A bijection between noncrossing and nonnesting partitions of types A and B

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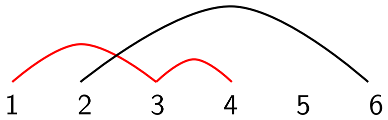
Noncrossing and nonnesting set partitions

A **set partition** of $[n] = \{1, \ldots, n\}$ is a collection of disjoint nonempty subsets of $[n]$, called blocks, whose union is $[n]$.

$\pi = \{\{1, 3, 4\}, \{2, 6\}, \{5\}\}$ is a partition of $[6]$ of type $(3, 2, 1)$



$op(\pi) = \{1, 2, 5\}, \quad cl(\pi) = \{4, 5, 6\}, \quad tr(\pi) = \{3\}$

$m(\pi) = (op(\pi), cl(\pi), tr(\pi))$

A **complete matching** of $[2n]$ is a set partition of $[2n]$ of type $(2, \ldots, 2)$

A **partial matching** of $[n]$ is a set partition of $[n]$ of type $(2, \ldots, 2, 1, \ldots, 1)$

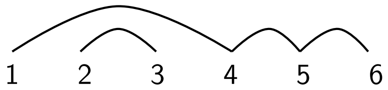
The triple $m(\pi) = (op(\pi), cl(\pi), tr(\pi))$ encodes some useful information about the set partition π:

- The number of blocks is $|op(\pi)| = |cl(\pi)|$;
- The number of singleton blocks is $|op(\pi) \cap cl(\pi)|$;
- π is a partial matching if and only if $tr(\pi) = \emptyset$;
- π is a complete matching if and only if $tr(\pi) = \emptyset$ and $op(\pi) \cap cl(\pi) = \emptyset$.

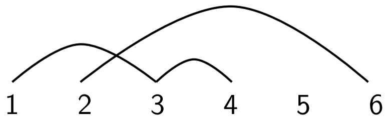
Noncrossing set partitions

A set partition π of $[n]$ is said **noncrossing** if whenever $a < b < c < d$ are such that a, c are contained in a block B and b, d are contained in a block B' of π, then $B = B'$.

The set partition $\{\{1, 4, 5, 6\}, \{2, 3\}\}$ is noncrossing:



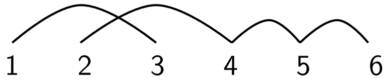
while the set partition $\{\{1, 3, 4\}, \{2, 6\}, \{5\}\}$ is not:



Nonnesting set partitions

A set partition π of $[n]$ is said **nonnesting** if whenever $a < b < c < d$ are such that a, d are contained in a block B and b, c are contained in a block B' of π, then $B = B'$.

The set partition $\{\{1,3\},\{2,4,5,6\}\}$ is nonnesting:



while the set partition $\{\{1,4,5,6\},\{2,3\}\}$ is not:

Absolute order

Let (W, S) be a finite Coxeter system with set of reflections T
Given $w \in W$, the **absolute length** $\ell_T(w)$ of w is the minimal integer k for which w can be written as the product of k reflections:

$$\ell_T(w) = \min\{k : w = t_1 \cdots t_k, \text{ for some } t_i \in T\}.$$

Definition

Define the **absolute order** on W by letting

$$v \leq_T w \text{ if and only if } \ell_T(w) = \ell_T(v) + \ell_T(v^{-1}w)$$

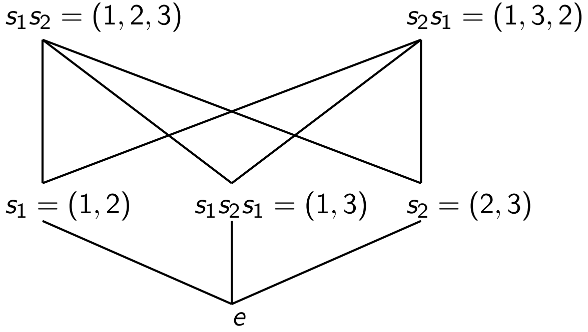
for all $v, w \in W$.

Proposition

Given $w, v \in W$, $v \leq_T w$ if and only if there is a shortest factorization of w as a product of reflections having as a prefix such a shortest factorization for v.

$W = S_3$, $S = \{s_1 = (1,2),\ s_2 = (2,3)\}$

$T = \{s_1, s_2, s_1 s_2 s_1 = (1,3)\}$



(W, S) finite Coxeter system, with $S = \{s_1, \ldots, s_n\}$

A **Coxeter element** of W is any element of the form

$$c = s_{\sigma(1)} \cdots s_{\sigma(n)},$$

for some permutation σ of the set $[n]$.

Proposition
(a) Any two Coxeter elements of W are conjugate.
(b) The Coxeter elements are a subclass of maximal elements in W.
(c) If c, c' are Coxeter elements, then $[e, c] \cong [e, c']$.

Noncrossing partitions

Definition

Let W be a finite reflection group and $c \in W$ a Coxeter element. The poset of noncrossing partitions of W is the interval

$$NC(W) := [e, c] = \{w \in W : e \leq_T w \leq_T c\}.$$

Theorem (Reiner, Bessis-Reiner)

Let W be a finite reflection group. Then,

$$|NC(W)| = Cat(W) := \prod_{i=1}^{n} \frac{d_i + h}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (d_i + h),$$

where

(i) n is the number of simple reflections in W,
(ii) h is the Coxeter number, and
(iii) $d_1, \ldots, d_n$ are the degrees of the fundamental invariants.

$Cat(W)$ for the finite irreducible Coxeter groups

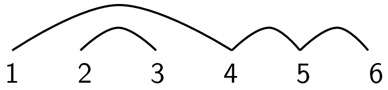
A_{n-1}	B_n	D_n	$I_2(m)$	H_3	H_4	F_4	E_6	E_7	E_8
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	$m+2$	32	280	105	833	4160	25080

Noncrossing partitions of type A_{n-1}

$c = (1, 2, \ldots, n)$ Coxeter element

$\pi \leq_T c$ iff all cycles in π are increasing and pairwise noncrossing

$$NC(A_{6-1}) \ni \pi = (1456)(23) \longleftrightarrow \pi = \{\{1,4,5,6\},\{2,3\}\} \in NC([6])$$



Noncrossing partitions of type B_n

B_n group of sign permutations π of $[\pm n] = \{\overline{1}, \overline{2}, \dots, \overline{n}, 1, 2, \dots, n\}$ such that $\pi(\overline{i}) = \overline{\pi(i)}$

$\pi = (\overline{5},\ 1,\ 2)(5,\ \overline{1},\ \overline{2})(3,4)(\overline{3},\overline{4}) \in B_5$ of type $(3,2)$

$m(\pi) = (op(\pi) = \{3\}, cl(\pi) = \{2,4,5\}, tr(\pi) = \{1\})$

$$\begin{aligned} B_n &\hookrightarrow A_{2n-1} \\ i &\mapsto i, \qquad \text{if } i \in [n] \\ i &\mapsto n - i, \ \text{if } i \in [\overline{1}, \dots, \overline{n}] \end{aligned}$$

$NC(B_n)$ is the subset of $NC([\pm n]) = NC([2n])$ consisting of all partitions that are invariant under the map $i \mapsto \overline{i}$

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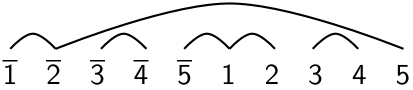


The root poset

Let W be a Weyl group with crystallographic root system Φ, and $\Delta \subseteq \Phi^+$ a set of simple roots

Definition

• For $\alpha, \beta \in \Phi^+$, we say that $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{Z}_{\geq 0}\Delta$. The pair $(\Phi^+, \leq)$ is called the root poset of W.

• An antichain in the root poset $(\Phi^+, \leq)$ is called a nonnesting partition of W. Let $NN(W)$ denote the set of nonnesting partitions of W.

Theorem

Let W be a Weyl group. Then,

$$|NC(W)| = |NN(W)| = Cat(W).$$

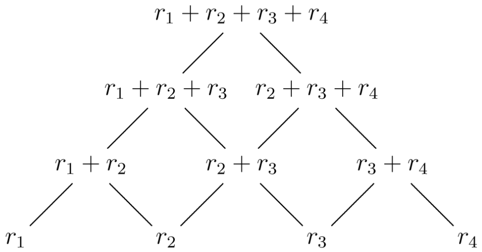
Nonnesting partitions of type A_{n-1}

$\{e_1, \ldots, e_n\}$ canonical basis of $\mathbb{R}^n$

$\Phi = \{e_i - e_j : n \geq i \neq j \geq 1\}, \qquad \Phi^+ = \{e_i - e_j : n \geq i > j \geq 1\}$

$\Delta = \{r_1 = e_2 - e_1, r_2 = e_3 - e_2, \ldots, r_{n-1} = e_n - e_{n-1}\}$

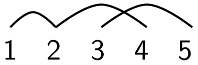
If $i > j$, then $e_i - e_j = r_j + \cdots + r_{i-1} \leftrightarrow (i, j) \in S_n$



Lemma

Let $\alpha = r_i + \cdots + r_j$ and $\beta = r_k + \cdots + r_\ell$ be two roots in Φ^+. Then, $\{\alpha, \beta\}$ is an antichain if and only if $i < k$ and $j < \ell$.

$$NN(A_4) \ni (r_1, r_2{+}r_3, r_3{+}r_4) \leftrightarrow (1,2)(2,4)(3,5) = (1,2,4)(3,5) \in NN([5])$$



- $supp(r_i + \cdots + r_j) = \{r_i, \ldots, r_j\}$

- An antichain $(\alpha_1, \ldots, \alpha_k)$ is connected if $supp(\alpha_i) \cap supp(\alpha_{i+1}) \neq \emptyset$ for $i = 1, \ldots, k-1$

- The connected components of an antichain π are the connected sub-antichains of π for which the supports of the union of the roots in any two distinct components are disjoint.

Nonnesting partitions of type B_n

$$\Phi = \{\pm e_i, 1 \le i \le n\} \cup \{\pm e_i \pm e_j : 1 \le i \ne j \le n\}$$

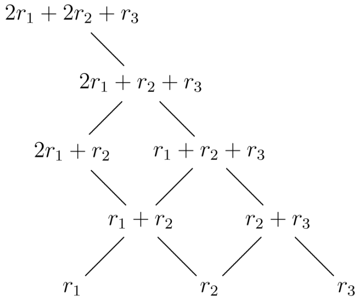
$$\Phi^+ = \{e_i : 1 \le i \le n\} \cup \{e_i \pm e_j : 1 \le j < i \le n\}$$

$$\Delta = \{r_1 = e_1, r_2 = e_2 - e_1, \ldots, r_n = e_n - e_{n-1}\}$$

$$e_i = \sum_{k=1}^{i} r_k \leftrightarrow (i, \bar{i})$$

$$e_i - e_j = \sum_{k=j+1}^{i} r_k \leftrightarrow (i, j)(\bar{i}, \bar{j})$$

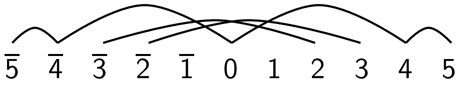
$$e_i + e_j = 2\sum_{k=1}^{j} r_k + \sum_{k=j+1}^{i} r_k \leftrightarrow (i, \bar{j})(\bar{i}, j)$$



Lemma

- $\{r_i + \cdots + r_j, r_k + \cdots + r_\ell\}$ is an antichain iff $i < k$ and $j < \ell$
- $\{2r_1 + \cdots + 2r_i + r_{i+1} + \cdots + r_j, r_k + \cdots + r_\ell\}$ is an antichain iff $1 < k$ and $j < \ell$
- $\{2r_1 + \cdots + 2r_i + r_{i+1} + \cdots + r_j, 2r_1 + \cdots + 2r_k + r_{k+1} + \cdots + r_\ell\}$ is an antichain iff $k < i$ and $j < \ell$

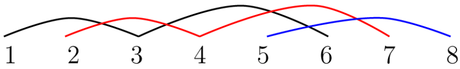
$$(2r_1 + 2r_2 + r_3, r_1 + r_2 + r_3 + r_4, r_5) \in NN(B_5)$$
$$\updownarrow$$
$$(2, \overline{3})(\overline{2}, 3)(\overline{5}, \overline{4}, 4, 5) \in NN([\pm n])$$



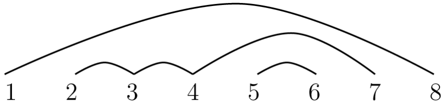
- $supp(2r_1 + 2r_2 + r_3) = \{r_1, r_2, r_3\}$, $supp(r_1 + r_2 + r_3 + r_4) = \{r_1, r_2, r_3, r_4\}$, $supp(r_5) = \{r_5\}$
- Connected components: $(2r_1 + 2r_2 + r_3, r_1 + r_2 + r_3 + r_4)$ and (r_5)

The bijection $f : NN(W) \to NC(W)$

$$\pi = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7)$$
$$= (1,3,6)(2,4,7)(5,8) \in NN(A_7)$$



$$\begin{aligned} f(\pi) &= (r_1 + \cdots + r_7) f(r_2, r_3, r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 f(r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 (r_4 + r_5 + r_6) r_5 \\ &= (1,8)(2,3,4,7)(5,6) \in NC(A_7), \qquad m(\pi) = m(f(\pi)) \end{aligned}$$



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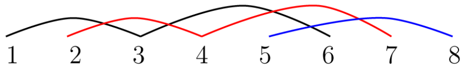

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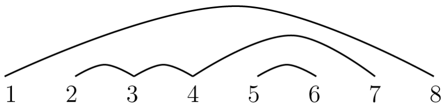
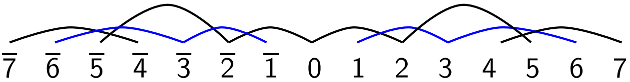


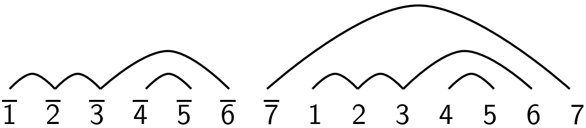
$F_{st} = (1 < 2 < 3 < 4 < 5), \qquad L_{st} = (2 < 3 < 5 < 6 < 7)$

$$\pi = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7)$$
$$= (1,3,6)(\overline{1},\overline{3},\overline{6})(4,7)(\overline{4},\overline{7})(\overline{5},\overline{2},2,5) \in NN(B_7)$$

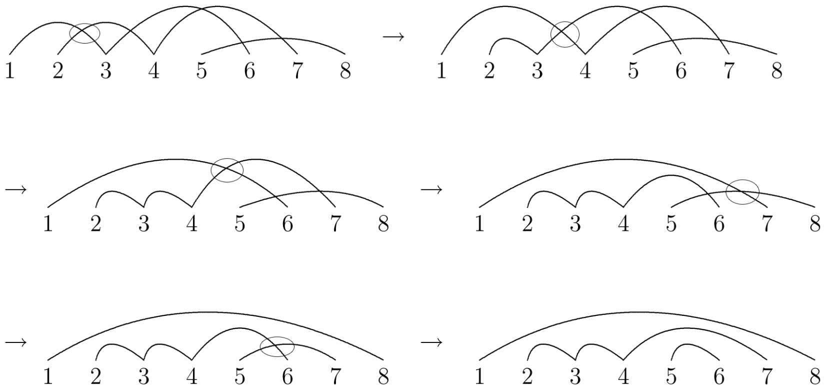


$$\begin{aligned} f(\pi) &= (r_1 + \cdots + r_7) f(r_2, r_3, r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 f(r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 (r_4 + r_5 + r_6) r_5 \\ &= (7,\overline{7})(1,2)(\overline{1},\overline{2})(2,3)(\overline{2},\overline{3})(3,6)(\overline{3},\overline{6})(4,5)(\overline{4},\overline{5}) \\ &= (7,\overline{7})(1,2,3,6)(\overline{1},\overline{2},\overline{3},\overline{6})(4,5)(\overline{4},\overline{5}) \in NC(B_7) \end{aligned}$$

$$m(\pi) = m(f(\pi)) = (\{1,4\},\{5,6,7\},\{2,3\})$$



$\pi = (1,3,6)(2,4,7)(5,8)$



$f(\pi) = (1,8)(2,3,4,7)(5,6)$

The bijection $f : NN(W) \to NC(W)$

$$\pi = (2r_1 + 2r_2 + 2r_3 + r_4,\ 2r_1 + 2r_2 + r_3 + r_4 + r_5,\ \ r_3 + r_4 + r_5 + r_6,\ \ r_4 + r_5 + r_6 + r_7,\ \ r_6 + r_7 + r_8)$$

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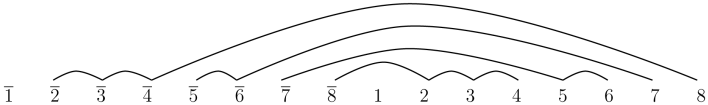
$$f(\pi) : (2r_1 + 2r_2 + r_3 + \cdots + r_8,\ 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7,\ r_3,\ f(r_4, r_6))$$

$$\to (2r_1 + 2r_2 + r_3 + \cdots + r_8,\ 2r_1 + 2r_2 + 2r_3 + 2r_4 + 2r_5 + r_6 + r_7,\ r_3,\ r_4,\ r_6)$$

$$\to (2,\overline{8})(\overline{2},8)(5,\overline{7})(\overline{5},7)(2,3)(\overline{2},\overline{3})(3,4)(\overline{3},\overline{4})(5,6)(\overline{5},\overline{6})$$

$$\to (2,3,4,\overline{8})(\overline{2},\overline{3},\overline{4},8)(5,6,\overline{7})(\overline{5},\overline{6},7) = f(\pi)$$



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$$\to (2r_1 + 2r_2 + r_3 + \cdots + r_8,\ 2r_1 + 2r_2 + \underbrace{2r_3 + 2r_4 + 2r_5 + r_6 + r_7,\ r_3}, r_4, r_6)$$

$D = (3 > 2)$, $F_{st} = (3 < 4 < 6)$

$L_{st} = (4 < 5 < 6 < 7 < 8)$

$$f(\pi) = (2,3,4,\overline{8})(\overline{2},\overline{3},\overline{4},8)(5,6,\overline{7})(\overline{5},\overline{6},7)$$

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Then

$$\pi = (2r_1 + 2r_2 + 2r_3 + r_4,\ 2r_1 + 2r_2 + r_3 + r_4 + r_5,\ r_3 + r_4 + r_5 + r_6,\ r_4 + r_5 + r_6 + r_7,\ r_6 + r_7 + r_8)$$

with $m(\pi) = m(f(\pi)) = (\{1\}, \{1,4,6,7,8\}, \{2,3,5\})$

Theorem

The map f is a bijection between the sets $NN(\Psi)$ and $NC(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$, that preserves the number of blocks and the triples $(op(\pi), cl(\pi), tr(\pi))$.

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Corollary

The map f establishes a bijection between nonnesting matching partitions of $[2n]$ and noncrossing matching partitions of $[2n]$.