w_2 & 2^{(q-k)} & \end{array} \right)$, otherwise. It is trivial to check that $s\partial_{21}^{(t+1)} = \mathrm{id}$. □

3. At most one triple overlap

In this section, we start our mapping cone approach to constructing the resolutions of three-rowed shapes. The basic tool underlying this construction is the following easy lemma.

Lemma 3.1. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence of modules, let $\mathbb{X}$ *and* $\mathbb{Y}$ *be acyclic left complexes over* A *and* B *respectively, and let* $F : \mathbb{X} \to \mathbb{Y}$ *be a map over* f. *Then the mapping cone of* F *is an acyclic left complex over* C.

3.1. **Skew-shapes.** We'll first handle the case where $r \leq t_1 + t_2 + 1$ and $r = t_2 + 1$. The short exact sequence (**S**) of Section 1 gives us two terms that we can resolve, and we have their resolutions and maps between them:

$$\begin{array}{cccc} 0 \to & D_{p+t_1+t_2+2} \otimes D_{q-t_1-1} \otimes D_0 & \xrightarrow{\partial_{21}^{(t_1+t_2+2)}} & D_p \otimes D_{q+t_2+1} \otimes D_0 \\ & \downarrow F(t_1,t_2) & & \downarrow \partial_{32}^{(t_2+1)} \\ 0 \to & D_{p+t_1+1} \otimes D_{q-t_1-1} \otimes D_r & \xrightarrow{\partial_{21}^{(t_1+1)}} & D_p \otimes D_q \otimes D_r, \end{array}$$

where

$$F(t_1,t_2) = \sum_{\beta=0}^{t_2+1} \binom{t_1+1+\beta}{t_1+1}^{-1} \partial_{21}^{(\beta)}\partial_{32}^{(\beta)}\partial_{31}^{(t_2+1-\beta)}.$$

This definition of $F(t_1,t_2)$ is pretty much forced by the need to make the diagram commute. To get the resolution of the original shape, $(p,q,r;t_1,t_2)$, we take the mapping cone and get:

$$0 \to D_{p+|t|+2} \otimes D_{q-t_1-1} \otimes D_0 \to \begin{array}{c} D_{p+t_1+1} \otimes D_{q-t_1-1} \otimes D_r \\ \oplus \\ D_p \otimes D_{q+t_2+1} \otimes D_0 \end{array} \to D_p \otimes D_q \otimes D_r,$$

which has the terms that the Lascoux resolution says we should have in this case. (We have written $|t|$ for the sum t_1+t_2.) The maps from the center summands are the obvious place polarizations, the map to the upper central summand is what we've called $-F(t_1,t_2)$, and the map to the lower central summand is again the obvious place polarization.

To go further, we'll now still insist that $r \leq t_1+t_2+1$, but let $r = t_2+2$. This case, while again being very special, is what indicates what the inductive procedure is in general.

Using the same short exact sequence:

$$0 \to (p, q+t_2+1, 1; t_1+t_2+1, 0) \to (p,q,r;t_1,t_2+1) \to (p,q,r;t_1,t_2) \to 0,$$

we get a map of the resolution of the left-most into that of the center one:

$$\begin{array}{ccccccc} 0 \to & D_{p+|t|+3} \otimes D_{q-t_1-1} \otimes D_0 \to & \begin{array}{c} D_{p+|t|+2} \otimes D_{q-t_1-1} \otimes D_1 \\ \oplus \\ D_p \otimes D_{q+t_2+2} \otimes D_0 \end{array} & \to & D_p \otimes D_{q+t_2+1} \otimes D_1 \\ & \downarrow (t_2+2) & \downarrow F(t_1,t_2) \oplus (t_2+2) & & \downarrow \partial_{32}^{(t_2+1)} \\ 0 \to & D_{p+|t|+3} \otimes D_{q-t_1-1} \otimes D_0 \to & \begin{array}{c} D_{p+t_1+1} \otimes D_{q-t_1-1} \otimes D_r \\ \oplus \\ D_p \otimes D_{q+t_2+2} \otimes D_0 \end{array} & \to & D_p \otimes D_q \otimes D_r, \end{array}$$

where the sum on the center arrow indicates that $F(t_1,t_2)$ applies to the upper term, and multiplication by (t_2+2) applies to the lower summand.

The proof that this diagram does really commute is trivial in the right square, but involves some calculation in the second square. It boils down to proving the identity:

$$F(t_1,t_2)F(|t|+1,0) = (t_2+2)F(t_1,t_2+1).$$

We defer the proof of this fact until later; for now, we continue with the rest of the procedure. First, though, we introduce some shorthand notation for the tensor product of divided powers, namely, instead of writing $D_{a_1} \otimes D_{a_2} \otimes \cdots \otimes D_{a_n}$, we will write this as $((a_1)|(a_2)|\cdots|(a_n))$.

Returning to the map of complexes above, we now form its mapping cone, and we get the exact complex:

$$0 \to \quad ((p+|t|+3)|(q-t_1-1)|(0)) \to \quad \begin{matrix} ((p+|t|+2)|(q-t_1-1)|(1)) \\ \oplus \\ ((p)|(q+t_2+2)|(0)) \\ \oplus \\ ((p+|t|+3)|(q-t_1-1)|(0)) \end{matrix} \quad \to$$

$$\begin{matrix} ((p)|(q+t_2+1)|(1)) \\ \oplus \\ ((p+t_1+1)|(q-t_1-1)|(r)) \\ \oplus \\ ((p)|(q+t_2+2)|(0)) \end{matrix} \quad \to \quad ((p)|(q)|(r)).$$

We note that we have the subcomplex:

$$0 \to ((p+|t|+3)|(q-t_1-1)|(0)) \to \begin{matrix} ((p)|(q+t_2+2)|(0)) \\ \oplus \\ ((p+|t|+3)|(q-t_1-1)|(0)) \end{matrix} \to ((p)|(q+t_2+2)|(0)) \to 0$$

and the quotient complex:

$$0 \to ((p+|t|+2)|(q-t_1-1)|(1)) \to \begin{matrix} ((p+t_1+1)|(q-t_1-1)|(r)) \\ \oplus \\ ((p)|(q+t_2+1)|(1)) \end{matrix} \to ((p)|(q)|(r)).$$

This last, of course, is the one that we want for our resolution, so we have to show that the subcomplex we're dividing out by is exact. But the exactness of that complex in characteristic zero is trivial to show.

Just to be explicit, the maps from the sum to $((p|q|r))$ are the usual place polarizations. The map from the tail to the sum is exactly what it was in the previous case: $-F(t_1,t_2) \oplus \partial_{21}^{(|t|+2)}$.

Proceeding in this way, we obtain the result for skew-shapes having no more than one triple overlap.

Theorem 3.2. *Under the hypothesis that* $r \leq |t|+1$, *the resolution in characteristic zero of* $(p,q,r;t_1,t_2)$ *is*

$$0 \to ((p+|t|+2)|(q-t_1-1)|(r-t_2-1)) \to \begin{matrix} ((p+t_1+1)|(q-t_1-1)|(r)) \\ \oplus \\ ((p)|(q+t_2+1)|(r-t_2-1)) \end{matrix} \to ((p|q|r)).$$

Or, more graphically,

$$0 \to ((p+|t|+2)|(q-t_1-1)|(r-t_2-1)) \begin{matrix} \overset{F(t_1,t_2)}{\nearrow} \\ \\ \underset{\partial_{21}^{(|t|+2)}}{\searrow} \end{matrix} \begin{matrix} ((p+t_1+1)|(q-t_1-1)|(r)) \\ \oplus \\ ((p)|(q+t_2+1)|(r-t_2-1)) \end{matrix} \to ((p|q|r)).$$

We now provide the proof of the fact that

$$F(t_1,t_2)F(|t|+1,0) = (t_2+2)F(t_1,t_2+1).$$

A heuristic "proof" might go like this.

We can use the basic fact that

$$(*) \quad \partial_{21}^{(t_1+1)} F(t_1,t_2) = \partial_{32}^{(t_2+1)} \partial_{21}^{(t_1+t_2+2)},$$

and apply $\partial_{21}^{(t_1+1)}$ to both sides of the equation $(**)$. This equality is then very easy to see. But unfortunately, since $\partial_{21}^{(t_1+1)}$ isn't a monomorphism, this can't serve as a proof, and we have to go through the calculations.

First, let

$$A = (t_2+2)F(t_1,t_2+1) = \sum_{t_2+2\geq\beta\geq 0} k_\beta \partial_{21}^{(\beta)} \partial_{32}^{(\beta)} \partial_{31}^{(t_2+2-\beta)},$$

and

$$B = F(t_1,t_2)F(|t|+1,0) = \sum_{t_2+1\geq\beta\geq 0} k_\beta \partial_{21}^{(\beta)} \partial_{32}^{(\beta)} \partial_{31}^{(t_2+1-\beta)} (\partial_{31} + \frac{1}{t_1+t_2+3} \partial_{21}\partial_{32}),$$

where $k_\beta = \binom{t_1+1+\beta}{t_1+1}^{-1}$.

Taking into account the different ranges of summation of the index β, we see that

$$A-B = \sum_{t_2+2\geq\beta\geq 1} \beta k_\beta \partial_{21}^{(\beta)} \partial_{32}^{(\beta)} \partial_{31}^{(t_2+2-\beta)} - \frac{1}{t_1+t_2+3} \sum_{\gamma\geq 0} k_\gamma \partial_{21}^{(\gamma)} \partial_{32}^{(\gamma)} \partial_{21}\partial_{32} \partial_{31}^{(t_2+1-\gamma)}.$$

We distinguish between the indices β and γ because they run over different limits.

We now want to bring the ∂_{21} past the $\partial_{32}^{(\gamma)}$ in the last expression so that we can collect our terms. That is, we use the fact that $\partial_{32}^{(\gamma)}\partial_{21} = \partial_{21}\partial_{32}^{(\gamma)} + \partial_{32}^{(\gamma-1)}\partial_{31}$. If we do that, and then collect the terms that modify a fixed power of ∂_{31}, we see that they add up to zero, and we're done.

3.2. **The almost skew case.** Recall that we use the notation $(p,q,r;t,s)$ for an almost skew-shape, with $0 < s \leq t$.

To say that the almost skew-shape has only one triple overlap is simply to say that $r = t-s+1$. If that is the case, then we have the exact sequence

$$0 \to (p+t-s+1, q, 0; s-1, 0) \to (p,q,r;t,0) \to (p,q,r;t,s) \to 0.$$

The left-most shape is a two-rowed skew-shape, and the middle term is a three-rowed skew shape with no triple overlaps. If we write down their resolutions and map the left one into the other, we get the following:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & & \longrightarrow ((p+t+2)|(q-t-1)|(r-1)) \\
\downarrow & & \downarrow -F(t,0) \oplus \partial_{21}^{(t+2)} \\
 & \overset{(-1)^{t-s+1}\partial_{32}^{(t-s+1)}}{\nearrow} & ((p+t+1)|(q-t-1)|(r))) \\
((p+t+1)|(q-s)|0)) & & \oplus \\
 & \underset{-G(s,t)}{\searrow} & ((p)|(q+1)|(r-1)) \\
\partial_{21}^{(s)} \downarrow & & \downarrow \partial_{32} \oplus \partial_{21}^{(t+1)} \\
((p+t-s+1)|(q)|(0)) & \overset{\partial_{31}^{(t-s+1)}}{\longrightarrow} & ((p)|(q)|(r)),
\end{array}$$

where we define

$$G(s,t) = \sum_{\beta>0} (-1)^{\beta} (\beta)^{-1} \partial_{32}^{(\beta-1)} \partial_{21}^{(s+\beta)} \partial_{31}^{(t-s+1-\beta)}.$$

Note that if $t = s$. then $G(s,s) = -\partial_{21}^{(s+1)}$.

The definition of $G(s,t)$ is inspired by the Capelli identity (as are the definitions of most of the other maps that have appeared or will appear in this article), which in this case is:

$$\partial_{21}^{(t+1)} \partial_{32}^{(\tau)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{32}^{(\tau-\alpha)} \partial_{21}^{(t+1-\alpha)} \partial_{31}^{(\alpha)}.$$

A discussion of the Capelli identities can be found in [1], pages 166 – 167.

All of this discussion then gives us the following theorem.

Theorem 3.3. *Let (p,q,r;t,s) be a three-rowed almost skew-shape, with $r = t - s + 1$. The resolution of the Weyl module corresponding to this shape is the mapping cone of the above map, namely, the complex:*

$$0 \to \begin{array}{c} ((p+t+1)|(q-s)|(0)) \\ \oplus \\ ((p+t+2)|(q-t-1)|(r-1)) \end{array} \xrightarrow{\delta_2} \begin{array}{c} ((p+t+1)|(q-t-1)|(r)) \\ \oplus \\ ((p+\tau)|(q)|(0)) \\ \oplus \\ ((p)|(q+1)|(r-1)) \end{array} \xrightarrow{\delta_1} ((p)|(q)|(r)).$$

where the maps δ_1 and δ_2 are defined as follows:

$$\delta_2 = \begin{pmatrix} (-1)^{\tau-1}\partial_{32}^{(\tau)} & \partial_{21}^{(s)} & G(s,t) \\ \\ -F(t,0) & 0 & \partial_{21}^{(t+2)} \end{pmatrix},$$

$$\delta_1 = \begin{pmatrix} \partial_{21}^{(t+1)} & \partial_{31}^{(\tau)} & \partial_{32} \end{pmatrix}$$

and where we have set $\tau = t - s + 1$.

4. The general case

In the previous section, the resolutions we obtained were both of length 2. For the skew-shapes, this was anticipated by the Jacobi-Trudi determinantal formula for Weyl modules associated to skew-shapes. Of course, we have no classical formula for the almost skew-shapes, so it is fortunate that even here the length of the resolution is, if not what we could anticipate, at least what we hoped would be the case.

As we would expect, once the number of triple overlaps in a skew-shape is at least two, we get a full-length resolution, namely one of length 3. The way we approach the general case is to use the same sort of induction procedure that we used before: we start with our skew-shape with a certain number, n, of triple overlaps, we look at the fundamental exact sequence (**S**) of Section 2, and notice that the skew-shape and almost skew-shape that occur there both have fewer triple overlaps ($n-1$, to be exact). Assuming we know the resolutions for all shapes (both skew and almost skew) having fewer than n, we take those resolutions, find a map between them, take the mapping cone, see that it's larger than the one we had in mind, and see if some reduction can be carried out to arrive at the conjectured form. This reduction process is not simply finding an acyclic subcomplex of the mapping cone as we've done in the previous section; for skew-shapes, it requires more work than that.

For almost skew-shapes we start out following the same pattern, using the exact sequence (**A**) of Section 2. But there, we run into the problem that if $s = 1$, the shapes that appear in the exact sequence are skew, not almost skew, so that their resolutions are not the same as the ones we would use for almost skew-shapes. As a result, when we apply our inductive step, we must consider the two cases, $s = 1$ and $s > 1$ separately.

Since the discussions of the skew- and almost skew-shapes require that we hypothesize the forms of their resolutions, we will state here the results. In each of the following subsections, we will indicate in more detail how we arrive at them.

Theorem 4.1. *Let* $(p, q, r; t_1, t_2)$ *be a three-rowed skew-shape. The terms of the resolution of the Weyl module associated to this shape are:*

$$0 \to ((p+|t|+2)|(q)|(r-|t|-2)) \xrightarrow{\partial_3} \begin{matrix} ((p+|t|+2)|(q-t_1-1)|(r-t_2-1)) \\ \oplus \\ ((p+t_1+1)|(q+t_2+1)|(r-|t|-2))) \end{matrix} \xrightarrow{\partial_2}$$

$$\begin{matrix} ((p)|(q+t_2+1)|(r-t_2-1)) \\ \oplus \\ ((p+t_1+1)|(q-t_1-1)|(r)) \end{matrix} \xrightarrow{\partial_1} ((p|q|r)),$$

where we have set $|t| = t_1 + t_2$.

The boundary maps, which we describe in matrix form, are the following:

$$\partial_3 = \begin{pmatrix} (-1)^{t_1}\partial_{32}^{(t_1+1)} & \partial_{21}^{(t_2+1)} \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} \partial_{21}^{(|t|+2)} & -F(t_1,t_2) \\ L(t_1,t_2) & (-1)^{t_1}\partial_{32}^{(|t|+2)} \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} \partial_{32}^{(t_2+1)} \\ \partial_{21}^{(t_1+1)} \end{pmatrix}$$

where $F(t_1,t_2)$ *has already been defined, and we define*

$$L(t_1,t_2) = \sum_{\beta \geq 0} (-1)^{\beta} \binom{t_2+1+\beta}{t_2+1}^{-1} \partial_{32}^{(\beta)} \partial_{21}^{(\beta)} \partial_{31}^{(t_1+1-\beta)}.$$

Note: When $r < |t| + 2$, we recover the resolution for skew-shapes with at most one triple overlap.

Theorem 4.2. *Let* $(p,q,r;t,s)$ *be a three-rowed almost skew-shape. The terms of the resolution of the Weyl module associated to this shape are:*

$$0 \to ((p+t+2)|(q-s)|(r-\tau-1)) \xrightarrow{\delta_3} \begin{matrix} ((p+t+2)|(q-t-1)|(r-1)) \\ \oplus \\ ((p+t+1)|(q-s)|(r-\tau))) \\ \oplus \\ ((p+\tau)|(q+1)|(r-\tau-1)) \end{matrix} \xrightarrow{\delta_2}$$

$$\begin{matrix} ((p+t+1)|(q-t-1)|(r)) \\ \oplus \\ ((p)|(q+1)|(r-1)) \\ \oplus \\ ((p+\tau)|(q)|(r-\tau)) \end{matrix} \xrightarrow{\delta_1} ((p|q|r)),$$

where, as usual, we have set $\tau = t - s + 1$.

The boundary maps, which we will again describe in matrix form, are the following:

$$\delta_3 = \begin{pmatrix} \frac{t+2}{s+1}\partial_{32}^{(\tau)} & -F(s-1,0) & \partial_{21}^{(s+1)} \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} -F(t,0) & \partial_{21}^{(t+2)} & 0 \\ (-1)^{\tau-1}\partial_{32}^{(\tau)} & G(s,t) & \partial_{21}^{(s)} \\ 0 & -\partial_{31}^{(\tau)} & \partial_{32} \end{pmatrix}$$

$$\delta_1 = \begin{pmatrix} \partial_{21}^{(t+1)} \\ \partial_{32} \\ \partial_{31}^{(\tau)} \end{pmatrix}.$$

Note: Here again we see easily that when $r \le \tau$, we recover our old resolution.

Before we get into the details to prove the above results, let us describe the simplification process that we will have to use.

Simplification Procedure

Suppose that you have a complex, $\mathbb{X}$ which, in each dimension, i, breaks up into a direct sum of modules, $X_i = A_i \oplus B_i$. We will write the boundary map of $\mathbb{X}$ as $\{\delta_i\}$, and the boundary map on its A and B components as $\delta_{A_iA_{i-1}}, \delta_{A_iB_{i-1}}, \delta_{B_i,B_{i-1}}, \delta_{B_i,A_{i-1}}$.

We are assuming that $\mathbb{X}$ is a complex, but we make no assumptions at this point about either A or B. Assume now that $B_0 = 0$, and that we have maps $\sigma_i : B_i \to A_i$ satisfying the following identities:

$$(*) \qquad \begin{array}{l} \delta_{A_1A_0}\sigma_1 = \delta_1\sigma_1 = \delta_{B_1A_0}; \\ \partial_i\sigma_i = \delta_{B_iA_{i-1}} + \sigma_{i-1}\delta_{B_iB_{i-1}} \text{ for } i \ge 2, \end{array}$$

where the newly-defined boundary maps on A, $\partial = \{\partial_i\}$, are defined by setting

$$\partial_i = \delta_{A_iA_{i-1}} + \sigma_{i-1}\delta_{A_iB_{i-1}}.$$

With this boundary, it is easy to see that A is turned into a complex (the identities were chosen to ensure this). Now define the boundary in B in a similar way, namely,

$$\partial_i' = \delta_{B_iB_{i-1}} - \delta_{A_iB_{i-1}}\sigma_i,$$

and you get an exact sequence of complexes:[2]

$$0 \to B \overset{-\sigma\oplus I}{\to} A \oplus B \overset{I+\sigma}{\to} A \to 0,$$

where I is the identity map. If you know that the complexes B and $\mathbb{X}$ are exact, you then have the exactness of A.

[2]The same set of identities, (*), serves to make both of the maps bona fide maps of complexes.

4.1. **Skew-shapes.** Here we describe the steps used to prove Theorem 4.1.

We use induction on the number of triple overlaps, that is, on the number $\omega = r - |t|$. We have the result for $\omega \leq 1$, so we assume $\omega = 2$. The exact sequence (**S**) of Section 2 shows us that the middle term is a skew-shape with $\omega = 1$, and the leftmost shape is an almost skew-shape also with the number of triple overlaps equal to 1. If we take the resolutions of these shapes, we can define a map, Ω, from the one into the other (in terms of matrices) as follows:

$$\Omega_2 = \begin{pmatrix} t_2+2 \\ 0 \end{pmatrix}$$

$$\Omega_1 = \begin{pmatrix} 0 & F(t_1,t_2) \\ t_2+2 & 0 \\ H(t_1,t_2) & (-1)^{t_1+1}\partial_{32}^{(|t|+2)} \end{pmatrix}$$

$$\Omega_0 = \begin{pmatrix} \partial_{32}^{(t_2+1)} \end{pmatrix}.$$

where

$$H(t_1,t_2) = (-1)^{t_1}\sum_{\beta>0}(-1)^{\beta}\binom{t_2+1+\beta}{t_2+2}^{-1}\partial_{32}^{(\beta-1)}\partial_{21}^{(\beta)}\partial_{31}^{(t_1+1-\beta)}.$$

The map $H(t_1,t_2)$ suggests itself in order to satisfy the following identity:

$$\partial_{32}^{(t_2+1)}\partial_{31}^{(t_1+1)} = (-1)^{t_1+1}\partial_{21}^{(t_1+1)}\partial_{32}^{(|t|+2)} + \partial_{32}^{(t_2+2)}H(t_1,t_2).$$

When we take the mapping cone of the map, Ω, we obtain a complex which, while a resolution of our skew-shape, has too many terms in it, that is, it isn't of the form described in Theorem 4.1. In fact, we see that this mapping cone, which we will denote by $\mathbb{X}$, is as a module the direct sum of two submodules, A and B (as in the simplification procedure), with the B part representing the "excess" that we want to get rid of. However, neither the A part nor the B part is a subcomplex, so here is where we have to apply our simplification procedure to make them into complexes by modifying the boundary maps by finding maps $\sigma_i : B_i \to A_i$.

To be precise, the B_1 piece consists of the term $((p)|(q+t_2+2)|(r-t_2-2))$ and we set the map $\sigma_1 = \frac{1}{t_2+2}\partial_{32}$.

The piece B_2 is the direct sum of two pieces:

$$((p)|(q+t_2+2)|(r-t_2-2)) \oplus ((p+|t|+3)|(q-t_1-1)|(r-t_2-2)).$$

We set σ_2 to be zero on the first piece, and equal to $-\frac{1}{t_2+2}F(|t|+1,0)$ on the second piece.

Finally, on B_3, which is $((p+|t|+3)|(q-t_1-1)|(r-t_2-2))$, we define σ_3 to be zero.

It is as a result of the modifications on the boundary maps to make A and B into complexes that we arrive at the announced form of the resolution. It should be pointed out that the complex B produced in this way is clearly homologically trivial. Since the mapping cone is known to be acyclic, we get the fact that the complex A is acyclic as well, and thus is a resolution.

As the reader can see, the inductive procedure is repetitive, but yields the stated result. That is, when the number of triple overlaps is greater than 2, we proceed in the same way, employing the fundamental short exact sequence to reduce the number of triple overlaps, we find the map corresponding to Ω (it is essentially the same map as in the described case), we take the mapping cone, and get rid of the excess.

4.2. **The almost skew case.** As we indicated, the almost skew case is treated inductively in the same way as the skew, but as a careful look at the exact sequence (**A**) in section 2 will convince you, we have to treat the case $s = 1$ separately (always using the induction hypothesis on skew- and almost skew-shapes on the number of triple overlaps), since the shapes that occur in that case are skew- not almost skew-shapes.

If we write down the resolutions of the two shapes that occur in (**A**) (using Theorem 4.1), we must define a map between them. We shall call this map Θ, and describe $\Theta_3, \Theta_2, \Theta_1$ and Θ_0 as matrices.

$$\Theta_3 = \begin{pmatrix} \binom{t+2}{2} \end{pmatrix}$$

$$\Theta_2 = \begin{pmatrix} (-1)^t \frac{t+2}{2} \partial_{32}^{(t)} & 0 \\ 0 & \binom{t+2}{2} \end{pmatrix}$$

$$\Theta_1 = \begin{pmatrix} \partial_{31}^{(t)} & 0 \\ -G(1,t) & (-1)^t \partial_{32}^{(t)} \end{pmatrix}$$

$$\Theta_0 = \begin{pmatrix} \partial_{31}^{(t)} \end{pmatrix}$$

In this case, we see that the superfluous terms of the mapping cone, $\mathbb{X}$, actually form a subcomplex of $\mathbb{X}$, namely:

$$0 \to \quad ((p+t+2)|(q)|(r-t-2)) \overset{\alpha_2}{\to} \begin{matrix} ((p+t+2)|(q)|(r-t-2)) \\ \oplus \\ ((p+t+1)|(q+1)|(r-t-2)) \end{matrix}$$

$$\overset{\alpha_1}{\to} ((p+t+1)|(q+1)|(r-t-2)) \to 0$$

where each of α_1 and α_2 have a component which involves simple multiplication by $\binom{t+2}{2}$. Therefore this complex is homologically trivial, and the factor complex is our desired resolution.

When $s > 1$, we again resort to the exact sequence, (**A**), and this time we use the resolutions of Theorem 4.2 to resolve the central and leftmost terms. The map between those resolutions, which we will call Ψ, is defined to be

$$\Psi_3 = \begin{pmatrix} \tau+1 \end{pmatrix}$$

$$\Psi_2 = \begin{pmatrix} (-1)^\tau \frac{t+2}{s+1}\partial_{32}^{(\tau)} & 0 & 0 \\ 0 & \tau+1 & 0 \\ 0 & 0 & \tau+1 \end{pmatrix}$$

$$\Psi_1 = \begin{pmatrix} \partial_{32}^{(\tau)} & G(t,s) & 0 \\ 0 & \partial_{31}^{\tau} & 0 \\ 0 & 0 & \tau+1 \end{pmatrix}$$

$$\Psi_0 = \begin{pmatrix} \partial_{31}^{(\tau)} \end{pmatrix}.$$

In this case, too, the mapping cone has superfluous terms, these terms form a homologically trivial subcomplex of it, and the quotient complex is the desired resolution. This concludes the discussion of the results Theorem 4.1 and Theorem 4.2.

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