Linear time equivalent Littlewood-Richardson coefficient symmetry maps

O. Azenhas, A. Conflitti, R. Mamede

CMUC, Centre for Mathematics, University of Coimbra

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Contents

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- Purbhoo has defined the operation *migration* on mosaics a sort of *jeu de taquin* moves on puzzles.

Linear time reductions

- Let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map. δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit–size of A.

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  - A map β is *linearly reducible* to α, write $\beta \hookrightarrow \alpha$, if there exist a finite α–based ps-circuit $\sqsupset$ which computes β. We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β, and β is linearly reducible to α.

Linear reduction of LR-symmetry maps

- Pak-Vallejo Theorem The following maps are linearly equivalent:

  (1) [PV] RSK correspondence.

  (2) [PV] Jeu de taquin map.

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- The conjugation symmetry on LR tableaux is any bijection

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- ϱ^{WHS}, ϱ^{BSS} and ϱ^{AZ} are identical.

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LR-coefficient conjugation symmetry map is linearly reducible to the Schützenberger involution

Partitions

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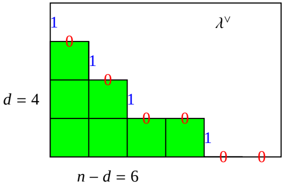
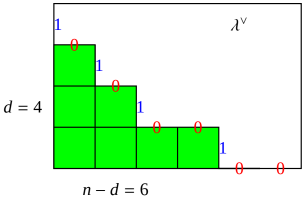
- $d = 4$ $n = 10$



$\lambda = (4, 2, 1, 0) \leftrightarrow 0010010101$

$\lambda^{\vee} = (6, 5, 4, 2) \leftrightarrow 1010100100$

Conjugate partitions



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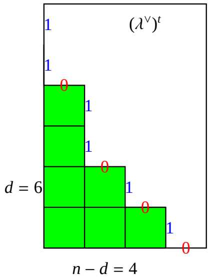
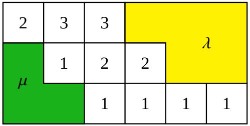


$\lambda^t = (3, 2, 1, 1, 0, 0)$ 1101101010

$(\lambda^\vee)^t = (4, 4, 3, 3, 2, 1)$ 0101011011

Littlewood-Richardson rule

- $c_{\mu\,\nu\,\lambda}$ is the number of semistandard Young tableaux with shape $\lambda^{\vee}/\mu$ and content ν, with the following property:
  - If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top, at any stage, the number of *i*'s encountered is at least as large as the number of $(i+1)$'s encountered, $\#1's \geq \#2's \ldots$.



$\nu = (5, 3, 2)$

Benkart-Sottile-Stroomer bijection ϱ^{BSS}

- $$\begin{array}{rcl} \varrho^{BSS} : LR(\mu,\nu,\lambda) & \longrightarrow & LR(\mu^t,\nu^t,\lambda^t) \\ T & \mapsto & \varrho(T) = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK} \end{array}$$

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- ▶ Facts: [Haiman] Consider two equivalence relations on a pair of tableaux. Two tableaux are Knuth equivalent if one can be obtained from the other by a sequence of (reverse) *jeu de taquin slides*. They are dual Knuth equivalent if such a (any) sequence results in tableaux of the same shape.

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  - Tableaux of the same (anti) normal shape are dual equivalent. A pair of tableaux that are both Knuth and dual Knuth equivalent must be equal.
    If $\mathcal{D}$ is a dual Knuth equivalence class and $\mathcal{K}$ is a Knuth equivalence class, both corresponding to the same straight shape. Then, there is a *unique* tableau in $\mathcal{D} \cap \mathcal{K}$.

ϱ^{BSS} bijection

- $LR(\mu\ \nu\ \lambda) \mapsto LR(\mu^t\ \lambda^t\ \nu^t)$

$$T = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 1 & 3 & \\ \hline & 2 & \\ \hline & & 1 \\ \hline \end{array} \longrightarrow \widehat{T} = \begin{array}{|c|c|c|}\hline 5 & & \\ \hline 1 & 4 & \\ \hline & 3 & \\ \hline & & 2 \\ \hline \end{array} \longrightarrow \widehat{T}^t = \begin{array}{|c|c|c|c|}\hline 2 & & & \\ \hline & 3 & 4 & \\ \hline & & 1 & 5 \\ \hline \end{array} \rightarrow$$

$$\widehat{T}^t \cup Y(\lambda^t)^{\mathrm{a}} = \begin{array}{|c|c|c|c|}\hline 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} \\ \hline & 3 & 4 & \bar{\mathbf{1}} \\ \hline & & 1 & 5 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline & \bar{\mathbf{1}} & \bar{\mathbf{2}} & 1 \\ \hline & & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ \hline \end{array} = Z \cup \widehat{Y}(\nu^t)^{\mathrm{a}} \rightarrow$$

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$T =$

4		
1	3	
	2	
		1

$\longrightarrow \widehat{T} =$

5		
1	4	
	3	
		2

$\longrightarrow \widehat{T}^t =$

2			
	3	4	
		1	5

$\rightarrow$

$\widehat{T}^t \cup Y(\lambda^t)^a =$

2	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$
	3	4	$\bar{\mathbf{1}}$
		1	5

$\rightarrow$

2	3	4	5
	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
		$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$

$= Z \cup \widehat{Y}(\nu^t)^a \rightarrow$

- $LR(\mu^t\ \lambda^t\ \nu^t) \mapsto LR(\mu^t\ \nu^t\ \lambda^t)$

$Z \cup Y(\nu^t)^a =$

1	1	1	2
	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
		$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$

$\rightarrow$

1	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$
	1	2	$\bar{\mathbf{1}}$
		1	1

$= \varrho^{BSS}(T) \cup Y(\lambda^t$

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$$\begin{array}{ccccc} LR(\mu,\nu,\lambda) & \stackrel{e}{\longrightarrow} & LR(\mu,\nu^*,\lambda) & \stackrel{\bullet}{\longrightarrow} & LR(\lambda,\nu,\mu) \\ T & \longrightarrow & T^e & \longrightarrow & T^{e\bullet} \end{array}$$

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-

$$\begin{array}{ccccccc} \varrho^{AZ} : LR(\mu,\nu,\lambda) & \xrightarrow{e} & LR(\mu,\nu^*,\lambda) & \xrightarrow{\bullet} & LR(\lambda,\nu,\mu) & \xrightarrow{\blacklozenge} & LR(\mu^t,\nu^t,\lambda^t) \\ T & \longrightarrow & T^e & \longrightarrow & T^{e\bullet} & \longrightarrow & T^{e\bullet\blacklozenge} \end{array}$$

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-

T =

1	1	3	3		
		2	2	2	
			1	1	1

$\xrightarrow{\blacklozenge}$ T$^{\blacklozenge}$ =

5		
4		
2	3	
1	2	3
	1	2
		1

Bijection ♦

- $LR(\mu,\ \nu,\ \lambda) \xrightarrow{\blacklozenge} LR(\lambda^t,\ \nu^t,\ \mu^t)$;
- $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$
- $\mathrm{T} =$

1	1	3	3		
		2	2	2	
			1	1	1

1112223311

$\xrightarrow{\blacklozenge}$ $\xrightarrow{\lozenge}$

$\mathrm{T}^{\blacklozenge} =$

5		
4		
2	3	
1	2	3
	1	2
		1

1231231245

Complexity of bijection ♦

Algorithm (Bijection ♦.)

Input: LR tableau T of skew shape λ/μ, with $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$, $\mu = (\mu_1 \geq \ldots \geq \mu_n)$, and filling $\nu = (\nu_1 \geq \ldots \geq \nu_n)$, having $A = \left(a_{i,j}\right) \in M_{n\times n}(\mathbb{N}) \quad (a_{i,j} = 0$ if $j > i)$ as (lower triangular) recording matrix.

Write $\widetilde{A}$, a copy of the matrix A.

For $j := n$ down to 2 do

For $i := 1$ to n do

Begin

If $i = j$ then $\widetilde{a}_{i,i} := \widetilde{a}_{i,i} + \lambda_1 - \lambda_i$

else

If $j > i$ then $\widetilde{a}_{i,j} = 0$ else $\widetilde{a}_{i,j} := \widetilde{a}_{i,j} + \widetilde{a}_{i,j+1}$.

End

So far the computational cost is $O\left(n^2\right) = O\left(\langle A\rangle\right)$.

Remark: *For all $1 \leq i \leq n$ and $0 \leq j \leq n-i+1$, we have*

$$\widetilde{a}_{i+j+1,i} - \widetilde{a}_{i+j,i} \geq a_{i+j+1,i}.$$

Complexity of bijection ♦ continued

Algorithm (Bijection ♦.)

Set a matrix $B = \left(b_{i,j}\right) \in M_{\lambda_1 \times \lambda_1}(\mathbb{N})$ *such that* $b_{i,j} = 0$ *for all* i, j.
For $i := 1$ *to* n *do*
Begin
Set $c := 0$.
For $j := 0$ *to* n *do*
Begin
$r := \widetilde{a}_{i+j,i} - a_{i+j,i}$.
For $t := 1$ *to* $a_{i+j,i}$ *do* $b_{r+t,c+t} := b_{r+t,c+t} + 1$.
$c := c + a_{i+j,i}$.
End
End
This part has total computational cost at most equal to

$$O\left(\sum_{1 \leq i.j \leq n} a_{i,j}\right) = O\left(|\lambda \setminus \mu|\right) = O\left(|\lambda| - |\mu|\right) = O\left(\langle T \rangle\right).$$

Output: B recording matrix of the output tableau.

Relative Complexity of map $\varrho = \varrho^{AZ} = \varrho^{BSS} = \varrho^{WHS}$

Theorem The conjugation symmetry maps ϱ^{BSS}, ϱ^{WHS} and ϱ^{AZ} are identical, and linear equivalent to the Schützenberger involution E,

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau \updownarrow & & \tau \updownarrow & & \\ P & \underset{E}{\overset{\text{evacuation}}{\longleftrightarrow}} & P^{E}. & & \end{array}$$

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Word of $T^{e\bullet\blacklozenge} = (\sigma_0 w)^{*\diamond}$.

$\sigma_0 = s_1 s_2 s_1$

$$w = 11(1(12)2)(1332) \longrightarrow 22(1(12)2)(1332) \longrightarrow 2211(2(213)3)2 \longrightarrow 3311(2(213)3)3$$
$$\longrightarrow 33(1(12)2)1333 \longrightarrow \sigma_0 w = 3311222333$$

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$$\overset{*}{\longrightarrow} 1112223311 \overset{\diamond}{\longrightarrow} 1231231245.$$

The $\mathbb{Z}_2 \oplus \underline{S}_3$-symmetries are linearly equivalent modulus the fundamental symmetry

$$c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$$

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$$\begin{array}{ccc} c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu} & & \\ c_{\mu\,\nu\,\lambda} & = & c_{\nu\,\mu\,\lambda} \\ c_{\nu\,\mu\,\lambda} = c_{\mu\,\lambda\,\nu} = c_{\lambda\,\nu\,\mu} & & \end{array}$$

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$$
\begin{array}{ccc}
c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu} & & \\
c_{\mu\,\nu\,\lambda} & = & c_{\nu\,\mu\,\lambda} \\
c_{\nu\,\mu\,\lambda} = c_{\mu\,\lambda\,\nu} = c_{\lambda\,\nu\,\mu} & & \\
 & = & c_{\mu^t\,\nu^t\,\lambda^t}
\end{array}
$$

Contents

Puzzle rule

- A puzzle of size n is a tiling of an equilateral triangle of side length n with puzzle pieces each of unit side length.
    - Puzzle pieces may be rotated in any orientation *but not reflected*, and wherever two pieces share an edge, the numbers on the edge must agree.

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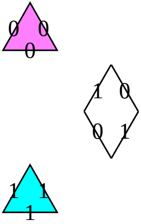
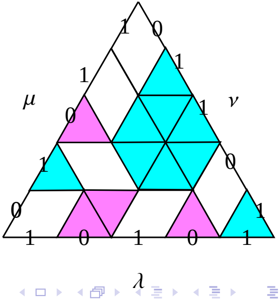
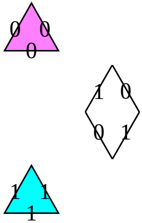
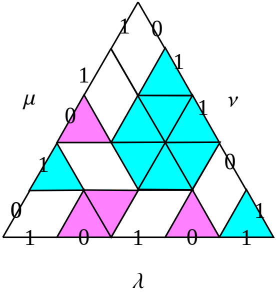




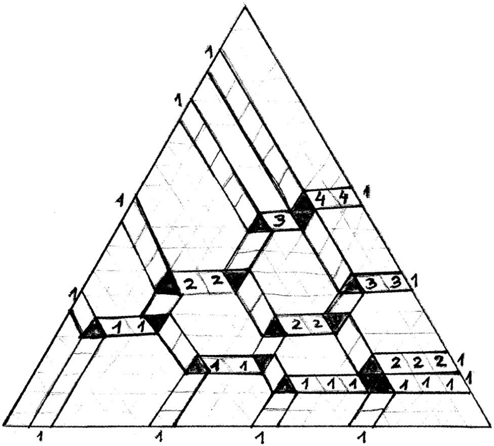
Puzzle rule

- (Knutson-Tao-Woodward) $c_{\mu\,\nu\,\lambda}$ is the number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary.





A bijection between Puzzles and LR tableaux: Tao's bijection

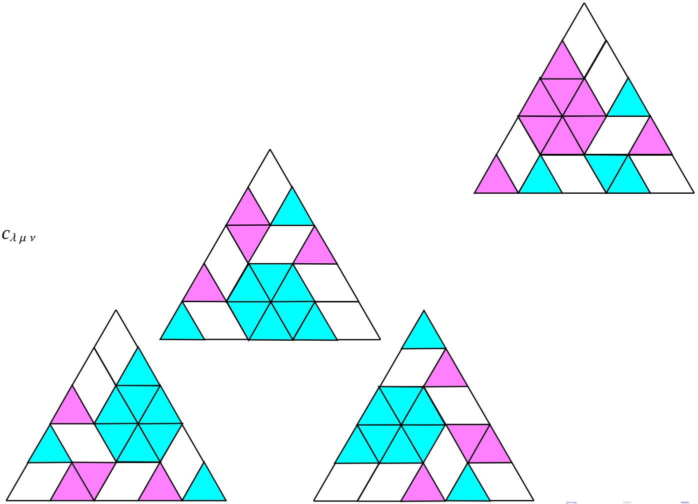


	1	1	2	2	3	4	4								
					1	1	2	2	3	3					
								1	1	1	2	2	2		
											1	1	1		

Rotation and reflection

- $c_{\mu\ \nu\ \lambda} = c_{\lambda\ \mu\ \nu} = c_{\nu\ \lambda\ \mu}$
- $c_{\mu\ \nu\ \lambda} = c_{\nu^t\ \mu^t\ \lambda^t} = c_{\lambda^t\ \nu^t\ \mu^t} = c_{\mu^t\ \lambda^t\ \nu^t}$

$c_{\mu\,\nu\,\lambda} = c_{\nu\,\lambda\,\mu} = c_{\lambda\,\mu\,\nu}$



- ϱ^{BSS} = rotation+reflection+fundamental symmetry
- $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\lambda^t\,\nu^t}$ $\qquad$ $c_{\mu^t\,\lambda^t\,\nu^t} = c_{\mu^t\,\nu^t\,\lambda^t}$

ϱ^{BSS} bijection

- $LR(\mu\ \nu\ \lambda) \mapsto LR(\mu^t\ \lambda^t\ \nu^t)$

$$T = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 1 & 3 & \\ \hline & 2 & \\ \hline & & 1 \\ \hline \end{array} \longrightarrow \widehat{T} = \begin{array}{|c|c|c|}\hline 5 & & \\ \hline 1 & 4 & \\ \hline & 3 & \\ \hline & & 2 \\ \hline \end{array} \longrightarrow \widehat{T}^t = \begin{array}{|c|c|c|c|}\hline 2 & & & \\ \hline & 3 & 4 & \\ \hline & & 1 & 5 \\ \hline \end{array} \rightarrow$$

$$\widehat{T}^t \cup Y(\lambda^t)^{\mathrm{a}} = \begin{array}{|c|c|c|c|}\hline 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} \\ \hline & 3 & 4 & \bar{\mathbf{1}} \\ \hline & & 1 & 5 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline & \bar{\mathbf{1}} & \bar{\mathbf{2}} & 1 \\ \hline & & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ \hline \end{array} = Z \cup \widehat{Y}(\nu^t)^{\mathrm{a}} \rightarrow$$

ϱ^{BSS} bijection

- $LR(\mu\ \nu\ \lambda) \mapsto LR(\mu^t\ \lambda^t\ \nu^t)$

$T =$

4		
1	3	
	2	
		1

$\longrightarrow \widehat{T} =$

5		
1	4	
	3	
		2

$\longrightarrow \widehat{T}^t =$

2			
	3	4	
		1	5

$\rightarrow$

$\widehat{T}^t \cup Y(\lambda^t)^{\mathrm{a}} =$

2	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$
	3	4	$\bar{\mathbf{1}}$
		1	5

$\rightarrow$

2	3	4	5
	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
		$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$

$= Z \cup \widehat{Y}(\nu^t)^{\mathrm{a}} \rightarrow$

- $LR(\mu^t\ \lambda^t\ \nu^t) \mapsto LR(\mu^t\ \nu^t\ \lambda^t)$

$Z \cup Y(\nu^t)^{\mathrm{a}} =$

1	1	1	2
	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
		$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$

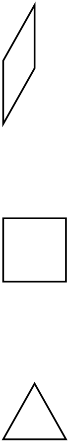
$\rightarrow$

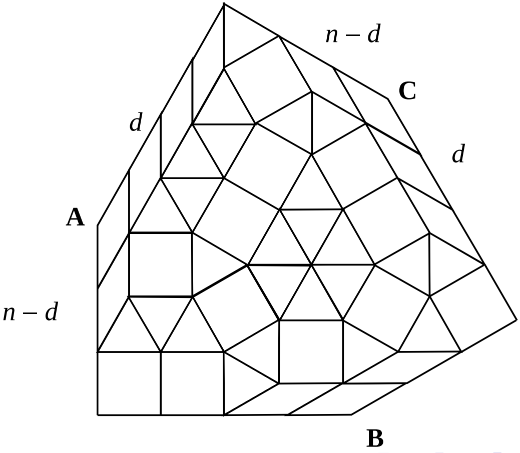
1	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$
	1	2	$\bar{\mathbf{1}}$
		1	1

$= \varrho^{BSS}(T) \cup Y(\lambda^t)$

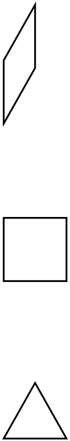
Purbhoo mosaics

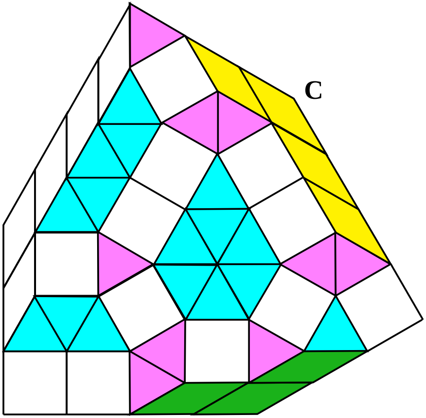
A mosaic is a tiling of a hexagon by the following three shapes such that all rhombi are packed into the three nests A,B, and C.





Mosaics are in bijection with puzzles





Migration

- Migration is an operation that takes a flock to a new nest. The rhombi must move in the standard order.(The standard order in a tableau is the numerical ordering of the entries with priority by the rule left=smaller, right=larger, in case of equality.)

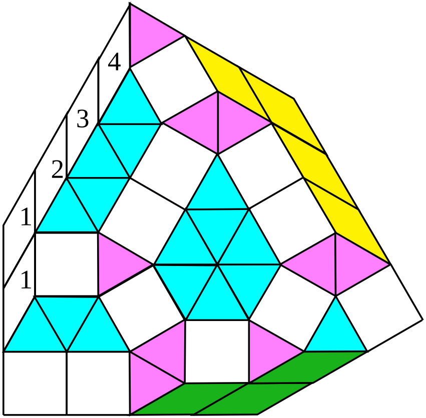
Migration

- Migration is an operation that takes a flock to a new nest. The rhombi must move in the standard order.(The standard order in a tableau is the numerical ordering of the entries with priority by the rule left=smaller, right=larger, in case of equality.)
- Choose the target nest. Rhombi move in the chosen direction of migration, inside a smallest hexagon in which $\diamondsuit$ is contained:

The move is such that the rhombus is either in its initial orientation, or its final orientation.

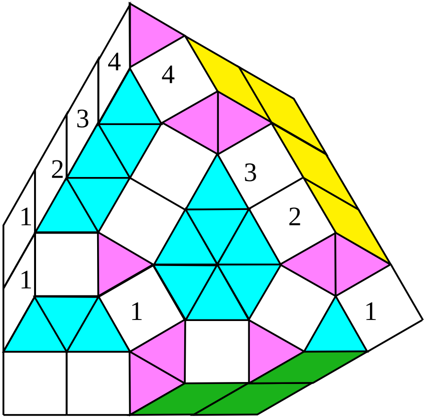
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

4	•	•
1	3	•
•	2	•
•	•	1



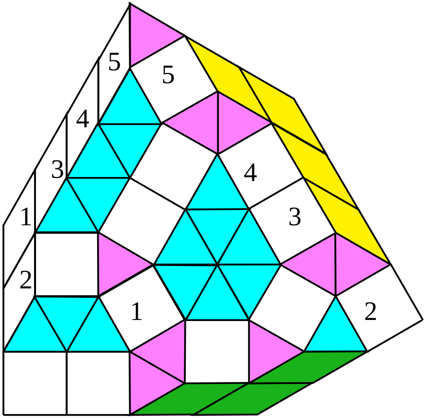
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

4	•	•
1	3	•
•	2	•
•	•	1



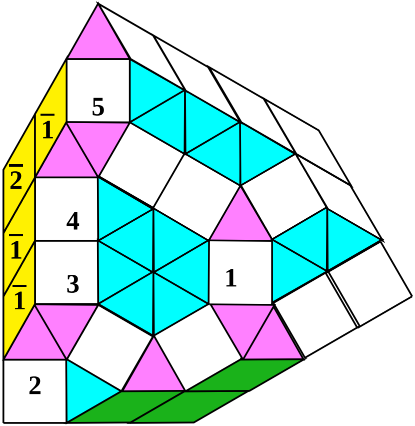
Migration($\equiv$ j.t.)/ϱ^{BSS}

5	•	•
1	4	•
•	3	•
•	•	2



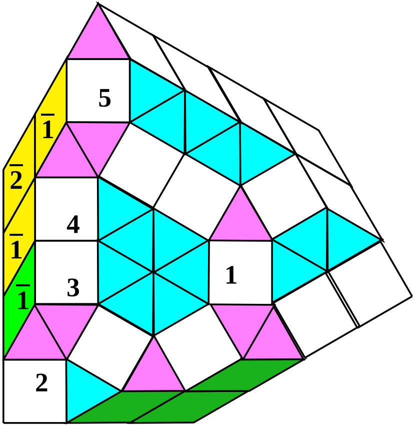
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
•	3	4	$\bar{1}$
•	•	1	5



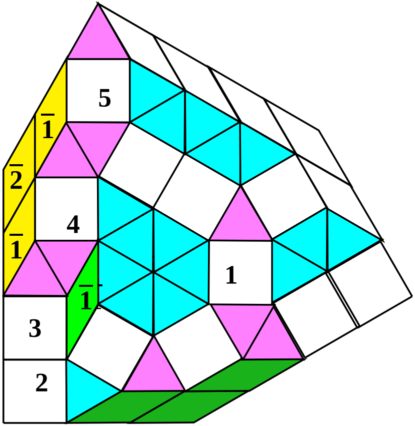
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	$\bar{1}$	$\bar{1}$	$\bar{2}$
•	3	4	$\bar{1}$
•	•	1	5



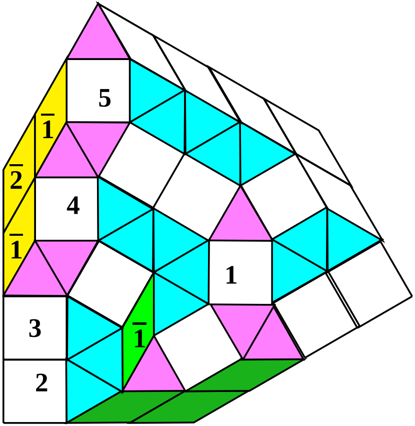
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$
•	$\bar{\mathbf{1}}$	4	$\bar{\mathbf{1}}$
•	•	1	5



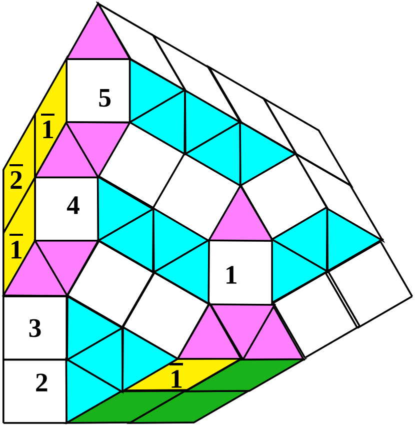
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



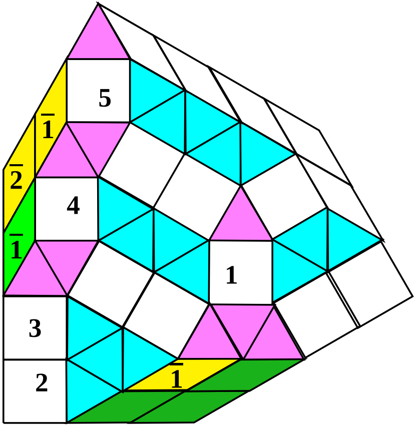
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



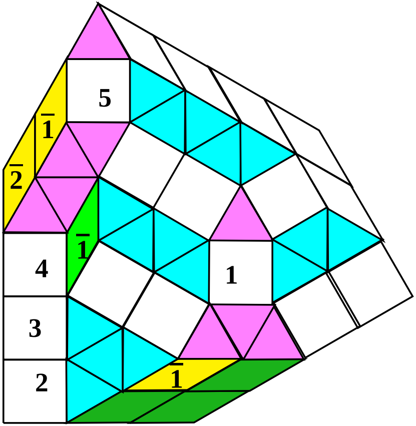
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	$\bar{1}$	$\bar{2}$
•	$\bar{1}$	4	$\bar{1}$
•	•	1	5



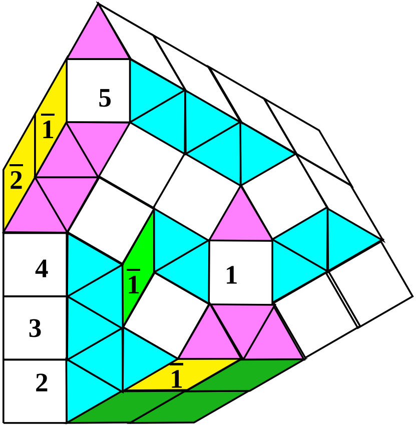
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	$\bar{1}$
•	•	1	5



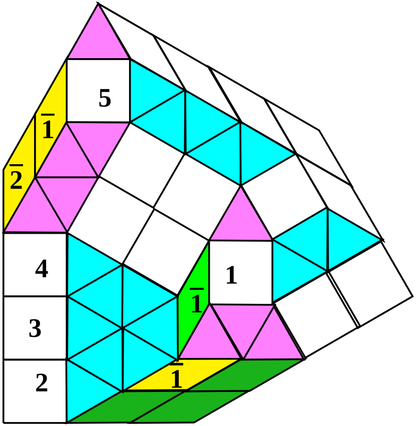
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	$\bar{1}$
•	•	1	5



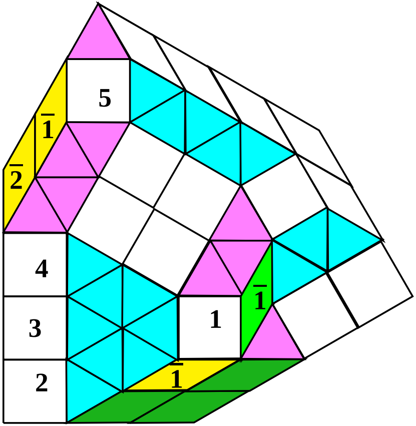
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{2}$
●	$\bar{1}$	$\bar{1}$	$\bar{1}$
●	●	1	5



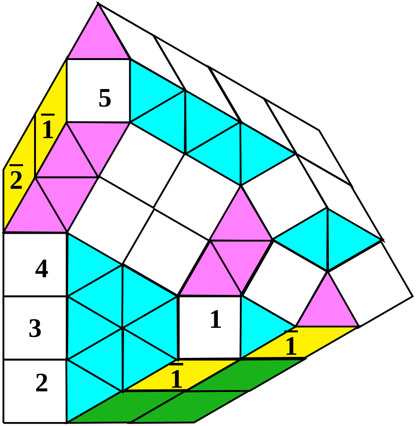
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
•	$\bar{1}$	$\bar{1}$	$\bar{1}$
•	•	1	5



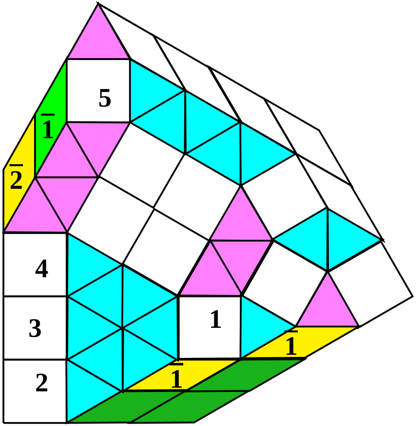
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{\mathbf{2}}$
•	$\bar{\mathbf{1}}$	1	$\bar{\mathbf{1}}$
•	•	$\bar{\mathbf{1}}$	5



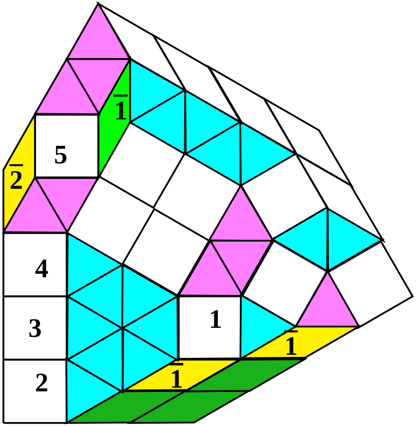
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	1	$\bar{1}$
•	•	$\bar{1}$	5



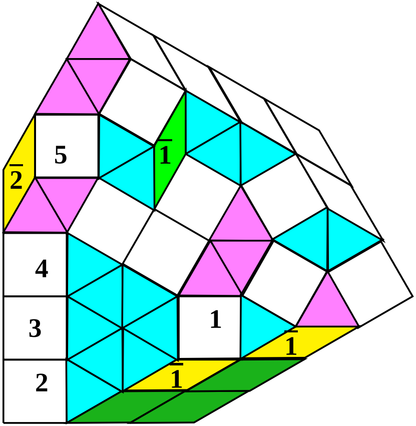
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



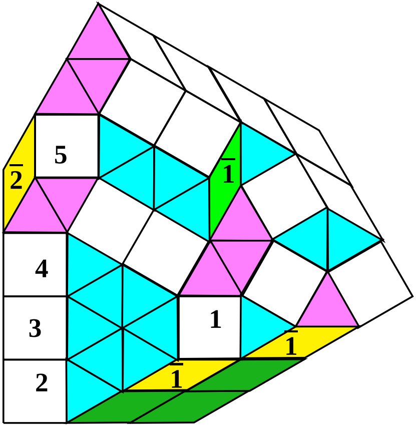
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



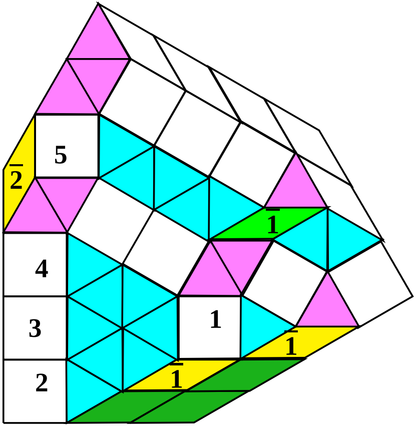
Migration($\equiv$ j.t.)/ϱ^{BSS}

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



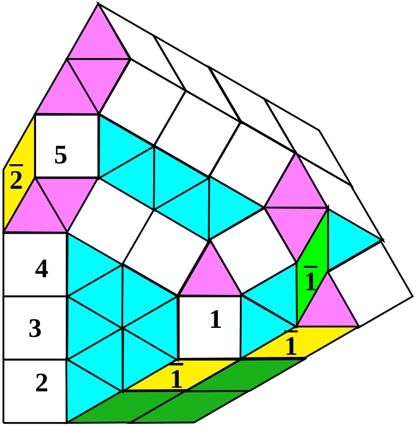
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



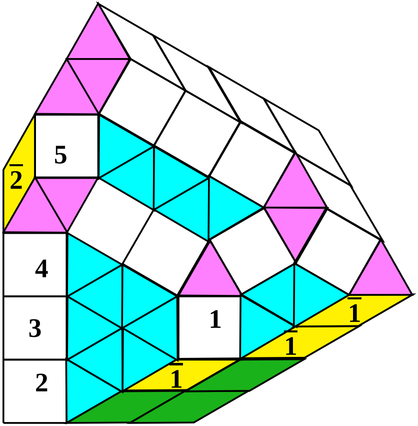
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



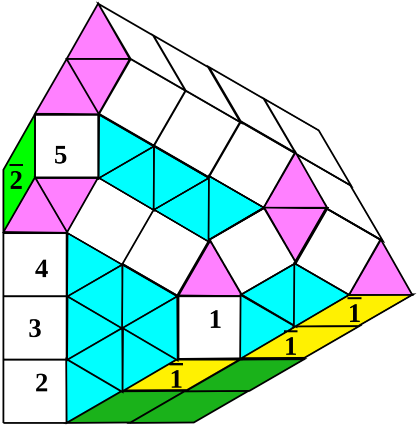
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
•	$\bar{1}$	1	5
•	•	$\bar{1}$	$\bar{1}$



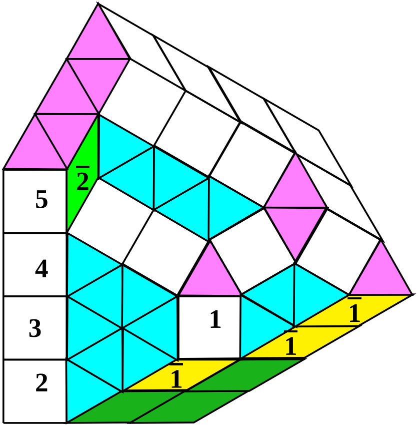
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	$\bar{2}$
●	$\bar{1}$	1	5
●	●	$\bar{1}$	$\bar{1}$



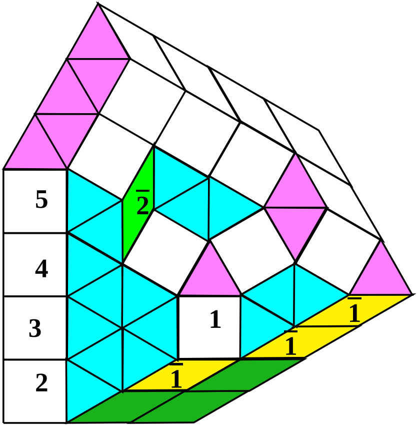
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
●	$\bar{\mathbf{1}}$	1	$\bar{\mathbf{2}}$
●	●	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$



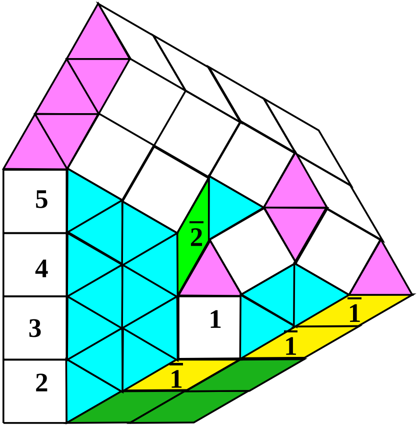
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
•	$\bar{1}$	1	$\bar{2}$
•	•	$\bar{1}$	$\bar{1}$



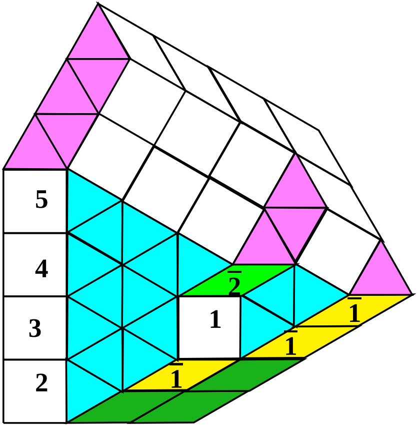
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
•	$\bar{1}$	1	$\bar{2}$
•	•	$\bar{1}$	$\bar{1}$



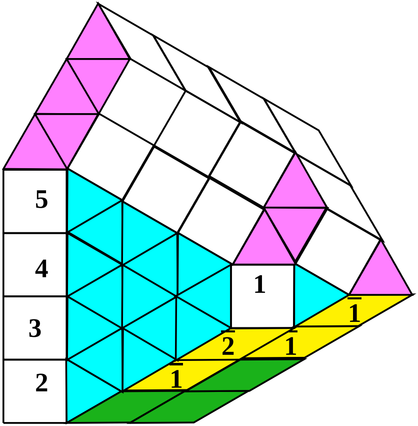
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
•	$\bar{\mathbf{1}}$	1	$\bar{\mathbf{2}}$
•	•	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$



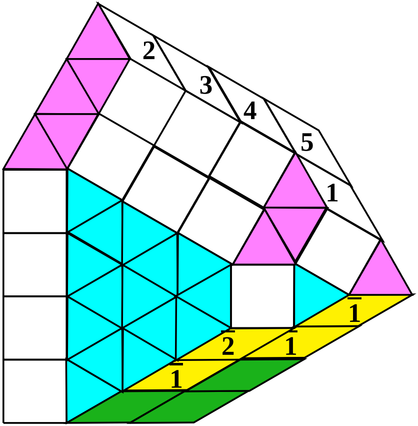
Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
•	$\bar{1}$	$\bar{2}$	1
•	•	$\bar{1}$	$\bar{1}$



Migration($\equiv$ j.t.)$/\varrho^{BSS}$

2	3	4	5
•	$\bar{1}$	$\bar{2}$	1
•	•	$\bar{1}$	$\bar{1}$



Migration($\equiv$ j.t.)$/\varrho^{BSS}$

1	1	1	2
•	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	1
•	•	$\bar{\mathbf{1}}$	$\bar{\mathbf{1}}$