Signed colorings of generalized permutation arrays

SLC 62, Heilsbronn

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joint work with J. A. Dias da Silva

Overview

- Generalized permutation arrays
- Colorings of generalized permutation arrays
- Open problems
- Applications

Generalized permutation arrays

A generalized permutation array

$$\Gamma = \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

with the properties

(1) $i_1 \leq i_2 \leq \ldots \leq i_m$;

(2) $\{i_1, \ldots, i_m\} = \{j_1, \ldots, j_m\} = \{1, \ldots, r\}$;

(3) $|\{k : \ i_k = 1\}| \geq |\{k : \ i_k = 2\}| \geq \ldots \geq |\{k : \ i_k = r\}|$.

(4) $|\{k : \ i_k = p\}| = |\{k : \ j_k = p\}|, \ p = 1, \ldots, r$.

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The sequence

$$\lambda = (|\{k: \ i_k = 1\}|, |\{k: \ i_k = 2\}|, \ldots, |\{k: \ i_k = r\}|)$$

is a partition of m and it is called the **multiplicity partition** of Γ.

The conjugate partition of λ is called the **rank partition** of Γ and it is denoted

$$\rho(\Gamma).$$

Example.

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 5 & 1 & 2 & 2 & 4 & 1 & 3 \end{pmatrix}$$

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Signed colorings of normal arrays

Let Γ be a normal array.

Let μ be a partition of m.

We say that Γ is μ-**colorable** if it is possible to fill the Young diagram $[\mu]$ with all the pairs

$$(i_k, j_k),\ \ k = 1, \ldots, m$$

in a way that there will be a bijection on every row of $[\mu]$.

The obtained Young tableau T^{μ} is called a μ-**coloring** of Γ.

A $\rho(\Gamma)$-coloring of Γ will be called a **full** coloring of Γ.

Theorem. If Γ is μ-colorable, then $\mu \preceq \rho(\Gamma)$.

In general, it is not true that every normal array admits a full coloring.

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If $T^{\rho(\Gamma)}$ is a full coloring of Γ, then on row v there is a permutation σ_v of the set $\{1, \ldots, \rho_v\}$, for every $v \in \{1, \ldots, \lambda_1\}$.

The **sign** of $T^{\rho(\Gamma)}$ is the product of the signs of the permutations $\sigma_1, \ldots, \sigma_{\lambda_1}$, lying on the rows of $T^{\rho(\Gamma)}$.

We say that a full coloring of Γ is **positive** (respectively **negative**) if its sign is 1 (respectively -1).

We denote

$$P(\Gamma)$$

the number of positive full colorings of Γ;

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Example

Let

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 5 & 1 & 2 & 2 & 4 & 1 & 3 \end{pmatrix}.$$

Then, $\rho(\Gamma)$ is the partition $(5, 3, 1)$ and

$$T = \begin{array}{|c|c|c|c|c|} \hline (1,5) & (2,2) & (3,4) & (4,1) & (5,3) \\ \hline (1,3) & (2,1) & (3,2) \\ \cline{1-3} (1,1) \\ \cline{1-1} \end{array}$$

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A partition λ is said to be **sign uniform** if, for every array Γ, with multiplicity partition λ, whether $N(\Gamma) = 0$ or $P(\Gamma) = 0$.

Theorem(Dias da Silva, MMT) : A partition λ is sign uniform if and only if its Young diagram does not contain the diagram

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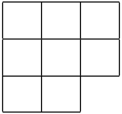

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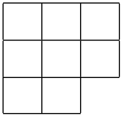

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Problem 1

Given a normal array Γ, find necessary and sufficient conditions for the existence of a full coloring of Γ.

We have established a necessary condition for the existence of a full coloring of Γ, using a graph theoretic approach.

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Problem 2

Given a normal array Γ, whose multiplicity partition is not sign uniform, find conditions for the equality of $P(\Gamma)$ and $N(\Gamma)$.

For normal arrays

$$\Gamma = \begin{pmatrix} i_1 & i_2 & \dots & i_{r^2} \\ j_1 & j_2 & \dots & j_{r^2} \end{pmatrix}$$

such that

$$\{(i_k, j_k) : \ k = 1, \dots, r^2\} = \{1, \dots, r\} \times \{1, \dots, r\}$$

there is a one-to-one correspondence between Latin squares of order r and full colorings of Γ.

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Applications to Multilinear Algebra

Let $V = \mathbb{C}^n$ and let $(e_1, \ldots, e_n)$ be a o.n. basis of V.
Let $m \in \mathbb{N}$.
Let $\Gamma_{m,n}$ be the set of the mappings from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.
Let χ be an irreducible character of S_m.

The χ-**symmetry class of tensors** on V is the span of the set of the decomposable symmetrized tensors e_α^χ

$$\left\{ \frac{\chi(id)}{m!} \sum_{\sigma \in S_m} \chi(\sigma) e_{\alpha\sigma^{-1}(1)} \otimes \ldots \otimes e_{\alpha\sigma^{-1}(m)} : \alpha \in \Gamma_{m,n} \right\}.$$

Grassmann space is the ε-symmetry class of tensors.

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The χ-**symmetry class of tensors** on V is the span of the set of the decomposable symmetrized tensors e_α^χ

$$\left\{ \frac{\chi(id)}{m!} \sum_{\sigma \in S_m} \chi(\sigma) e_{\alpha\sigma^{-1}(1)} \otimes \ldots \otimes e_{\alpha\sigma^{-1}(m)} : \ \alpha \in \Gamma_{m,n} \right\}.$$

Grassmann space is the ε-symmetry class of tensors.

The inner product of two symmetrized decomposable tensors e^χ_α and e^χ_β is zero whenever α and β are not congruent modulo S_m.

Otherwise, it is given by the formula

$$\frac{\chi(id)}{m!}\sum_{\sigma\in S_\alpha}\chi(\tau^{-1}\sigma)$$

where $\beta = \alpha\tau$ and S_α is the stabilizer of α.

It is important to have conditions for the orthogonality of two symmetrized decomposable tensors.

Without loss of generality, we can suppose that α is weakly increasing and $|\alpha^{-1}(1)| \geq \ldots \geq |\alpha^{-1}(n)|$.

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It is easy to see that, under the previous conditions, α and β are congruent modulo S_m if and only if

$$\Gamma = \begin{pmatrix} \alpha(1) & \alpha(2) & \dots & \alpha(m) \\ \beta(1) & \beta(2) & \dots & \beta(m) \end{pmatrix}$$

is a normal array.

Theorem. (Dias da Silva, MMT) If the multiplicity partition of Γ is equal to χ, then

$$e_\alpha^\chi \text{ and } e_\beta^\chi \text{ are orthogonal if and only if } N(\Gamma) = P(\Gamma).$$

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Example

Let $\chi = (3, 2^2, 1^2)$, $u = e_\alpha^\chi$, $v = e_\beta^\chi$ and

$$\Gamma = \left(\begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 5 & 1 & 2 & 2 & 4 & 1 & 3 \end{array} \right).$$

Since χ is sign uniform and there is a full coloring of Γ, we know that $N(\Gamma) \neq P(\Gamma)$, so u and v are not orthogonal.

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