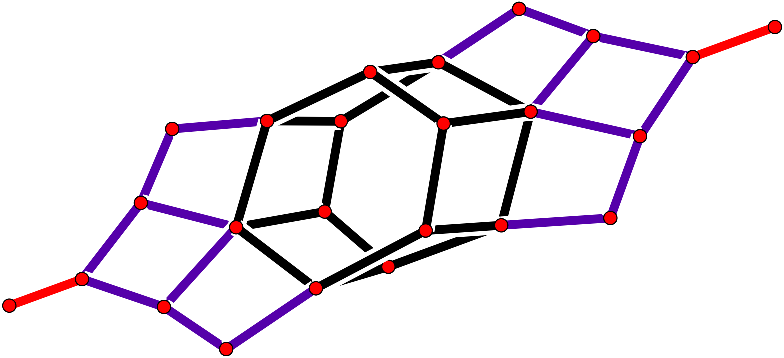
Splitting Polytopes

Sven Herrmann (joint work with Michael Joswig)

62ème Séminaire Lotharingien de Combinatoire

TECHNISCHE UNIVERSITÄT DARMSTADT



Subdivisions and Splits of Convex Polytopes

Regular Subdivisions and Secondary Polytopes

Properties of Splits

Application: Tropical Geometry

Generalizations/Outlook

Subdivisions

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Definition

A subdivision of P is a collection Σ of polytopes (faces) such that

- $\bigcup_{F \in \Sigma} = P$,
- $F \in \Sigma \implies$ all faces of F are in Σ,
- $F_1, F_2 \in \Sigma \implies F_1 \cap F_2$ is a face of both,
- F 0-dimensional $\implies F$ is a vertex of P.

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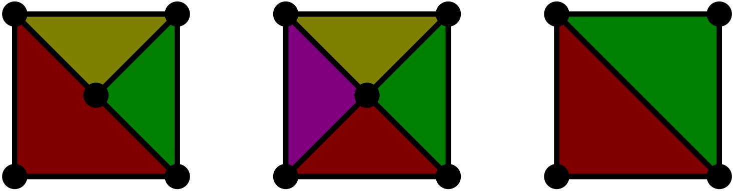

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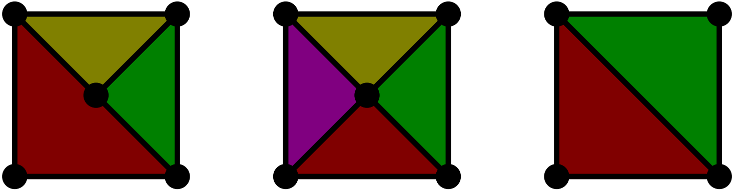

Refinements of Subdivisions

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Definition

- Σ' is a refinement of Σ if each face of Σ' is contained in a face of Σ.
- The common refinement of two subdivisions Σ, Σ' of P is the subdivision

$$\{S \cap S' \mid S \in \Sigma, S' \in \Sigma'\}.$$

- The refinement defines a partial order on the set of all subdivisions of P.
- A finest subdivision (minimal element) is a triangulation.

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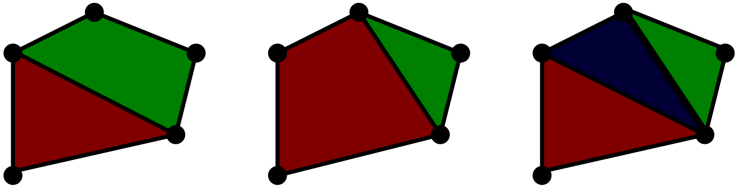

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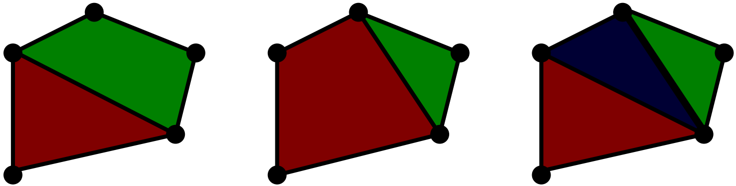

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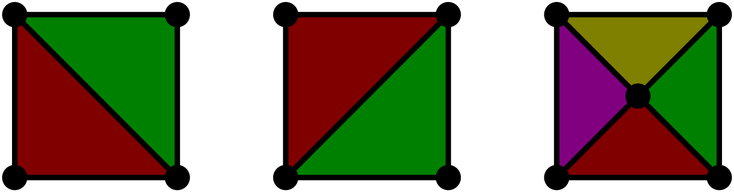

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Definition

A split S of a polytope P is a subdivision of P with exactly two maximal faces.

- A splits S is defined by a hyperplane H_S.
- A hyperplane H (that meets the interior of P) defines a split if and only if H does not cut any edge of P.
- $\Longrightarrow$ The splits of P only depend on the combinatorics (oriented matroid) of P, not on the realization.
- Example: v a vertex of P such that all neighbors of v lie in a common Hyperplane H_v: vertex split for v.

Splits of Convex Polytopes

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Example: Hypersimplices

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- $\Delta(k,n) \;:=\; \operatorname{conv}\left\{ \sum_{i\in I} e_i \;\middle|\; I \in \binom{\{1,\dots,n\}}{k} \right\} \subset \mathbb{R}^n,$
- n-dimensional unit cube cut with the hyperplane $\sum_i x_i = k$,
- For a partition (A,B) of $\{1,\dots,n\}$ define the $(A,B;\mu)$-hyperplane by

$$\sum_{i\in A} x_i \;=\; \mu\,.$$

Satz (Joswig, H. 08)

The splits of $\Delta(k,n)$ correspond to the $(A,B;\mu)$-hyperplanes with $k-\mu+1 \le |A| \le n-\mu-1$ and $1 \le \mu \le k-1$.

Theorem (Josiwg, H. 08)

The number of splits of $\Delta(k,n)$ equals $(k-1)\left(2^n-(n-1)\right)-\sum_{i=2}^{k-1}(k-i)\binom{n}{i}$.

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Subdivisions and Splits of Convex Polytopes

Regular Subdivisions and Secondary Polytopes

Properties of Splits

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Generalizations/Outlook

Regular Subdivisions

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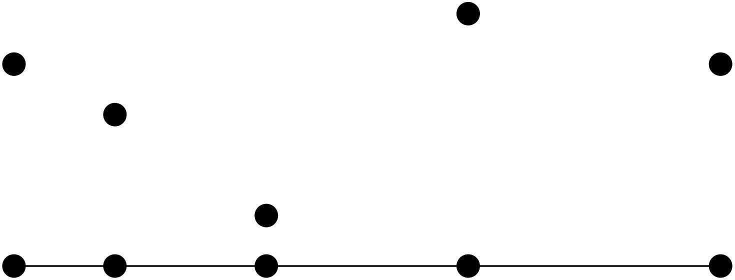
- $w : \text{vert}\, P \to \mathbb{R}$ weight function,
- consider $\text{conv}\{(v, w(v)) \mid v \in \text{vert}\, P\}$,
- project the lower convex hull down to P,
- the resulting subdivision $\Sigma_w(P)$ is called regular.



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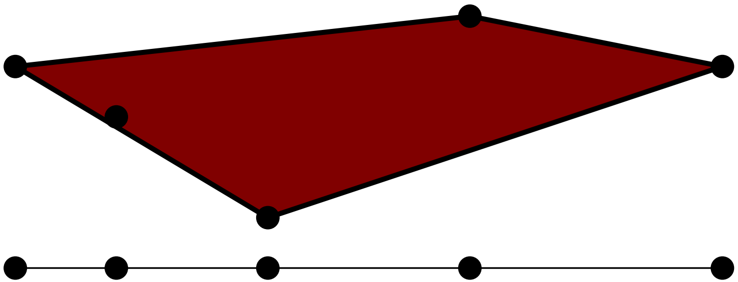

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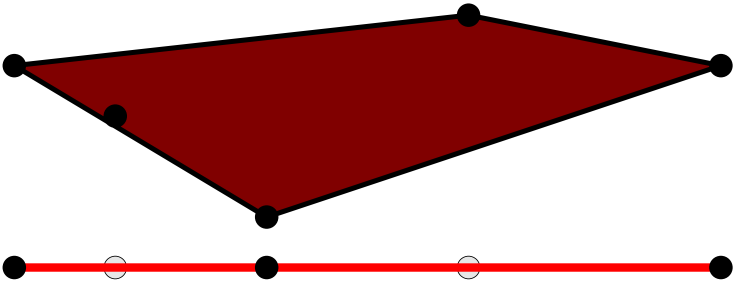

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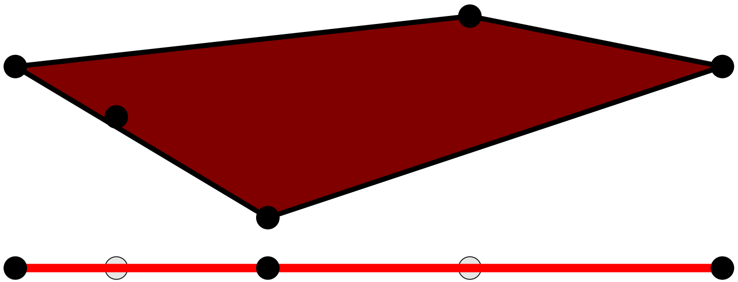

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Lemma

Splits are regular.

The Secondary Polytope

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- P d-dimensional polytope in $\mathbb{R}^d$ with n vertices $v_1, \ldots, v_n$,

Theorem (Gel′fand, Kapranov, Zelevinsky 90)

There exists an $(n - d - 1)$-dimensional polytope $\mathrm{SecPoly}(P)$ (secondary polytope of P) whose face lattice is isomorphic to the poset of all regular subdivisions of P.

- Vertices of $\mathrm{SecPoly}(P)$ correspond to triangulations Σ: $x_i^\Sigma = \sum_{v_i \in S \in \Sigma} \mathrm{vol}(S)$.
- Facets of $\mathrm{SecPoly}(P)$ correspond to coarsest regular subdivisions.
- The intersection of two faces corresponds to the common refinement of the subdivisions corresponding to the faces.

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Splits and Secondary Polytopes

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- Splits are facets of $\mathsf{SecPoly}(P)$, they define an approximation $\mathsf{SplitPoly}(P) \supset \mathsf{SecPoly}(P)$.
- This is a common approximation for all polytopes with the same oriented matroid.

Theorem (Joswig, H. 09)

$\mathsf{SecPoly}(P) = \mathsf{SplitPoly}(P)$ *if and only if P is a simplex, polygon, cross polytope, prism over a simplex, or a (possible multiple) join of these polytopes.*

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(Weakly) Compatible Split Systems

Definition

Let $\mathcal{S}$ be a set of splits (split system) of a polytope P.

- We call $\mathcal{S}$ weakly compatible if the subdivisions $S \in \mathcal{S}$ have a common refinement (without new vertices).
- We call $\mathcal{S}$ compatible if none of the split defining hyperplanes meet in the interior of P.

- Example: Vertex splits are (weakly) compatible if and only if the corresponding vertices are not connected by and edge.
- Stable set of the edge graph of a polytope yields a compatible split system.

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The Split Complex

Definition

The split complex of a polytope P is the simplicial complex

$$\mathsf{Split}(P) := \{\mathcal{S} \mid \mathcal{S} \text{ set of compatible splits}\} .$$

- The weak split complex is defined in the same way (but is in general not simplicial).
- These complexes can be seen as (kind of) subcomplexes of the dual complex of the secondary polytope of P.

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(Weakly) Compatible Split Systems

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Satz (Joswig, H. 08)

- *The dual graph of a compatible split system is a tree.*
- *The dual graph of a weakly compatible split system is bipartite.*

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Compatibility for Hypersimplices

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Satz (Joswig, H. 08)

Two splits $(A, B; \mu)$ and $(C, D; \nu)$ of $\Delta(k, n)$ are compatible if and only if one of the following holds:

$$|A \cap C| \leq k - \mu - \nu\,, \qquad |A \cap D| \leq \nu - \mu\,,$$
$$|B \cap C| \leq \mu - \nu\,, \qquad \text{or } |B \cap D| \leq \mu + \nu - k\,.$$

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Split Decomposition

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- A decomposition $w + w'$ of weight functions is called coherent if $\Sigma_w(P)$ and $\Sigma_{w'}(P)$ have a common refinement ($\Sigma_{w+w'}(P)$).
- A weight function w is called split prime if $\Sigma_w(P)$ does not refine any split.

Theorem (Bandelt, Dress 92; Hirai 06; Joswig, H. 08)

Each weight function w for a polytope P has a coherent decomposition

$$w = w_0 + \sum_{S \in \mathcal{S}} \alpha^w_{w_S} w_S ,$$

where $\mathcal{S}$ is some weakly compatible set of splits and w_0 is split prime. This decomposition is unique.

Split Decomposition

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- A decomposition $w + w'$ of weight functions is called coherent if $\Sigma_w(P)$ and $\Sigma_{w'}(P)$ have a common refinement ($\Sigma_{w+w'}(P)$).
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The Second Hypersimplex and Metric Spaces

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- $\Delta(2, n) \;=\; \operatorname{conv}\{e_i + e_j \;|\, 1 \leq i < j \leq n\}$.
- Lifting functions of $\Delta(2, n)$ correspond to (pseudo-)metrics on n points.
- Splits of $\Delta(2, n)$ are in bijection with partitions (A, B) of $\{1, \ldots, n\}$ where each part has at least two elements.
- Originally, these were the splits of finite metric spaces defined by Bandelt and Dress (92) for applications in biology.
- Two splits (A, B) and (C, D) of $\Delta(2, n)$ are compatible if and only if one of the four sets $A \cap C$, $A \cap D$, $B \cap C$, and $B \cap D$ is empty.
- This is the original definition of compatibility for splits of finite metric spaces.
- There is also a combinatorial condition for weak compatibility.

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TECHNISCHE UNIVERSITÄT DARMSTADT

Definition

- A subdivision Σ of $\Delta(k,n)$ is called a matroid subdivision if all edges of Σ are edges of $\Delta(k,n)$.
- (Equivalently: Each face of Σ is a matroid polytope $P_{\mathcal{M}}$, i.e. each vertex of $P_{\mathcal{M}}$ corresponds to a basis of $\mathcal{M}$.)
- The Dressian is the polyhedral complex

$$\mathrm{Dr}(k,n) := \left\{ w \in \mathbb{R}^{\binom{n}{k}} \,\middle|\, \Sigma_w(\Delta(k,n)) \text{ is a matroid subdivision} \right\} \cap \mathbb{S}^{\binom{n}{k}-1}.$$

- Elements of $\mathrm{Dr}(k,n)$ are the tropical Plücker vectors (Speyer 08).
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Hypersimplices, Dressians, and Tropical Grassmannians

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The Split Complex and the Dressian

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$k = 2$:

- Lifting functions of $\Delta(2, n)$ correspond to (pseudo-)metrics on n points.
- $Gr(2, n) = \mathrm{Dr}(2, n) \cong \mathrm{Split}(\Delta(2, n))$ is the space of metric trees (Bunemann 74; Billera, Holmes & Vogtmann 01).

Theorem (Joswig, H. 08)

$\mathrm{Split}(\Delta(k, n))$ *is a subcomplex of* $\mathrm{Dr}(k, n)$.

- Proof idea:
- Splits are matroid subdivisions.
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Subdivisions and Splits of Convex Polytopes

Regular Subdivisions and Secondary Polytopes

Properties of Splits

Application: Tropical Geometry

Generalizations/Outlook

Further Coarsest Subdivisions

- Split: one combinatorial type, dual graph is an edge.
- Three maximal faces: one combinatorial type, regular.
- Four maximal faces: three combinatorial types.
- More than four: gets complicated...

Definition

A subdivision of Σ of P is called *k-split*, if the tight span of Σ is a $(k-1)$-dimensional simplex.

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k-splits are regular.

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Thanks for your attention!

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