On non-crossing and non-nesting set partitions in types A, B and C

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62ème Séminaire Lotharingien de Combinatoire

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Overview

Overview

Non-crossing and non-nesting set partitions

Non-crossing set partitions of types B and C

Non-nesting set partitions of type C

Non-nesting set partitions of type B

A counterexample in type D

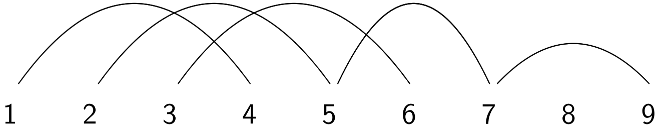
Generalizations

Set partitions

Let $\mathcal{B} \vdash [n] = \{1, \ldots, n\}$ be a **set partition**.

Example

$\mathcal{B} = \big\{\{1,4\},\{2,5,7,9\},\{3,6\},\{8\}\big\} \vdash [9]$:



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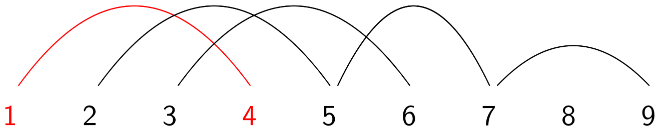

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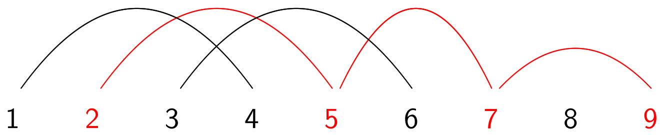

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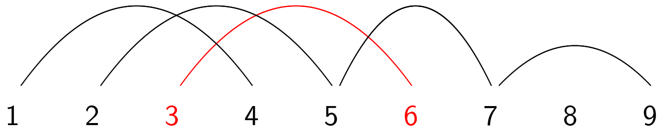

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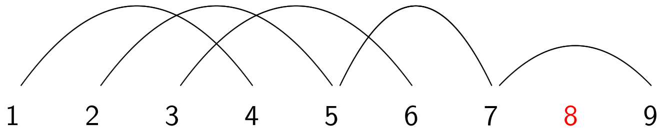

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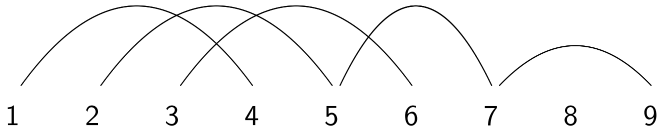
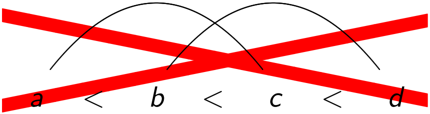


Non-crossing set partitions

A set partition $\mathcal{B} \vdash [n]$ is called

- **non-crossing**, if for $a < b < c < d$ such that a, c are contained in a block B of $\mathcal{B}$, while b, d are contained in a block B' of $\mathcal{B}$, then $B = B'$:



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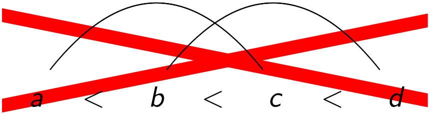
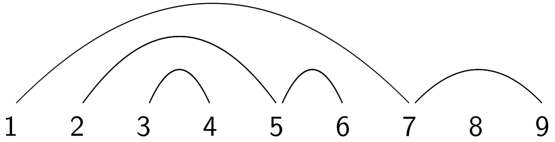


Example

$\mathcal{B} = \{\{1,7,9\},\{2,5,6\},\{3,4\},\{8\}\} \vdash [9]$ is non-crossing:



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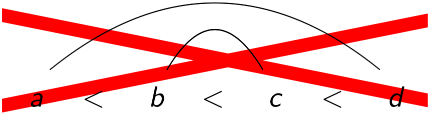

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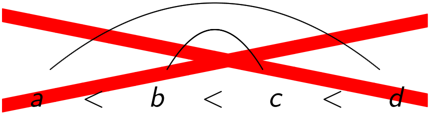

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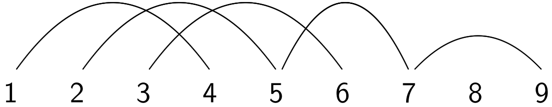

Bijections between non-crossing and non-nesting set partitions

There exist several bijections different between non-crossing and non-nesting set partitions, e.g.:

- A bijection preserving the **type** (C.A. Athanasiadis),
- a bijection sending the sum of the **major index** and the **inverse major index** to the **area statistic** (St.),
- a bijection preserving **openers** and **closers** and thereby the # of **blocks** (A. Kasraoui & J. Zeng, C. Krattenthaler).

Bijections between non-crossing and non-nesting set partitions preserving **openers** and **closers**

Openers and **closers** of a set partition on a totally ordered set S are defined by

$$\begin{aligned} \mathrm{op}(\mathcal{B}) &:= S \setminus \{ \max(B) : B \in \mathcal{B} \}, \\ \mathrm{cl}(\mathcal{B}) &:= S \setminus \{ \min(B) : B \in \mathcal{B} \}, \end{aligned}$$

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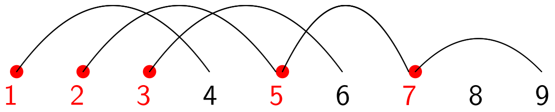

This gives $\mathrm{op}(\mathcal{B}) = \{1,2,3,5,7\}$.

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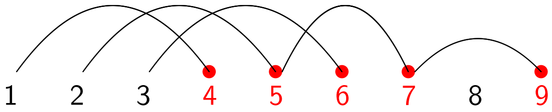

This gives $\mathrm{cl}(\mathcal{B}) = \{4,5,6,7,9\}$.

Bijections between non-crossing and non-nesting set partitions preserving **openers** and **closers**

Observation

Let $O, C \subseteq S$ for some finite, totally ordered set S. Then there exists a **unique** non-crossing set partition $\mathcal{B}$ and a **unique** non-nesting set partition $\mathcal{B}'$ on S with

$$\mathrm{op}(\mathcal{B}) = \mathrm{op}(\mathcal{B}') = O \quad , \quad \mathrm{cl}(\mathcal{B}) = \mathrm{cl}(\mathcal{B}') = C$$

if and only if $|O| = |C|$ and for $i \in \{1, \ldots, |S|\}$,

$$\big|O \cap \{s_1, \ldots, s_{i-1}\}\big| \geq \big|C \cap \{s_1, \ldots, s_i\}\big|.$$

Idea (following A. Kasraoui, J. Zeng)

- NC: connect the i-th closer to the **last** unused opener,
- NN: connect the i-th closer to the **first** unused opener.

Bijections between non-crossing and non-nesting set partitions preserving **openers** and **closers**

Example (Non-crossing, i-th closer – last unused opener)

Let $\mathrm{op}(\mathcal{B}) := \{1,2,3,5,7\}, \mathrm{cl}(\mathcal{B}) := \{4,5,6,7,9\} \subseteq [9]$. Then

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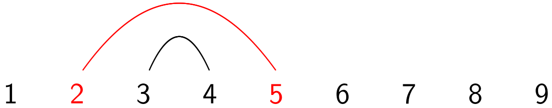

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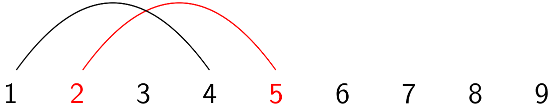

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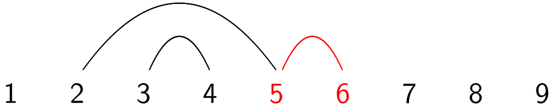

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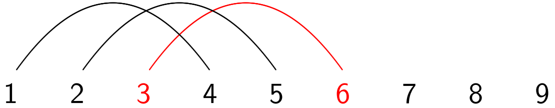

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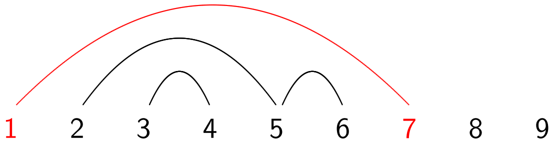

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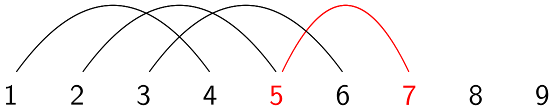

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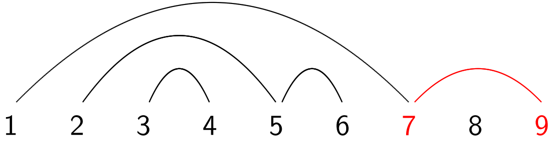

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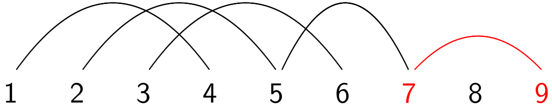

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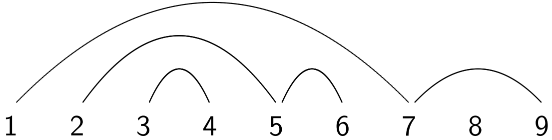

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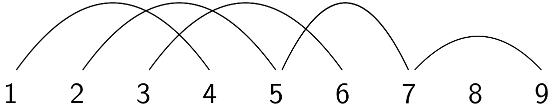


Bijections between non-crossing and non-nesting set partitions preserving **openers** and **closers**

Theorem

There exists a **unique** *bijection between non-crossing and non-nesting set partitions preserving openers and closers.*

Corollary

- *The bijection by A. Kasraoui and J. Zeng which interchanges crossings and nestings preserves openers and closers,*
- *the bijection by C. Krattenthaler between k-crossing and k-nesting set partitions preserves openers and closers.*

$\Rightarrow$ *For non-crossing and non-nesting set partitions both bijections coincide.*

Generalizations of non-crossing and non-nesting set partitions

Non-crossing and non-nesting set partitions were generalized to other (classical) reflection groups:

- non-crossing: as intersection lattices of **Coxeter arrangements** (V. Reiner),
- non-nesting: as anti-chains in the **root poset** (A. Postnikov),

and later reinterpreted in terms of

- set partitions on $[\pm n]$ or $[\pm n] \cup \{0\}$ (C.A. Athanasiadis).

Remark

- Recently, A. Fink and B.I. Giraldo generalized Athanasiadis' **type**-preserving bijection to all classical reflection groups.
- The bijection sending the **area** to the sum of the **major** and the **inverse major index** can be generalized to types B and C but **fails** to exist in type D.

Overview

Non-crossing set partitions of types B and C

A **non-crossing set partition of type** B and **of type** C is a set partition $\mathcal{B}$ on $[\pm n] := \{1, 2, \ldots, n, -1, -2, \ldots, -n\}$ such that

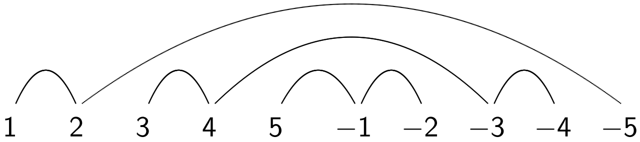
$$B \in \mathcal{B} \Leftrightarrow -B \in \mathcal{B},$$

and which is non-crossing in the **crossing order**

$$1 < 2 < \ldots < n < -1 < -2 < \ldots < -n.$$

Example

$\mathcal{B} = \{\{1, 2, -5\}, \{3, 4, -3, -4\}, \{5, -1, -2\}\} \vdash [\pm 5]$ is non-crossing of types B and C:



Openers and **closers** for non-crossing set partitions of types B and C

Observation

Let $O, C \subseteq [n]$. Then there exists a **unique** non-crossing set partition $\mathcal{B}$ of types B and C on $[\pm n]$ with $\mathrm{op}(\mathcal{B}) \cap [n] = O$ and $\mathrm{cl}(\mathcal{B}) \cap [n] = C$ if and only if for all i,

$$\left|O \cap \{s_1, \ldots, s_i\}\right| \geq \left|C \cap \{s_1, \ldots, s_i\}\right|.$$

Idea

1. complete the "positive part"
2. reflect it to the "negative part" and
3. connect both parts.

Openers and **closers** for non-crossing set partitions of types B and C

Example

Let

$$\begin{aligned} \mathrm{op}(\mathcal{B}) \cap [n] := \{1,2,3,4,5\} &\subseteq [5], \\ \mathrm{cl}(\mathcal{B}) \cap [n] := \{2,4\} &\subseteq [5]. \end{aligned}$$

Then we get

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5$

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Openers and **closers** for non-crossing set partitions of types B and C

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$$\mathrm{op}(\mathcal{B}) := \{1,2,3,4,5,-1,-3\} \subseteq [\pm 5],$$
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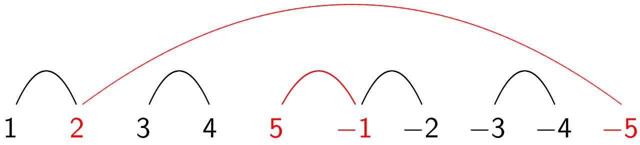
$$\mathrm{cl}(\mathcal{B}) := \{2, 4, -1, -2, -3, -4, -5\} \subseteq [\pm 5].$$

Then we get



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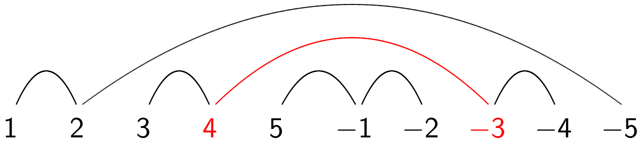

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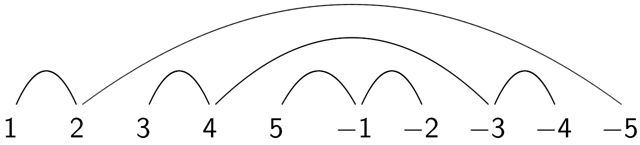

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A counterexample in type D

Generalizations

Non-nesting set partitions of type C

A **non-nesting set partition of type** C is a set partition $\mathcal{B}$ on $[\pm n] := \{1, 2, \ldots, n, -n, \ldots -2, -1\}$ such that

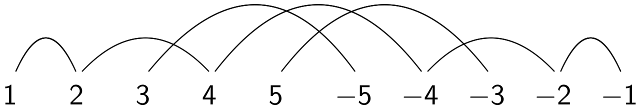
$$B \in \mathcal{B} \Leftrightarrow -B \in \mathcal{B},$$

and which is non-nesting in the **nesting order**

$$1 < 2 < \ldots < n < -n < \ldots < -2 < -1.$$

Example

$\mathcal{B} = \big\{\{1, 2, 4, -4, -2, -1\}, \{3, -5\}, \{5, -3\}\big\} \vdash [\pm 5]$ is non-nesting of type C:



Openers and **closers** for non-nesting partitions of type C

Observation

Let $O, C \subseteq [n]$. Then there exists a **unique** non-nesting set partition $\mathcal{B}$ of type C on $[\pm n]$ with $\mathrm{op}(\mathcal{B}) \cap [n] = O$ and $\mathrm{cl}(\mathcal{B}) \cap [n] = C$ if and only if for all i,

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Idea

1. reflect the "positive part" to the "negative part" and
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$$\begin{aligned} \mathrm{op}(\mathcal{B}) \cap [n] := \{1,2,3,4,5\} &\subseteq [5], \\ \mathrm{cl}(\mathcal{B}) \cap [n] := \{2,4\} &\subseteq [5]. \end{aligned}$$

Then we get

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1$

Openers and **closers** for non-nesting partitions of type C

Example

Let

$$\mathrm{op}(\mathcal{B}) := \{1,2,3,4,5,-4,-2\} \quad \subseteq \quad [\pm 5],$$
$$\mathrm{cl}(\mathcal{B}) := \{2,4,-5,-4,-3,-3,-1\} \quad \subseteq \quad [\pm 5].$$

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Openers and **closers** for non-nesting partitions of type C

Example

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$$\mathrm{op}(\mathcal{B}) := \{1, 2, 3, 4, 5, -4, -2\} \quad \subseteq \quad [\pm 5],$$
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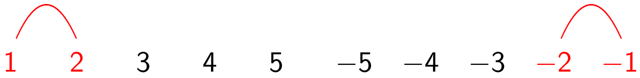
Then we get



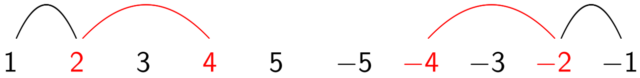
Openers and **closers** for non-nesting partitions of type C

Example

Let

$$\begin{aligned} \mathrm{op}(\mathcal{B}) &:= \{1, \color{red}{2}\color{black}, 3, 4, 5, \color{red}{-4}\color{black}, -2\} &\subseteq& \quad [\pm 5], \\ \mathrm{cl}(\mathcal{B}) &:= \{2, \color{red}{4}\color{black}, -5, -4, -3, \color{red}{-2}\color{black}, -1\} &\subseteq& \quad [\pm 5]. \end{aligned}$$

Then we get



Openers and **closers** for non-nesting partitions of type C

Example

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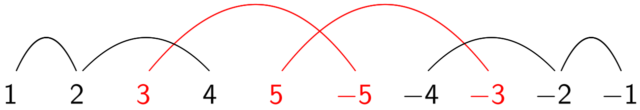

Openers and **closers** for non-nesting partitions of type C

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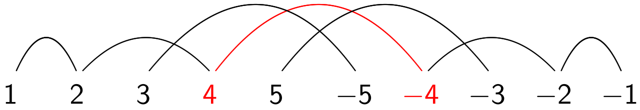

Openers and **closers** for non-nesting partitions of type C

Example

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$$\mathrm{op}(\mathcal{B}) := \{1,2,3,4,5,-4,-2\} \subseteq [\pm 5],$$
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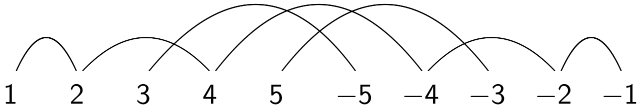


Overview

Non-nesting set partitions of type B

A **non-nesting set partition of type** B is a set partition $\mathcal{B}$ on $[\pm n] \cup \{0\} := \{1, 2, \ldots, n, 0, -n, \ldots -2, -1\}$ such that

$$B \in \mathcal{B} \Leftrightarrow -B \in \mathcal{B},$$

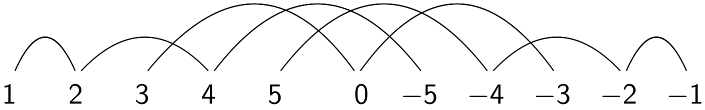
$$B = -B \Leftrightarrow 0 \in B$$

and which is non-nesting in the **nesting order**

$$1 < 2 < \ldots < n < 0 < -n < \ldots < -2 < -1.$$

Example

$\mathcal{B} = \big\{\{1, 2, 4, -5\}, \{3, 0, -3\}, \{5, -4, -2, -1\}\big\} \vdash [\pm 5] \cup \{0\}$ is non-nesting of type B:



Openers and **closers** for non-nesting partitions of type B

Observation

Let $O, C \subseteq [n]$. Then there exists a **unique** non-nesting set partition $\mathcal{B}$ of type B on $[\pm n] \cup \{0\}$ with $\mathrm{op}(\mathcal{B}) \cap [n] = O$ and $\mathrm{cl}(\mathcal{B}) \cap [n] = C$ if and only if for all i,

$$\big|O \cap \{s_1, \ldots, s_i\}\big| \geq \big|C \cap \{s_1, \ldots, s_i\}\big|.$$

Idea

1. reflect the “positive part” to the “negative part”
2. if $|O| - |C|$ is odd, insert 0 to the set of openers and closers and
3. complete the set partition.

Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\begin{aligned} O \cap [n] := \{1,2,3,4,5\} &\subseteq [5], \\ \mathrm{cl}(\mathcal{B}) \cap [n] := \{2,4\} &\subseteq [5]. \end{aligned}$$

Then we get

1 2 3 4 5 0 −5 −4 −3 −2 −1

Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\begin{aligned} \mathrm{op}(\mathcal{B}) &:= \{1,2,3,4,5,-4,-2\} \quad \subseteq \quad [\pm 5], \\ \mathrm{cl}(\mathcal{B}) &:= \{2,4,-5,-4,-3,-3,-1\} \quad \subseteq \quad [\pm 5]. \end{aligned}$$

Then we get

$1 \quad 2 \quad 3 \quad 4 \quad 5 \qquad 0 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1$

Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\begin{aligned} \mathrm{op}(\mathcal{B}) &:= \{1,2,3,4,5,0,-4,-2\} &\subseteq& \quad [\pm 5], \\ \mathrm{cl}(\mathcal{B}) &:= \{2,4,0,-5,-4,-3,-3,-1\} &\subseteq& \quad [\pm 5]. \end{aligned}$$

Then we get

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Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\mathrm{op}(\mathcal{B}) := \{1, 2, 3, 4, 5, 0, -4, -2\} \quad \subseteq \quad [\pm 5],$$
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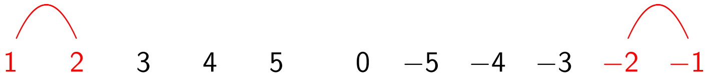
Then we get



Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\begin{aligned} \mathrm{op}(\mathcal{B}) &:= \{1, 2, 3, 4, 5, 0, -4, -2\} \quad \subseteq \quad [\pm 5], \\ \mathrm{cl}(\mathcal{B}) &:= \{2, 4, 0, -5, -4, -3, -2, -1\} \quad \subseteq \quad [\pm 5]. \end{aligned}$$

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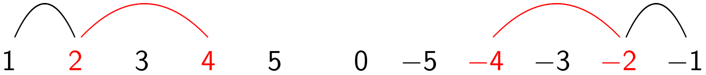

Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\text{op}(\mathcal{B}) := \{1, 2, 3, 4, 5, 0, -4, -2\} \quad \subseteq \quad [\pm 5],$$
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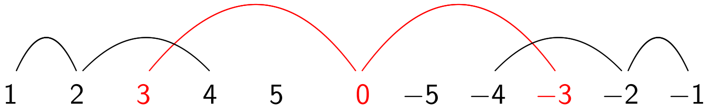
Then we get



Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\mathrm{op}(\mathcal{B}) := \{1, 2, 3, 4, 5, 0, -4, -2\} \quad \subseteq \quad [\pm 5],$$
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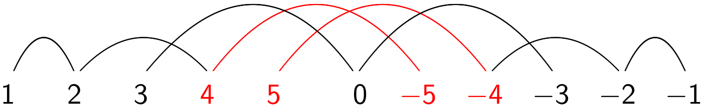

Openers and **closers** for non-nesting partitions of type B

Example

Let

$$\mathrm{op}(\mathcal{B}) := \{1,2,3,4,5,0,-4,-2\} \quad \subseteq \quad [\pm 5],$$
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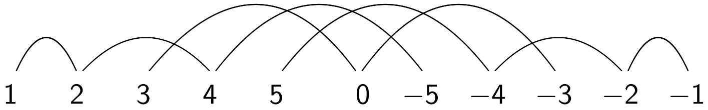


Bijections preserving **openers** and **closers** in types B, C

Theorem

The presented bijection between

- *non-crossing set partitions in types B and C,*
- *non-nesting set partitions in type B and*
- *non-nesting set partitions in type C*

is the **unique** *bijection preserving openers and closers on $[n]$.*

Corollary

- *The presented bijection preserves openers and closers on $[n]$,*
- *the bijection by R. Mamede, we will be introduced to in a second, preserves openers and closers on $[n]$.*

$\Rightarrow$ *Both bijections coincide.*

Overview

Non-crossing and non-nesting set partitions

Non-crossing set partitions of types B and C

Non-nesting set partitions of type C

Non-nesting set partitions of type B

A counterexample in type D

Generalizations

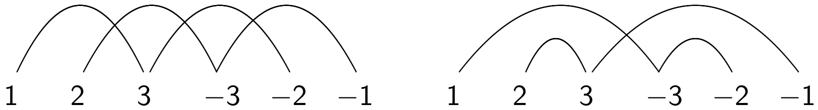
A counterexample for non-nesting partitions of type D

Remark

The previous observation is false in type D: the anti-chains

$$\{e_1 - e_3, e_2 + e_3\} \quad , \quad \{e_2 - e_3, e_1 + e_3\}$$

belong to the non-nesting set partitions



which have the same sets of openers and closers,

$$\mathrm{op}(\mathcal{B}) \cap [3] = \{1, 2, 3\} \quad , \quad \mathrm{cl}(\mathcal{B}) \cap [3] = \{3\}.$$

Overview

Generalizations and future work

We have seen in the classical case that the bijection preserving openers and closers have generalizations in two different directions:

- k-crossings – k-nestings (C. Krattenthaler),
- # of crossings – # of nestings (A. Kasraoui, J. Zeng).

- the k-crossing – k-nesting generalization was done in type C by M. Rubey using **growth diagrams**,

Current work:

- k-crossing – k-nesting generalization in type B (joint work with M. Rubey),
- # of crossing – # of nestings generalization in types B and C (joint work with M. Rubey).

Remark

Here, the definitions of **crossings** and **nestings** are different!

Thank you very much!