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THE DESCENT STATISTIC ON 123-AVOIDING PERMUTATIONS

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Abstract. We exploit Krattenthaler's bijection between 123-avoiding permutations and Dyck paths to determine the Eulerian distribution over the set $S_n(123)$ of 123-avoiding permutations in S_n. In particular, we show that the descents of a permutation correspond to valleys and triple ascents of the associated Dyck path. We get the Eulerian numbers of $S_n(123)$ by studying the joint distribution of these two statistics on Dyck paths.

1. Introduction

A permutation $\sigma \in S_n$ *avoids a pattern* $\tau \in S_k$ if σ does not contain a subsequence that is order-isomorphic to τ. The subset of S_n of all permutations avoiding a pattern τ is denoted by $S_n(\tau)$. Pattern avoiding permutations have been intensively studied in recent years from many points of view (see e.g. [7], [4], [1], and references therein).

In the case $\tau \in S_3$, it has been shown that the cardinality of $S_n(\tau)$ equals the n-th Catalan number, for every pattern τ (see e.g. [3] and [7]), and hence the set $S_n(\tau)$ is in bijection with the set of Dyck paths of semilength n. Indeed, the six patterns in S_3 are related as follows:

- $321 = 123^{rev}$,
- $231 = 132^{rev}$,
- $312 = 132^{c}$,
- $213 = (132^{c})^{rev}$,

where rev and c denote the usual reverse and complement operations. Hence, in order to determine the distribution of the descent statistic over $S_n(\tau)$, for every $\tau \in S_3$, it is sufficient to examine the distribution of descents over two sets, say $S_n(132)$ and $S_n(123)$.

In both cases, the two bijections due to Krattenthaler [4] (see also [2]) allow to translate the descent statistic into some appropriate statistics on Dyck paths.

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In the case $\tau = 132$, the descents of a permutation are in one-to-one correspondence with the valleys of the associated Dyck path (see [8] and [9]).

In this paper we investigate the case $\tau = 123$. In particular, we exploit Krattenthaler's map to translate the descents of a permutation $\sigma \in S_n(123)$ into peculiar subconfigurations of the associated Dyck path, namely, valleys and triple ascents.

For that reason, we study the joint distribution of valleys and triple ascents over the set $\mathcal{P}_n$ of Dyck paths of semilength n, and we give an explicit expression for its trivariate generating function

$$A(x, y, z) = \sum_{n\geq 0} \sum_{\mathscr{D}\in\mathcal{P}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} a_{n,p,q} x^n y^p z^q,$$

where $v(\mathscr{D})$ denotes the number of valleys in $\mathscr{D}$ and $ta(\mathscr{D})$ denotes the number of triple ascents in $\mathscr{D}$. This series specializes to some well known generating functions, such as the generating function of Catalan numbers, Motzkin numbers, Narayana numbers, and sequence A092107 in [8] (see also [5]).

2. Dyck paths

A *Dyck path* of semilength n is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of n up-steps $U = (1, 1)$ and n down steps $D = (1, -1)$, never passing below the x-axis.

A *return* of a Dyck path is a down step ending on the x-axis, not counting the last step of the Dyck path. An *irreducible* Dyck path is a Dyck path with no return.

We note that a Dyck path $\mathscr{D}$ can be decomposed according to its last return (*last return decomposition*) into the juxtaposition of a (possibly empty) Dyck path $\mathscr{D}'$ of shorter length and an irreducible Dyck path $\mathscr{D}''$.

For example, the Dyck path $\mathscr{D} = U^5D^2UD^4UDU^3DUD^3$ decomposes into $\mathscr{D}' \bigoplus \mathscr{D}''$, where $\mathscr{D}' = U^5D^2UD^4UD$ and $\mathscr{D}'' = U^3DUD^3$, as shown in Figure 1.

3. Krattenthaler's bijection

In [4], Krattenthaler describes a bijection between the set $S_n(123)$ and the set $\mathcal{P}_n$ of Dyck paths of semilength n.

Let $\sigma = \sigma(1) \ldots \sigma(n)$ be a 123-avoiding permutation. Recall that a *right-to-left maximum* of σ is an element $\sigma(i)$ which is larger than $\sigma(j)$ for all j with $j > i$ (note that the last entry $\sigma(n)$ is a right-to-left

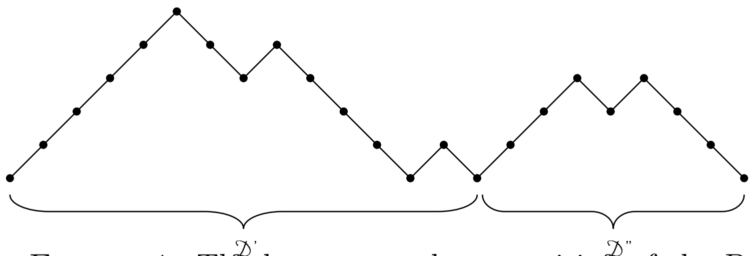

FIGURE 1. The last return decomposition of the Dyck path $\mathscr{D} = U^5D^2UD^4UDU^3DUD^3$.

maximum). Let $x_s, \ldots, x_1$ be the right-to-left maxima in σ. Then, we can write

$$\sigma = w_s\, x_s\, \ldots\, w_1\, x_1, \tag{1}$$

where w_i are (possibly empty) words. Moreover, since σ avoids 123, the word $w_s\, w_{s-1}\, \ldots\, w_1$ must be decreasing.

In order to construct the Dyck path $\kappa(\sigma)$ corresponding to σ, read the decomposition (1) from right to left. Any right-to-left maximum x_i is translated into $x_i - x_{i-1}$ up steps (with the convention $x_0 = 0$) and any subword w_i is translated into $l_i + 1$ down steps, where l_i denotes the number of elements in w_i. Then, reflect the constructed path in a vertical line.

For example, the permutation $\sigma = 6\,4\,7\,3\,2\,5\,1$ in $S_7(123)$ corresponds to the path in Figure 2.

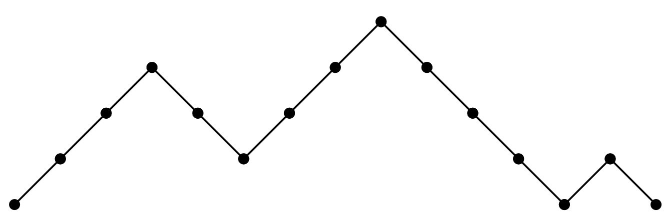

FIGURE 2. The Dyck path $\kappa(\sigma)$, with $\sigma = 6\,4\,7\,3\,2\,5\,1$.

4. The descent statistic

We say that a permutation σ has a *descent* at position i if $\sigma(i) > \sigma(i+1)$. We denote by $\mathrm{des}(\sigma)$ the number of descents of the permutation σ.

In this section we determine the generating function

$$E(x,y) = \sum_{n\geq 0} \sum_{\sigma \in S_n(123)} x^n y^{\mathrm{des}(\sigma)} = \sum_{n\geq 0}\sum_{k\geq 0} e_{n,k} x^n y^k,$$

where $e_{n,k}$ denotes the number of permutations in $S_n(123)$ with k descents.

Proposition 1. *Let σ be a permutation in $S_n(123)$, and $\mathscr{D} = \kappa(\sigma)$. The number of descents of σ is*

$$des(\sigma) = v(\mathscr{D}) + ta(\mathscr{D}),$$

where $v(\mathscr{D})$ is the number of valleys (the number of occurrences of DU) in $\mathscr{D}$ and $ta(\mathscr{D})$ is the number of triple ascents *(the number of occurrences of UUU) in $\mathscr{D}$.*

Proof. Let $\sigma = w_s\, x_s \, \ldots \, w_1\, x_1$ be a 123-avoiding permutation. The descents of σ occur precisely in the following positions:

1. between two consecutive symbols in the same word w_i (we have $l_i - 1$ of such descents),
2. after every right-to-left maximum x_i, except for the last one.

The proof is completed as soon as we observe that:

1. every word w_i is mapped into an ascending run of $\kappa(\sigma)$ of length l_i+1. Such an ascending run contains l_i-1 triple ascents, these in their turn are in bijection with the descents contained in w_i,
2. every right-to-left maximum x_i with $i \geq 2$ corresponds to a valley in $\kappa(\sigma)$.

□

The preceding result implies that we can switch our attention from permutations in $S_n(123)$ with k descents to Dyck paths of semilength n with k valleys and triple ascents. Hence, we study the joint distribution of valleys and triple ascents over $\mathcal{P}_n$, namely, we analyze the generating function

$$A(x,y,z) = \sum_{n\geq 0}\sum_{\mathscr{D}\in \mathcal{P}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} a_{n,p,q} x^n y^p z^q.$$

We determine the relation between the function $A(x,y,z)$ and the generating function

$$B(x,y,z) = \sum_{n\geq 0}\sum_{\mathscr{D}\in \mathcal{IP}_n} x^n y^{v(\mathscr{D})} z^{ta(\mathscr{D})} = \sum_{n,p,q\geq 0} b_{n,p,q} x^n y^p z^q$$

of the same joint distribution over the set $\mathcal{IP}_n$ of irreducible Dyck paths in $\mathcal{P}_n$.

Proposition 2. *For every $n > 2$, we have:*

$$b_{n,p,q} = a_{n-1,p,q-1} - a_{n-2,p-1,q-1} + a_{n-2,p-1,q}. \tag{2}$$

Proof. An irreducible Dyck path of semilength n with p valleys and q triple ascents can be obtained by prepending U and appending D to a Dyck path of semilength $n-1$ of one of the two following types:

1. a Dyck path with p valleys and q triple ascents, starting with UD,
2. a Dyck path with p valleys and $q-1$ triple ascents, which does not start with UD.

We observe that:

1. The paths of the first kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and q triple ascents, enumerated by $a_{n-2,p-1,q}$.

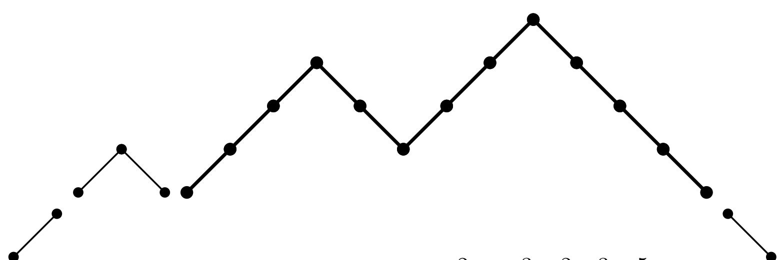

FIGURE 3. The Dyck path $U^2DU^3D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by prepending UD to the path $U^3D^2U^3D^4$ with 1 valley and 2 triple ascents, and then elevating.

2. In order to enumerate the paths of the second kind we have to subtract from the integer $a_{n-1,p,q-1}$ the number of Dyck paths of semilength $n-1$ with p valleys and $q-1$ triple ascents, starting with UD. Dyck paths of this kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and $q-1$ triple ascents, enumerated by $a_{n-2,p-1,q-1}$.

□

Proposition 3. *For every $n > 0$, we have:*

$$a_{n,p,q} = b_{n,p,q} + \sum_{i=1}^{n-1} \sum_{j,s \geq 0} b_{i,j,s} a_{n-i,p-j-1,q-s}. \tag{3}$$

FIGURE 4. The Dyck path $U^3DU^2D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by elevating the path $U^2DU^2D^2U^3D^4$ with 2 valleys and 1 triple ascent.

Proof. Let $\mathscr{D}$ be a Dyck path of semilength n and consider its last return decomposition $\mathscr{D} = \mathscr{D}' \bigoplus \mathscr{D}''$. If $\mathscr{D}'$ is empty, then $\mathscr{D}$ is irreducible. Otherwise, we have

- $v(\mathscr{D}) = v(\mathscr{D}') + v(\mathscr{D}'') + 1$,
- $ta(\mathscr{D}) = ta(\mathscr{D}') + ta(\mathscr{D}'')$.

□

Identities (2) and (3) yield the following relations between the two generating functions $A(x, y, z)$ and $B(x, y, z)$.

Proposition 4. *We have*

$$B(x, y, z) = (A(x, y, z) - 1)(xz + x^2y - x^2yz) + 1 + x + x^2 - x^2z \tag{4}$$

and

$$A(x, y, z) = B(x, y, z) + y(B(x, y, z) - 1)(A(x, y, z) - 1). \tag{5}$$

Proof. Note that recurrence (2) holds for $n > 2$. This fact gives rise to the correction terms of x-degree less than 3 in Formula (4). □

Combining Formulae (4) and (5) we obtain the following result.

Theorem 5. *We have:*

$$\begin{aligned} A(x, y, z) = \frac{1}{2xy(xyz - z - xy)} \Big(& -1 + xy + 2x^2y \\ & - 2x^2y^2 + xz - 2xyz - 2x^2yz + 2x^2y^2z \\ & + \sqrt{1 - 2xy - 4x^2y + x^2y^2 - 2xz + 2x^2yz + x^2z^2} \Big). \end{aligned} \tag{6}$$

This last result allows us to determine the generating function $E(x, y)$ of the Eulerian distribution over $S_n(123)$. In fact, the previous arguments show that

$$E(x, y) = A(x, y, y).$$

Hence, we obtain the following explicit expression for $E(x,y)$.

Theorem 6. *We have:*

$$E(x,y)=\frac{-1+2xy+2x^2y-2xy^2-4x^2y^2+2x^2y^3+\sqrt{1-4xy-4x^2y+4x^2y^2}}{2xy^2(xy-1-x)}.$$

The first values of the sequence $e_{n,d}$ are shown in the following table:

n/d	0	1	2	3	4	5	6
0	1						
1	1						
2	1	1					
3	0	4	1				
4	0	2	11	1			
5	0	0	15	26	1		
6	0	0	5	69	57	1	
7	0	0	0	56	252	120	1

Needless to say, the series $A(x,y,z)$ specializes to some well known generating functions. In particular, $A(x,1,1)$ is the generating function of Catalan numbers, $A(x,1,0)$ the generating function of Motzkin numbers, $yA(x,y,1)$ the generating function of Narayana numbers, and $A(x,1,z)$ the generating function of seq. A092107 in [8].

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