Combinatorics of q-Charlier and q-Laguerre polynomials*

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*1. Kim-Stanton-Zeng: The Combinatorics of the Al-Salam-Chihara q-Charlier Polynomials, SLC 54 (2006), Article B54i.
2. Kasraoui-Stanton-Zeng: The Combinatorics of the Al-Salam-Chihara q-Laguerre Polynomials, preprint, 2008.

Garsia and Remmel (Europ. J. Combinatorics (1980) 1, 47-59):

"It has becoming increasingly apparent since the work of
Foata-Schützenberger and recent works of
the Lotharingien school of combinatorics,
see for instance Viennot, Dumont and Flajolet's works,
that the special functions and identities of classical mathematics
are gravid with combinatorial information."

- Combinatorics of OP and their generating functions. Labeled structures and exponential generating functions. Combinatorial proof of Mehler's formula. Foata, Joyal in 1970's and 1980's.

- Formal theory of OP and lattice path interpretation for polynomials and moments. Flajolet, Viennot in 1980's.

- Linearization formula of some classical orthogonal polynomials. In particular, Ismail-Stanton-Viennot in '87 gave a q-analogue of linearization formula for Hermite polynomials and a combinatorial evaluation of Askey-Wilson integral.

Rogers q-Hermite polynomials:

$$xH_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x|q).$$

Here q is (say) in $(-1, 1)$, and

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$

Orthogonal with respect to

$$d\mu_q(x) = \frac{1}{\pi}\sqrt{1-q}\,\sin(\theta)(q;q)_\infty \left|(qe^{2i\theta;q})_\infty\right|^2 dx,$$

for $x = \frac{2}{\sqrt{1-q}}\cos(\theta)$, $\theta \in [0, \pi]$, and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

Moments

$$\mu_n = \int x^n d\mu_q(x) = \sum_{\pi} q^{\mathsf{cr}\pi},$$

where the summation is over all perfect matchings of $\{1, 2, \ldots, n\}$.

The polynomials have the combinatorial interpretation:

$$H_n(x|q) = \sum_{\alpha} (-1)^{p(\alpha)} x^{n-2p(\alpha)} q^{s(\alpha)},$$

where the summation is over all matchings α of K_n.

Linearization coefficients

$$\int H_{n_1}(x|q)\ldots H_{n_k}(x|q)d\mu_q(x) = \sum_{\pi \in \Pi(n_1,n_2,\ldots,n_k)} q^{\mathrm{cr}\pi}.$$

(Ismail, Stanton, Viennot '87).
For $k = 4$ it gives a combinatorial evaluation of Askey-Wilson integral.

For $k = 3$ it gives the linearization formula

$$H_{n_1}(x|q)\ H_{n_2}(x|q) = \sum_{l=0}^{\min(n_1,n_2)} \begin{bmatrix} n_1 \\ l \end{bmatrix}_q \begin{bmatrix} n_2 \\ l \end{bmatrix}_q l!_q\ H_{n_1+n_2-2l}(x|q).$$

What's the next?

Askey's scheme of hepergeometric orthogonal polynomials

Wilson Racah

Continuous dual Hahn Continuous Hahn Hahn Dual Hahn

Meixner-Pollaczek Jacobi Meixner Krawtchouk

q–Laguerre q–Charlier

q–Hermite

The Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomials are defined by

$$\begin{cases} Q_{n+1}(x) = (2x - (\alpha+\beta)q^n)Q_n(x) - (1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x), \\ Q_0(x) = 1, \quad Q_{-1}(x) = 0, \end{cases}$$

$$\begin{aligned} Q_n(x;\alpha,\beta|q) &= \frac{(\alpha\beta;q)_n}{\alpha^n}\, {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix}\right| q; q\right), \\ &= (\alpha u;q)_n u^{-n}\, {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1}q^{-n+1}u^{-1} \end{matrix}\right| q; \alpha^{-1}qu\right), \\ &= (\beta u^{-1};q)_n u^{n}\, {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, \alpha u \\ \beta^{-1}q^{-n+1}u \end{matrix}\right| q; \beta^{-1}qu^{-1}\right), \end{aligned}$$

where $x = \frac{u+u^{-1}}{2}$ or $x = \cos\theta$ if $u = e^{i\theta}$.

The generating function of the latter is

$$\sum_{n=0}^{\infty} \frac{Q_n(x;\alpha,\beta)}{(q;q)_n} t^n = \frac{(\alpha t, \beta t; q)_\infty}{(e^{i\theta}t, e^{-i\theta}t; q)_\infty}, \qquad x = \cos\theta. \tag{1}$$

The orthogonality reads as follows:

$$\frac{(q,\alpha\beta;q)_\infty}{2\pi} \int_0^\pi \frac{Q_m(x)Q_n(x)(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta};q)_\infty(\beta e^{i\theta}, \beta e^{-i\theta};q)_\infty} d\theta$$
$$= (q;q)_n(\alpha\beta;q)_n\delta_{m,n}.$$

The new q-Charlier polynomials

Moments (Biane '97)

$$\mu_n(a,q) = \sum_{\pi \in \Pi_n} q^{\mathsf{cr}\pi} a^{|\pi|},$$

where $\mathsf{cr}\pi$ = number of crossings of the partition π.

Polynomials (Anshelevich '05):

$$xC_{q,n}(x,a) = C_{q,n+1}(x,a) + (a+[n]_q)C_{q,n}(x,a) + a[n]_q C_{q,n-1}(x,a).$$

Linearization coefficients

$$\mathcal{L}_q\left(C_{q,n_1}(x,a)\dots C_{q,n_k}(x,a)\right) = \sum_{\pi \in \Pi(n_1,n_2,\dots,n_k)} q^{\mathsf{cr}\pi} a^{|\pi|}.$$

A partition of $[n]$:=$\{1, 2, \cdots, n\}$ is a collection of disjoint nonempty subsets of $[n]$, called blocks, whose union is $[n]$.
A perfect matching of $[2n]$ is a partition of $[2n]$ in n two-element blocks.

$$\Pi_n = \text{set of partitions of} \quad [n]$$
$$\mathcal{M}_n = \text{set of matchings of} \quad [n].$$

Standard form of a partition

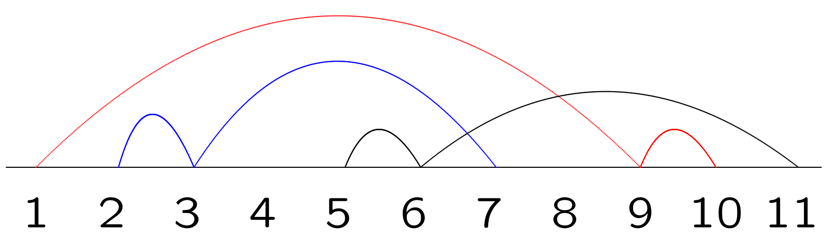
$$\pi = \{1, 9, 10\} - \{2, 3, 7\} - \{4\} - \{5, 6, 11\} - \{8\}$$

graph of partition on the vertex set $[n]$:

there is an edge e joining i and j if and only if i and j are consecutive elements in a same block.

The graph of the partition

$$\pi = \{1, 9, 10\} - \{2, 3, 7\} - \{4\} - \{5, 6, 11\} - \{8\}$$

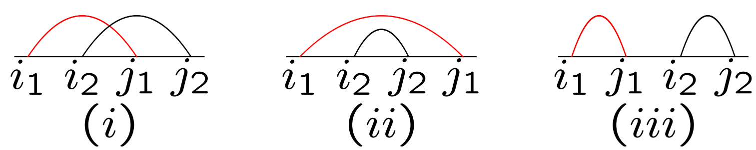


Given a partition π of $[n]$, two edges $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ of π are said to form:

(i) a *crossing* with e_1 as the *initial edge* if $i_1 < i_2 < j_1 < j_2$;

(ii) a *nesting* with e_2 as *interior edge* if $i_1 < i_2 < j_2 < j_1$;

(iii) an *alignment* with e_1 as *initial edge* if $i_1 < j_1 \leq i_2 < j_2$.



Theorem 1 (Kasraoui-Z.) *The n^{th}-moment of the q-Charlier polynomials $C_n(x, a; q)$ is*

$$\mu_n(a) := \mathcal{L}_q(x^n) = \sum_{\pi \in \Pi_n} a^{|\pi|} q^{\mathrm{rc}(\pi)} = \sum_{\pi \in \Pi_n} a^{|\pi|} q^{\mathrm{rn}(\pi)},$$

where Π_n denotes the set of partitions of $[n] := \{1, \ldots, n\}$.

The first values of $\mu_n(a)$ are as follows:

$$\mu_1(a) = a, \quad \mu_2(a) = a + a^2, \quad \mu_3(a) = a + 3a + a^3,$$
$$\mu_4(a) = a + (6 + q)a^2 + 6a^3 + a^4.$$

The new q-Charlier polynomials are a re-scaled version of the Al-Salam-Chihara polynomials $Q_n(x, \alpha, \beta; q)$:

$$C_n(x, a; q) = \left(\frac{a}{1-q}\right)^{n/2}$$
$$\times Q_n\left(\frac{1}{2}\sqrt{\frac{1-q}{a}}\left(x - a - \frac{1}{1-q}\right), \frac{-1}{\sqrt{a(1-q)}}, 0; q\right).$$

Since the generating function of the Al-Salam-Chihara polynomials is known, we derive that

$$\sum_{n=0}^{\infty} C_n(x, a; q)\frac{t^n}{n!_q} = \frac{(-t; q)_\infty}{(\sqrt{a(1-q)}te^{i\theta}, \sqrt{a(1-q)}te^{-i\theta}; q)_\infty},$$

where $n!_q = [n]_q[n-1]_q \dots [2]_q[1]_q$ and

$$\cos\theta = \frac{1}{2}\sqrt{\frac{1-q}{a}}\left(x - a - \frac{1}{1-q}\right).$$

Theorem 2 *We have*

$$C_n(x, a; q) = \sum_{(B,\sigma)} (-1)^{n-\mathrm{cyc}(\sigma)} a^{|B|} x^{\mathrm{cyc}(\sigma)} q^{w(B,\sigma)}.$$

where $B \subset [n]$ and σ is a permutation on $[n] \setminus B$.

For example, if $(B, \sigma) = [9](8\,1\,5\,3)(7\,4)(6)[2]$ has the weight

$$(-1)^{9-3} x^3 a^2 q^{0+3+1+1} = a^2 q^5 x^3,$$

because $n = 9$, three cycles of label x, two cycles of label a, five Charlier-inversions, i.e. $(5, 3), (5, 4), (5, 2), (3, 2), (4, 2)$.

Theorem 3 (Anshelevich) *The linearization coefficients of q-Charlier polynomials are the generating functions of the inhomogeneous partitions:*

$$\mathcal{L}_q\left(C_{n_1}(x,a;q)\cdots C_{n_k}(x,a;q)\right) = \sum_{\pi\in\Pi(n_1,n_2,\ldots,n_k)} q^{\mathrm{rc}(\pi)} a^{|\pi|}. \qquad (2)$$

For example, if $k = 3$ and $n_1 = n_2 = 2$ and $n_3 = 1$, then $\Pi(2,2,1)$ is

$$\{\{(1,3,5)(2,4)\},\{(1,4,5)(2,3)\},\{(2,3,5)(1,4)\},\{(2,4,5)(1,3)\}\}.$$

It is easy to see that the corresponding generating function in (2) is

$$a^2q^2 + a^2 + a^2q + a^2q = a^2(1+q)^2.$$

If $k = 2$, equation (2) gives the orthogonality relation.

When $k = 3$, there is an explicit formula for the generating function in (2).

Theorem 4 *We have*

$$\sum_{\pi \in \Pi(n_1,n_2,n_3)} q^{\mathrm{rc}(\pi)} a^{|\pi|}$$

$$= \sum_{l \geq 0} \frac{n_1!_q n_2!_q n_3!_q \, a^{l+n_3} q^{\binom{n_1+n_2-n_3-2l}{2}}}{l!_q (n_3 - n_1 + l)!_q (n_3 - n_2 + l)!_q (n_1 + n_2 - n_3 - 2l)!_q}.$$

Derangement

A derangement is a permutation in which none of the objects appear in their "natural" (i.e., ordered) place. For example, the only derangements of (1,2,3) are (2,3,1) and (3,1,2), so $d_3 = 2$.

$$\begin{aligned} d_n &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\ &= \int_0^\infty (x-1)^n e^{-x} dx. \end{aligned}$$

A generalization is the following problem:

How many anagrams with no fixed letters of a given word are there?

For instance, for a word made of only two different letters, say n letters A and m letters B,

$$w = \underbrace{A \dots A}_{n} \underbrace{B \dots B}_{m}$$

the answer is, of course, 1 or 0 according whether $n = m$ or not, for the only way to form an anagram without fixed letters is to exchange all the A with B, which is possible if and only if $n = m$.

In the general case, for a word with n_1 letters X_1, n_2 letters X_2, ..., n_r letters X_r:

$$w = \underbrace{X_1 \dots X_1}_{n_1} \underbrace{X_2 \dots X_2}_{n_2} \cdots \underbrace{X_r \dots X_r}_{n_r}$$

it turns out (after a proper use of the inclusion-exclusion formula) that the answer has the form:

$$\int_0^\infty P_{n_1}(x) P_{n_2}(x) \cdots P_{n_r}(x) e^{-x}\, dx,$$

where the P_n's are the Laguerre polynomials (up to a sign that is easily decided). The case $r = 2$ gives an orthogonality relation.

The Laguerre polynomials.

The monic simple Laguerre polynomials $L_n(x)$:

$$L_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} x^k, \tag{3}$$

or by the three-term recurrence relation

$$L_{n+1}(x) = (x - (2n+1)) L_n(x) - n^2 L_{n-1}(x). \tag{4}$$

The moments are

$$\mu_n = \mathcal{L}(x^n) = \int_0^\infty x^n e^{-x} dx = n!. \tag{5}$$

The linearization formula reads as follows:

$$L_{n_1}(x) L_{n_2}(x) = \sum_{n_3} C_{n_1 n_2}^{n_3} L_{n_3}(x),$$

where

$$C_{n_1 n_2}^{n_3} = \mathcal{L}(L_{n_1}(x) L_{n_2}(x) L_{n_3}(x)) / \mathcal{L}(L_{n_3}(x) L_{n_3}(x))$$

Equivalently we have

$$\begin{aligned} &\mathcal{L}(L_{n_1}(x) L_{n_2}(x) L_{n_3}(x)) \\ &= \sum_{s \geq 0} \frac{n_1!\, n_2!\, n_3!\, 2^{n_1+n_2+n_3-2s}\, s!}{(s-n_1)!(s-n_2)!(s-n_3)!(n_1+n_2+n_3-2s)!}. \end{aligned}$$

We have

$$\mathcal{L}(L_{n_1}(x) \ldots L_{n_k}(x)) = \sum_{\sigma \in \mathcal{D}_n} 1.$$

Combinatorial q-analogues

MacMahon:

$$\sum_{\sigma \in \mathcal{S}_n} q^{maj\,\sigma} = [n]_q!.$$

Garsia-Remmel, Wachs, Gessel-Reutenauer, ...:

$$\sum_{\sigma \in \mathcal{D}_n} q^{maj\,\sigma} = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}}.$$

where

$$[n]_q! = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

A q-analogue of MacMahon's Master theorem?

Postnikov '03, Williams '04, Corteel '06 studied some statistics on permutation tableaux.

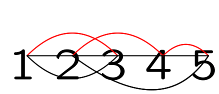
For $\sigma \in \mathcal{S}_n$ the crossing number of σ is defined by

$$\mathrm{cr}(\sigma) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \#\{\mathrm{j} | \mathrm{j} < \mathrm{i} \leq \sigma(\mathrm{j}) < \sigma(\mathrm{i})\} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \#\{\mathrm{j} | \mathrm{j} > \mathrm{i} > \sigma(\mathrm{j}) > \sigma(\mathrm{i})\},$$

while the number of weak excedances of σ is defined by

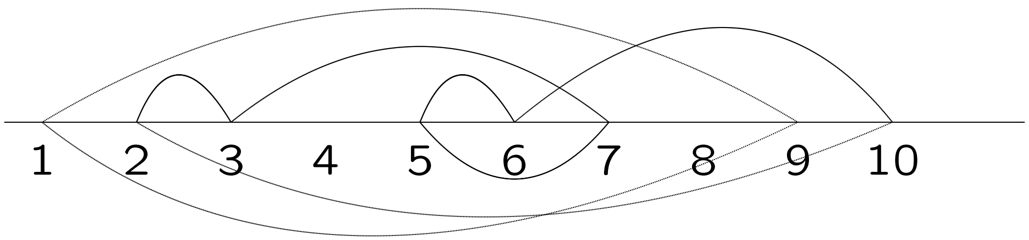
$$wex(\sigma) = \#\{i | 1 \leq i \leq n \text{ and } i \leq \sigma(i)\}.$$

(1 3)(2 4 5)



$y^3 q^3$

Permutation diagram



$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 3 & 7 & 4 & 6 & 10 & 5 & 8 & 1 & 2 \end{pmatrix}$$

Introduce the enumerating polynomial (moments):

$$\mu_n^{(\ell)}(y,q) := \sum_{\sigma\in S_n} y^{wex(\sigma)} q^{\mathsf{cr}(\sigma)}.$$

Randrianarivony '94 and Corteel '06 then proved the following continued fraction expansion:

$$E(y,q,t) := \sum_{n\geq 0} \mu_n^{(\ell)}(y,q)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{\ddots}}}, \tag{6}$$

where $b_n = y[n+1]_q + [n]_q$ and $\lambda_n = y[n]_q^2$.

The new q-Laguerre polynomials

We define the new q-Laguerre polynomials $L_n(x;\, q)$ by the recurrence:

$$L_{n+1}(x;\, q) = (x - y[n+1]_q - [n]_q)L_n(x;\, q) - y[n]_q^2 L_{n-1}(x;\, q).$$

These are re-scaled Al-Salam-Chihara polynomials:

$$L_n(x;\, q) = \left(\frac{\sqrt{y}}{q-1}\right)^n Q_n\left(\frac{(q-1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q|q\right).$$

We derive then the explicit formula for $L_n(x)$:

$$L_n(x;\, q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} \left(x - (1 - yq^{-j})[j]_q\right).$$

Thus

$$
\begin{aligned}
L_1(x;\,q) &= x - y,\\
L_2(x;\,q) &= x^2 - (1 + 2y + qy)x + (1 + q)y^2,\\
L_3(x;\,q) &= x^3 - (q^2y + 3\,y + q + 2 + 2\,qy)x^2\\
&\quad + (q^3y^2 + yq^2 + q + 2\,qy + 3\,q^2y^2 + 1 + 4\,qy^2 + 2\,y + 3\,y^2)x\\
&\qquad - (2\,q^2 + 2\,q + q^3 + 1)y^3
\end{aligned}
$$

Theorem 5 *The q-Laguerre polynomials have the following interpretation:*

$$L_n(x;\, q) = \sum_{A \subset [n],\, f: A \to [n]} (-1)^{|A|} x^{n-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where f is injective.

Proof. This is the $a = 1$, $s = u = 1$ and $r = t = q$ special case of the quadrabasic Laguerre polynomials of Simion-Stanton.

Theorem 6 (Kasraoui-Stanton-Z.) *The linearization coefficients of the q-Laguerre polynomials are*

$$\mathcal{L}_q(L_{n_1}(x;\, q)\dots L_{n_k}(x;\, q)) = \sum_{\sigma\in\mathcal{D}(n_1,\dots,n_k)} y^{wex(\sigma)} q^{\mathsf{cr}(\sigma)}.$$

We first show that the above result is true for $(n_1,\dots,n_k) = (1,\dots,1)$.

Lemma 7 *Then*

$$\mathcal{L}_q((x-y)^n) = \sum_{\sigma\in\mathcal{D}_n} y^{wex(\sigma)} q^{\mathsf{cr}(\sigma)}.$$

Proof. Note that

$$\mathcal{L}_q((x-y)^n) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} y^{n-k}\mu_k(y,q).$$

By binomial inversion, it suffices to prove that

$$\mu_n(y,q) = \sum_{k=0}^{n} \binom{n}{k} y^k \sum_{\sigma\in\mathcal{D}_{n-k}} y^{wex(\sigma)} q^{\mathrm{cr}(\sigma)}$$

But the latter identity is obvious.

The invariance of $\sum_{\sigma\in\mathcal{D}(n_1,n_2,\ldots,n_k)} y^{wex(\sigma)} q^{\mathrm{cr}(\sigma)}$ by permutating the $n_i's$ is also a consequence of Theorem 4, but for our proof we need to first prove it as a lemma.

Lemma 8 *For* $i = 1, \ldots, k-1$ *we have*

$$\sum_{\sigma\in\mathcal{D}(\ldots,n_i,n_{i+1},\ldots)} y^{wex(\sigma)} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma\in\mathcal{D}(\ldots,n_{i+1},n_i,\ldots)} y^{wex(\sigma)} q^{\mathrm{cr}(\sigma)}.$$

Proof of Theorem 2. Note that

$$(x-y)L_n(x) = L_{n+1}(x) + (yq+1)[n]_q L_n(x) + y[n]_q^2 L_{n-1}(x).$$

Let $w(\pi) = y^{wex(\sigma)} q^{\mathrm{cr}(\sigma)}$. It then suffices to show that

$$\begin{aligned}\sum_{\sigma\in\mathcal{D}(1,n,n_2,\ldots,n_k)} w(\pi) = & \sum_{\sigma\in\mathcal{D}(n+1,n_2,\ldots,n_k)} w(\pi) \\ & +(yq+1)[n]_q \sum_{\sigma\in\mathcal{D}(n,n_2,\ldots,n_k)} w(\pi) \\ & +y[n]_q^2 \sum_{\sigma\in\mathcal{D}(n-1,n_2,\ldots,n_k)} w(\pi).\end{aligned}$$