ON CONJUGACY CLASSES OF NILPOTENT MATRICES

RICCARDO BIAGIOLI (LYON),
SARA FARIDI (HALIFAX)
& MERCEDES ROSAS (SEVILLA),

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"THE DEFINING IDEALS OF CONJUGACY CLASSES OF

Origin of the problem

Partitions parametrize nilpotent matrices.

A nilpotent matrix of size n has Jordan canonical form

$$\begin{pmatrix} J_{\lambda_1} & 0 & 0 & 0 \\ 0 & J_{\lambda_2} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_{\lambda_m} \end{pmatrix}$$

where J_{λ_i} is Jordan blocks of size λ_i, and λ is a partition of n.

Let C_λ = Conjugacy class of nilpotent matrices with Jordan block sizes given by the partition λ.

Question. Describe the ideal $\mathcal{J}_\lambda$ of polynomial functions vanishing on C_λ.

J. Weyman [*The equations of conjugacy classes of nilpotent matrices. Invent. Math., (1989)*] conjectured a minimal set of generators for $\mathcal{J}_\lambda$.

Our strategy

Look at a related problem.

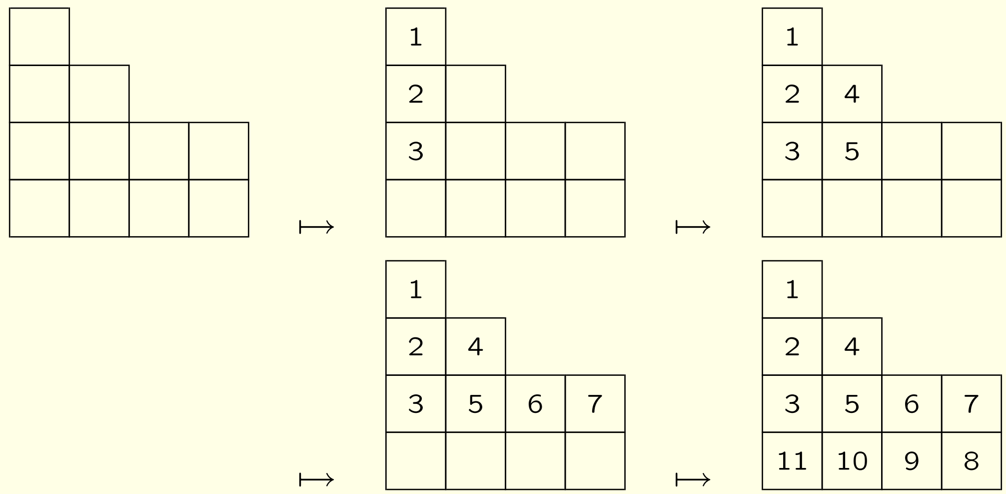
$\mathcal{I}_\lambda =$ *ideal of scheme theoretic intersection of* $\overline{C_\lambda}$ *with the set of diagonal matrices.*

Example 2×2 matrices.

De Concini and Procesi produced a specific generating set for $\mathcal{I}_\lambda$, and Tanisaki gave a simpler generating set, motivating much subsequent work.

first result, a simpler generating set for $\mathcal{I}_\lambda$.

Computing the generating set for $\mathcal{I}_\lambda$ when $\lambda = (4, 4, 2, 1) \vdash 11$.



Notation For an alphabet X, let $f(X)$ denote the function f evaluated at f. Moreover, for a given integer n, let $f(n)$ denote the union of all $f(X)$ for $|X| = n$.

Our generating set :

$$e_1(11), e_2(11), e_3(11), e_4(10), e_6(9), e_7(8)$$

Another presentation for our generating set :

The work of Weyman

Weyman uses the representation theory of the general linear group to construct and study two generating sets for the ideal $\mathcal{J}_\lambda$ of polynomial functions vanishing on the conjugacy class $\mathcal{C}_\lambda$.

The generators in the first family, denoted by V_λ, are expressed as sums of minors, and come from reducible representations of $GL(n)$.

The ideals V_λ are a sum of ideals $V_{i,p}$ where the points (i, p) are easily read from the Weyman diagram of λ.

The second set of generators U_λ arises from the irreducible representations of $GL(n)$.

Again, the sets U_λ are a sum of ideals $U_{i,p}$ for some points determined by the Weyman diagram of λ.

Weyman's diagrams

Weyman diagram for $\lambda = (4, 4, 2, 1)$.

$$\begin{array}{cccc} \boxed{1} & & & \\ \boxed{2} & \boxed{4} & & \\ \boxed{3} & \boxed{5} & \boxed{6} & \boxed{7} \\ \boxed{11} & \boxed{10} & \boxed{9} & \boxed{8} \end{array} = \begin{array}{ccccc} p=1 & \boxed{X} & & & \\ p=2 & \boxed{X} & & & \\ p=3 & \boxed{X} & & & \\ p=4 & \boxed{X} & \boxed{X} & & \\ p=5 & X & X & & \\ p=6 & X & X & \boxed{X} & \\ p=7 & X & X & X & \boxed{X} \\ p=8 & X & X & X & X \\ p=9 & X & X & X & \\ p=10 & X & X & & \\ p=11 & X & & & \\ i & 0 & 1 & 2 & 3 \end{array}$$

We need generators

$$\begin{array}{rr} U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}, & 0^{th} \text{ column} \\ U_{1,4} & 1^{st} \text{ column} \\ U_{2,6} & 2^{nd} \text{ column} \end{array}$$

Weyman's Conjecture, '89

Let λ be a partition. The set consisting of $U_{0,p}$ for $1 \leq p \leq \ell(\lambda)$, and $U_{i,p}$, where (i,p) labels a top cell of the i-th row (in the Weyman diagram of λ), such that there are no X's to the right of or on the line segment joining (i,p) with $(0,1)$, is a minimal set of generators $\mathcal{W}_\lambda$ of $\mathcal{J}_\lambda$.

$$
\begin{array}{cccccc}
1 & \boxed{\underline{X}} & & & & \\
2 & \underline{X} & & & & \\
3 & \underline{X} & & & & \\
4 & \underline{X} & \boxed{\underline{X}} & & & \\
5 & X & X & & & \\
6 & X & X & & & \\
7 & X & X & \blacksquare & & \\
8 & X & X & X & & \\
9 & X & X & X & & \\
10 & X & X & X & \blacksquare & \\
11 & X & X & X & X & \\
12 & X & X & X & X & \underline{X} \\
13 & X & X & X & X & \\
14 & X & X & X & & \\
15 & X & X & & & \\
16 & X & & & & \\
i = & 0 & 1 & 2 & 3 & 4
\end{array}
$$

The shaded cells are redundant.

Our generating set, an example

1				
2	3	4	5	
10	9	8	7	6

Let $\lambda = (5, 4, 1)$,

$e_3(9) = h_3(1) = m_{(3)}$: Add $x_1^3, \cdots, x_{10}^3$.

$e_4(8) = h_4(2) = m_{(3,1)}(2) + m_{(4)}(2) + m_{(2)}(2)$.
But $m_{(3,1)}(2)$ and $m_{(4)}(2)$ are in the ideal and $m_{(2,2)}(2) = (x_i x_j)^2$, with $i \neq j$ is not. Add $m_{(2,2)}(2)$ to the generating set.

$e_5(7) = h_5(3)$. Partitions : $(2, 2, 1)$, $(3, 1, 1)$, $(3, 2)$, $(4, 1)$, (5). They are already in the ideal.

$e_6(6) = h_6(4)$. Partitions : $(2, 2, 1, 1)$, $(3, 1, 1, 1)$, $(2, 2, 2)$, $(3, 2, 1)$ ... They are already in the ideal.

$p = 1$	$\underline{X}$				
$p = 2$	$\underline{X}$				
$p = 3$	$\underline{X}$	$\underline{X}$			
$p = 4$	X	X	$\underline{X}$		
$p = 5$	X	X	X	$\blacksquare$	
$p = 6$	X	X	X	X	$\blacksquare$
$p = 7$	X	X	X	X	
$p = 8$	X	X	X		
$p = 9$	X	X			

Theorem. [The case of partially-rectangular partitions]
Let λ be a partition of n, and let $k > 2$ be any integer. If columns $0, 1, \ldots, k-1$ of the Young diagram have the same height, then in the generating set for the ideal $\mathcal{I}_\lambda$ generators coming from columns $2, \ldots, k$ are redundant.

1	$g+1$	$*$	$*$	$*$		
					$*$	
g						
n	$n-1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

(Only the first two columns are relevant !)

Diagram for $(5, 4, 4, 3)$

1	$\boxed{\underline{X}}$				
2	$\underline{X}$				
3	$\underline{X}$				
4	$\underline{X}$	$\boxed{\underline{X}}$			
5	X	X			
6	X	X			
7	X	X			
8	X	X	X		
9	X	X	X		
10	X	X	X		
11	X	X	X	X	
12	X	X	X	X	X

An infinite family of counterexamples

Theorem. If $\lambda = (u^a, (u-1)^c, 1)$ with $u \geq 3$ and $g = a + c > 1$, then $\mathcal{I}_\lambda$ is generated by

$$e_1(n), \ldots, e_g(n), x_1^{g+1}, \ldots, x_n^{g+1},$$
$$(x_1x_2)^g, (x_1x_3)^g, \ldots, (x_{n-1}x_n)^g.$$

Only the first three columns are relevant.

Diagram for $(5, 5, 5, 1)$.

1	X				
2	$\boxed{\underline{X}}$				
3	X				
4	X	$\boxed{\underline{X}}$			
5	X	X			
6	X	X	$\boxed{\underline{X}}$		
7	X	X	X		
8	X	X	X	$\blacksquare$	
9	X	X	X	X	
10	X	X	X	X	$\blacksquare$
11	X	X	X	X	X
12	X	X	X	X	X
13	X	X	X	X	
14	X	X	X		
15	X	X			
16	X				

A second infinite family of counterexamples

Theorem. If $\lambda = (u^a, (u-1)^c, 1, 1)$ with $u \geq 4$ and $g = a + c + 1 > 2$, then $\mathcal{I}_\lambda$ is generated by

$e_1(n), \ldots, e_g(n)$, $x_1^{g+1}, \ldots, x_n^{g+1}$, $(x_i + x_j)(x_i x_j)^{g-1}$ for all $i \neq j$, $(x_i x_j x_k)^{g-1}$ for all $i < j < k$.

Diagram for $(5, 5, 1, 1)$.

1	$\underline{X}$				
2	$\underline{X}$				
3	$\boxed{\underline{X}}$				
4	$\underline{X}$	$\boxed{\underline{X}}$			
5	X	X	$\boxed{\underline{X}}$		
6	X	X	X	$\boxed{\underline{X}}$	
7	X	X	X	X	$\blacksquare$
8	X	X	X	X	X
9	X	X	X	X	
10	X	X	X		
11	X	X			
12	X				
$i =$	0	1	2	3	4

Can Weyman's conjecture be fixed ? Let λ be a partition and draw the Weyman diagram of λ.

If the $X's$ at the top of columns $1, 2, \ldots, r$ are collinear, and the line containing them passes through the point $(0, k)$, then the generators coming from columns $k+1, \ldots, r$ redundant.

- We know this holds for $k = 1, 2$, and 3.

- For $k = 4$, we used Macaulay2 to verify whether the statement is still true for the smallest possible member of this family, the partition (6,5,1,1,1)

  The Weyman diagram for $(6, 5, 1, 1, 1)$:

$$
\begin{array}{llllllll}
1 & \underline{X} & & & & & \\
2 & \underline{X} & & & & & \\
3 & \underline{X} & & & & & \\
4 & \boxed{\underline{X}} & & & & & \\
5 & \underline{X} & \boxed{\underline{X}} & & & & \\
6 & X & X & \boxed{\underline{X}} & & & \\
7 & X & X & X & \boxed{\underline{X}} & & \\
8 & X & X & X & X & \boxed{\underline{X}} & \\
9 & X & X & X & X & X & \blacksquare \\
10 & X & X & X & X & X & \\
11 & X & X & X & X & &
\end{array}
$$

Counterexample to Weyman's original conjecture

Consider the partition $(4,3,1)$, the points $(1,3)$, $(2,4)$ and $(3,5)$ are collinear, but the line that contains them does not pass through $(0,1)$. So according to Weyman's conjecture, all these cells contribute generators to a minimal generating set of $\mathcal{J}_{(4,3,1)}$.

However, our previous results suggest that the generators coming from cell $(3,5)$ may be redundant.

$$
\begin{array}{ccccc}
1 & \underline{X} & & & \\
2 & \boxed{\underline{X}} & & & \\
3 & \underline{X} & \boxed{\underline{X}} & & \\
4 & X & X & \boxed{\underline{X}} & \\
5 & X & X & X & \blacksquare \\
6 & X & X & X & \\
7 & X & X & & \\
8 & X & & & \\
i= & 0 & 1 & 2 & 3
\end{array}
$$

Using Macaulay 2, we computed the minimal generating set for $\mathcal{J}_{(4,3,1)}$ and verified that this is indeed the case. We conclude that $(4, 3, 1)$ is a counterexample to Weyman's original conjecture.

Degrees	**Weyman's conjecture**	**Minimal number of generators**
1	1	1
2	1	1
3	64	64
4	720	720
5	2352	redundant
Total	3138	786

To summarize, in this particular case, Weyman's conjecture predicts that we need 3138 generators, but only 786 of them are really necessary.

We end the presentation with a natural question.