Equality Of Multiplicity Free Skew Characters

Christian Gutschwager

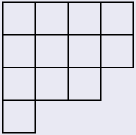
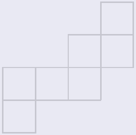
23.02.2009

Outline

Partitions

Diagram

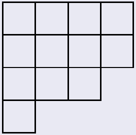
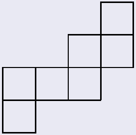
$$\lambda = (4^2, 3, 1) \qquad \mu = (3, 2)$$

$\lambda =$  $\Longrightarrow \lambda/\mu =$ 

Partitions

Skew-diagram

$$\lambda = (4^2, 3, 1) \qquad \mu = (3, 2)$$

$\lambda =$  $\Longrightarrow \lambda/\mu =$ 

Littlewood-Richardson Coefficients

- Semistandard (weakly increasing among rows from left to right, strictly increasing among collumns from top to bottom)
- Tableauword w is a lattice permutations.

Semistandard

semistandard:

			1	1	1
2	2	3	3		
4	4	4	4		

not semistandard:

			1	1	1
2	2	1	1		
2	2				

Littlewood-Richardson Coefficients

- Semistandard (weakly increasing among rows from left to right, strictly increasing among collumns from top to bottom)
- Tableauword w is a lattice permutation.

Lattice permutation

			1	1
3	3	2	2	

$w = (112233)$

lattice permutation

			1	1
1	2	2	2	

$w = (112221)$

no lattice permutation

Littlewood-Richardson Coefficients

- Semistandard (weakly increasing among rows from left to right, strictly increasing among collumns from top to bottom)
- Tableauword w is a lattice permutations.

Definition

LR-coeffizient $c(\lambda;\mu,\nu)$ *equals the number of tableaus of shape* λ/μ *with content* $\nu = (\nu_1, \nu_2, \nu_3, \dots)$ *satisfying the above conditions.*

Skew characters

Skew characters

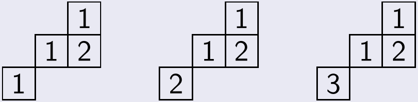
$$[\lambda/\mu] = \sum_{\nu} c(\lambda; \mu, \nu)[\nu]$$

Skew characters

Skew characters

$$[\lambda/\mu] = \sum_{\nu} c(\lambda; \mu, \nu)[\nu]$$

Example $\lambda = (3, 3, 1), \mu = (2, 1)$:



Skew characters

Skew characters

$$[\lambda/\mu] = \sum_{\nu} c(\lambda; \mu, \nu)[\nu]$$

Example $\lambda = (3, 3, 1), \mu = (2, 1)$:

		1
	1	2
1		

		1
	1	2
2		

		1
	1	2
3		

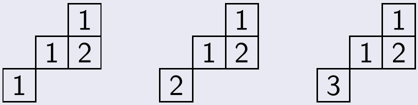
$$[(3, 3, 1)/(2, 1)] = [3, 1] + [2, 2] + [2, 1, 1]$$

Skew characters

Skew characters

$$[\lambda/\mu] = \sum_{\nu} c(\lambda;\mu,\nu)[\nu]$$

Example $\lambda = (3,3,1), \mu = (2,1)$:



$$[(3,3,1)/(2,1)] = [3,1] + [2,2] + [2,1,1]$$

Skew Schur functions

$$s_{\lambda/\mu} = \sum_{\nu} c(\lambda;\mu,\nu) s_{\nu}$$

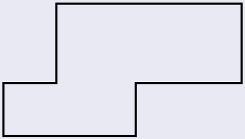
Multiplicity free skew characters $[\lambda/\mu]$

For fixed λ, μ all $c(\lambda; \mu, \nu) \in \{0, 1\}$.

Multiplicity free skew characters $[\lambda/\mu]$

Multiplicity free skew characters $[\lambda/\mu]$

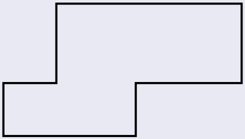
1.



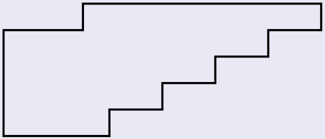
Multiplicity free skew characters $[\lambda/\mu]$

Multiplicity free skew characters $[\lambda/\mu]$

1.



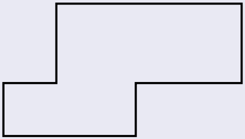
2. $s_{in} = 1$



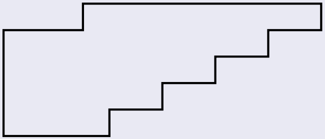
Multiplicity free skew characters $[\lambda/\mu]$

Multiplicity free skew characters $[\lambda/\mu]$

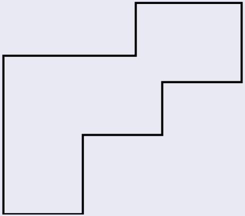
1.



2. $s_{in} = 1$



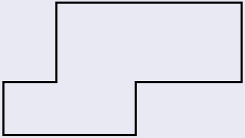
3. $s_{in} = 2$



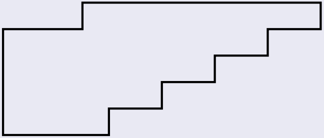
Multiplicity free skew characters $[\lambda/\mu]$

Multiplicity free skew characters $[\lambda/\mu]$

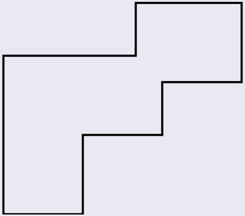
1.



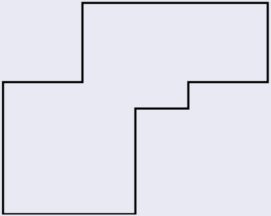
2. $s_{in} = 1$



3. $s_{in} = 2$



4. $s_{out} = 1$



Equality of skew characters

- Trivial: Translation of the skew diagram
- Trivial: Rotation of the skew diagram
- Nontrivial results by
  - Billera, Thomas, van Willigenburg
  - Reiner, Shaw, van Willigenburg
  - McNamara, van Willigenburg

Example

Staircase partition $\lambda = (l, l-1, l-2, \ldots, 2, 1)$, μ *arbitrary.*

$$[\lambda/\mu] = [(\lambda/\mu)^{\text{conjugate}}]$$

Equality of skew characters

- Trivial: Translation of the skew diagram
- Trivial: Rotation of the skew diagram
- Nontrivial results by
  - Billera, Thomas, van Willigenburg
  - Reiner, Shaw, van Willigenburg
  - McNamara, van Willigenburg

Example

Staircase partition $\lambda = (l, l-1, l-2, \ldots, 2, 1)$, μ *arbitrary.*

$$[\lambda/\mu] = [(\lambda/\mu)^{\text{conjugate}}]$$

Equality of skew characters

- Trivial: Translation of the skew diagram
- Trivial: Rotation of the skew diagram
- Nontrivial results by
  - Billera, Thomas, van Willigenburg
  - Reiner, Shaw, van Willigenburg
  - McNamara, van Willigenburg

Example

Staircase partition $\lambda = (l, l-1, l-2, \ldots, 2, 1)$, μ *arbitrary.*

$$[\lambda/\mu] = [(\lambda/\mu)^{\text{conjugate}}]$$

Equality of skew characters

- Trivial: Translation of the skew diagram
- Trivial: Rotation of the skew diagram
- Nontrivial results by
  - Billera, Thomas, van Willigenburg
  - Reiner, Shaw, van Willigenburg
  - McNamara, van Willigenburg

Example

Staircase partition $\lambda = (l, l-1, l-2, \ldots, 2, 1)$, μ *arbitrary.*

$$[\lambda/\mu] = [(\lambda/\mu)^{\text{conjugate}}]$$

Results

Theorem

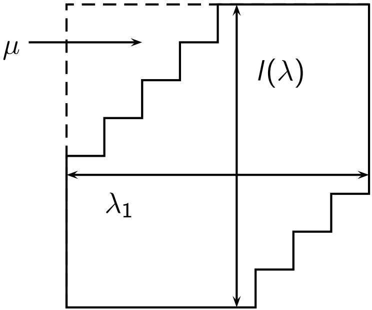
Let $[\lambda/\mu] = [\alpha/\beta]$ *be multiplicity free.*
Then up to rotation and/or translation

- $\lambda/\mu = \alpha/\beta$ *or*
- $\lambda = \alpha = (l, l-1, l-2, \ldots)$ *and* $\mu = \beta^{\text{conjugate}}$

Theorem

Let $[\mathcal{A}^1] = [\mathcal{A}^2]$ *with* $\mathcal{A}^1$ *being an arbitrary skew diagram* $\mathcal{A}^1 = \lambda/\mu$ *having a part* λ_1 *and a height* $l(\lambda)$.
Then $\mathcal{A}^1 = \mathcal{A}^2$ *or* $\mathcal{A}^1 = (\mathcal{A}^2)^{\text{rotated}}$.

$\mathcal{A}^1$:



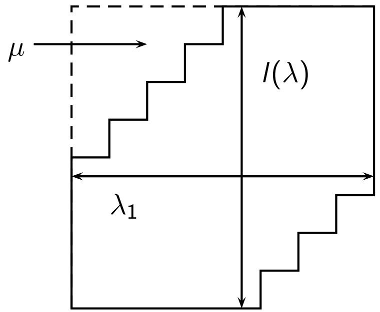
Proof

The corresponding product of Schubert classes is in this case a product of Schur functions.

Theorem

Let $[\mathcal{A}^1] = [\mathcal{A}^2]$ *with* $\mathcal{A}^1$ *being an arbitrary skew diagram* $\mathcal{A}^1 = \lambda/\mu$ *having a part* λ_1 *and a height* $l(\lambda)$*.*
Then $\mathcal{A}^1 = \mathcal{A}^2$ *or* $\mathcal{A}^1 = (\mathcal{A}^2)^{\text{rotated}}$*.*

$\mathcal{A}^1$:



Proof

The corresponding product of Schubert classes is in this case a product of Schur functions.

Bijection

Take an arbitrary LR-Tableau which contains in every column a box filled with the entry 1.
Removing all boxes filled with the entry 1 and replacing afterwards every entry i ($i > 1$) by $i - 1$ yields another LR-Tableau.

Bijection

This gives a bijection between the characters $[\nu] \in [\lambda/\mu]$ with maximal first part and arbitrary characters $[\xi] \in [\hat{\lambda}/\mu]$ with $\hat{\lambda}/\mu$ the skew diagram obtained by removing the top boxes in every column of λ/μ.

Theorem

$$[\lambda/\mu] = [\alpha/\beta] \Rightarrow [\hat{\lambda}/\mu] = [\hat{\alpha}/\beta]$$

Bijection

Take an arbitrary LR-Tableau which contains in every column a box filled with the entry 1.
Removing all boxes filled with the entry 1 and replacing afterwards every entry i ($i > 1$) by $i - 1$ yields another LR-Tableau.

Bijection

This gives a bijection between the characters $[\nu] \in [\lambda/\mu]$ with maximal first part and arbitrary characters $[\xi] \in [\hat{\lambda}/\mu]$ with $\hat{\lambda}/\mu$ the skew diagram obtained by removing the top boxes in every column of λ/μ.

Theorem

$$[\lambda/\mu] = [\alpha/\beta] \Rightarrow [\hat{\lambda}/\mu] = [\hat{\alpha}/\beta]$$

Bijection

Take an arbitrary LR-Tableau which contains in every column a box filled with the entry 1.
Removing all boxes filled with the entry 1 and replacing afterwards every entry i ($i > 1$) by $i - 1$ yields another LR-Tableau.

Bijection

This gives a bijection between the characters $[\nu] \in [\lambda/\mu]$ with maximal first part and arbitrary characters $[\xi] \in [\hat{\lambda}/\mu]$ with $\hat{\lambda}/\mu$ the skew diagram obtained by removing the top boxes in every column of λ/μ.

Theorem

$$[\lambda/\mu] = [\alpha/\beta] \Rightarrow [\hat{\lambda}/\mu] = [\hat{\alpha}/\beta]$$

Example $\lambda/\mu = (\lambda_1^a, \lambda_2^b)/(\mu_1^m)$

Theorem

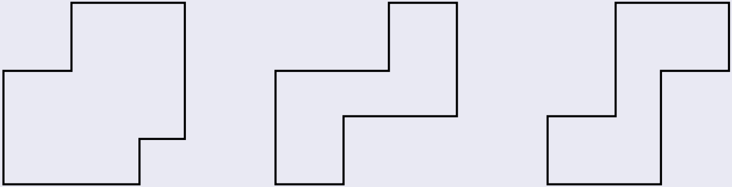
Let $\lambda/\mu = (\lambda_1^a, \lambda_2^b)/(\mu_1^m)$ *and* $[\lambda/\mu] = [\alpha/\beta]$.

Then $\lambda/\mu = \alpha/\beta$ *or* $\lambda/\mu = \alpha/\beta^{\text{rotated}}$.

Example $\lambda/\mu = (\lambda_1^a, \lambda_2^b)/(\mu_1^m)$

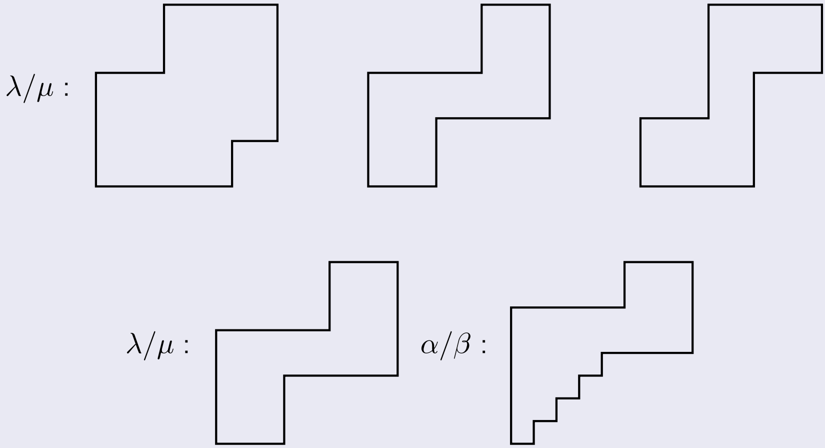
Proof: $\lambda/\mu = \alpha/\beta$ or $\lambda/\mu = \alpha/\beta^{\text{rotated}}$

λ/μ :



Example $\lambda/\mu = (\lambda_1^a, \lambda_2^b)/(\mu_1^m)$

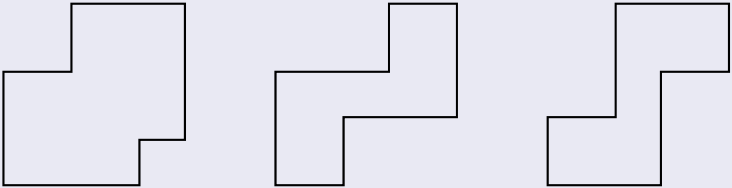
Proof: $\lambda/\mu = \alpha/\beta$ or $\lambda/\mu = \alpha/\beta^{\text{rotated}}$



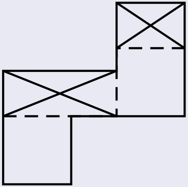
Example $\lambda/\mu = (\lambda_1^a, \lambda_2^b)/(\mu_1^m)$

Proof: $\lambda/\mu = \alpha/\beta$ or $\lambda/\mu = \alpha/\beta^{\text{rotated}}$

λ/μ :



λ/μ :



α/β :