Dyck partitions, quasi-minuscule quotiens and Kazhdan-Lusztig polynomials

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partly based on a joint work with

Francesco Brenti and **Mario Marietti**

1. Background

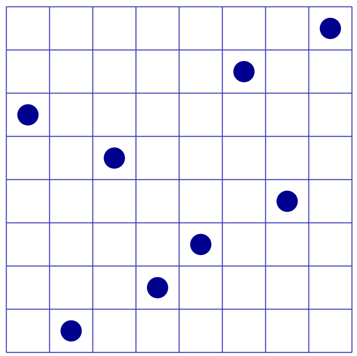
1.1 The symmetric group

$\mathbf{P} = \{1, 2, 3, \ldots\}$, $\quad [n] = \{1, 2, \ldots, n\} \quad (n \in \mathbf{P})$,

Symmetric group: $\quad S_n = \{v : [n] \to [n] \text{ bijection}\}$.

We denote $v \in S_n$ by the word $v(1)\,v(2)\ldots v(n)$ and by its *diagram*.

Example. $\quad v = 61523748 \in S_8$ has diagram



S_n is a Coxeter group, with generators the simple transpositions:

$$S = \{(1,2),(2,3),\ldots,(n-1,n)\}.$$

When we refer to these generators, the transposition $(i, i+1)$ is simply denoted by i. With this convention, the set of generators of S_n is

$$S = [n-1].$$

Let $J \subseteq [n-1]$. The quotient of S_n by J is

$$(S_n)^J = \{v \in S_n : v^{-1}(r) < v^{-1}(r+1) \text{ for all } r \in J\}.$$

The *maximal quotients* of S_n are obtained by taking

$$J = [n-1] \setminus \{i\} \quad (i \in [n-1]).$$

The *quasi-minuscule quotients* of S_n are obtained by taking

$$J = [n-1] \setminus \{i-1, i\} \quad (2 \leqslant i \leqslant n-1)$$

or

$$J = [n-1] \setminus \{1, n-1\}.$$

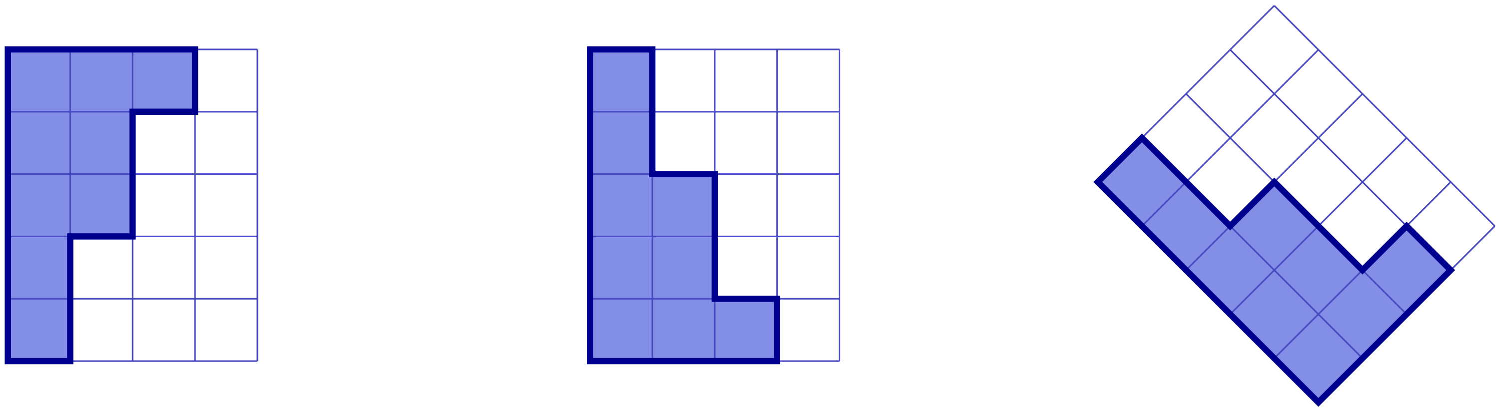
In this talk we study the quasi-minuscule quotiens of S_n.

1.2 Partitions and lattice paths

We identify a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (n^m)$ with its *diagram*:

$$\{(i,j) \in \mathbf{P}^2 : 1 \leqslant i \leqslant k \text{ and } 1 \leqslant j \leqslant \lambda_i\}.$$

Example. $\lambda = (3,2,2,1,1) \subseteq (4^5)$.



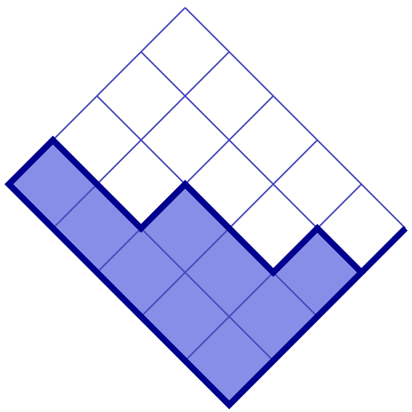
English notation | French notation | Our notation (Russian)

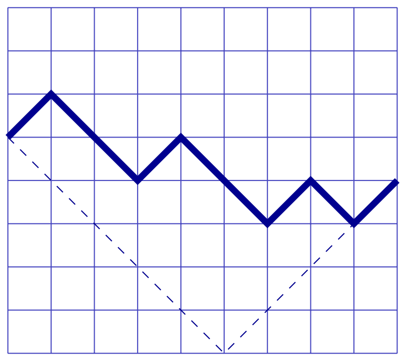
Given a partition $\lambda \subseteq (n^m)$, the *path* associated with λ is the lattice path from $(0, m)$ to $(n+m, n)$, with steps $(1,1)$ (up steps) and $(1,-1)$ (down steps) which is the upper border of the diagram of λ:

$$\text{path}(\lambda) = x_1 x_2 \ldots x_{n+m}, \quad x_k \in \{\text{U}, \text{D}\},$$

Note that path(λ) has exactly n U's and m D's.

Example. $\lambda = (3, 2, 2, 1, 1) \subseteq (4^5)$.

$\lambda =$ 

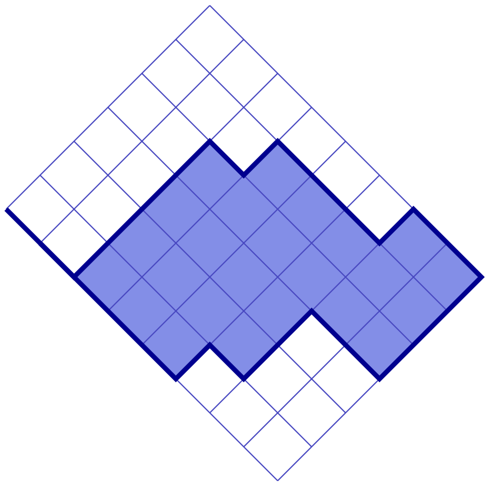
path(λ) =  = UDDUDDUDU

Let $\lambda, \mu \in \mathcal{P}$, with $\mu \subseteq \lambda$. Then we call $\lambda \setminus \mu$ a *skew partition*.

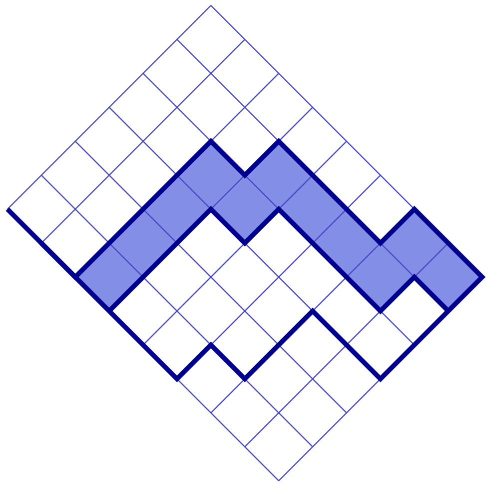
A skew partition is a *border strip* (also called a *ribbon*) if it contains no 2×2 square of cells. For brevity, we call a connected (by which we mean "rookwise connected") border strip a *cbs*.

The *outer border strip* θ of $\lambda \setminus \mu$ is the set of cells of $\lambda \setminus \mu$ such that the cell directly above it is not in $\lambda \setminus \mu$.

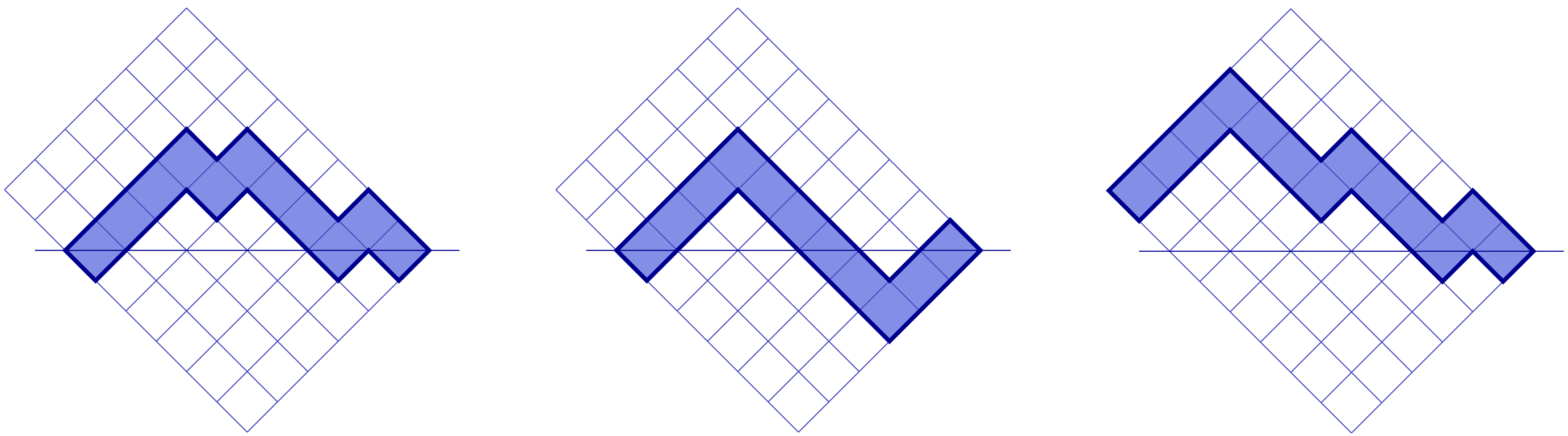
skew partition $\lambda \setminus \mu$



outer border strip of $\lambda \setminus \mu$



A cbs $\theta \subset \mathbf{P}^2$ is called a *Dyck cbs* if it is a “Dyck path”, which means that no cell of θ has level strictly less than that of either the leftmost or the rightmost of its cells. (In particular, in a Dyck cbs the leftmost and rightmost cells have the same level.)



Dyck non-Dyck non-Dyck

Let $\lambda \setminus \mu \subset \mathbf{P}^2$ be a skew partition.

Recall that $\lambda \setminus \mu$ is defined to be *Dyck* in the following inductive way:

(1) the empty partition is *Dyck*,

(2) if $\lambda \setminus \mu$ is connected, then $\lambda \setminus \mu$ is *Dyck* if and only if
(a) its outer border strip θ is a Dyck cbs,
(b) $(\lambda \setminus \mu) \setminus \theta$ is Dyck,

(3) if $\lambda \setminus \mu$ is not connected, then $\lambda \setminus \mu$ is *Dyck* if and only if all of its connected components are Dyck.

Let $\lambda \setminus \mu \subset \mathbf{P}^2$ be a skew partition (not necessarily Dyck).

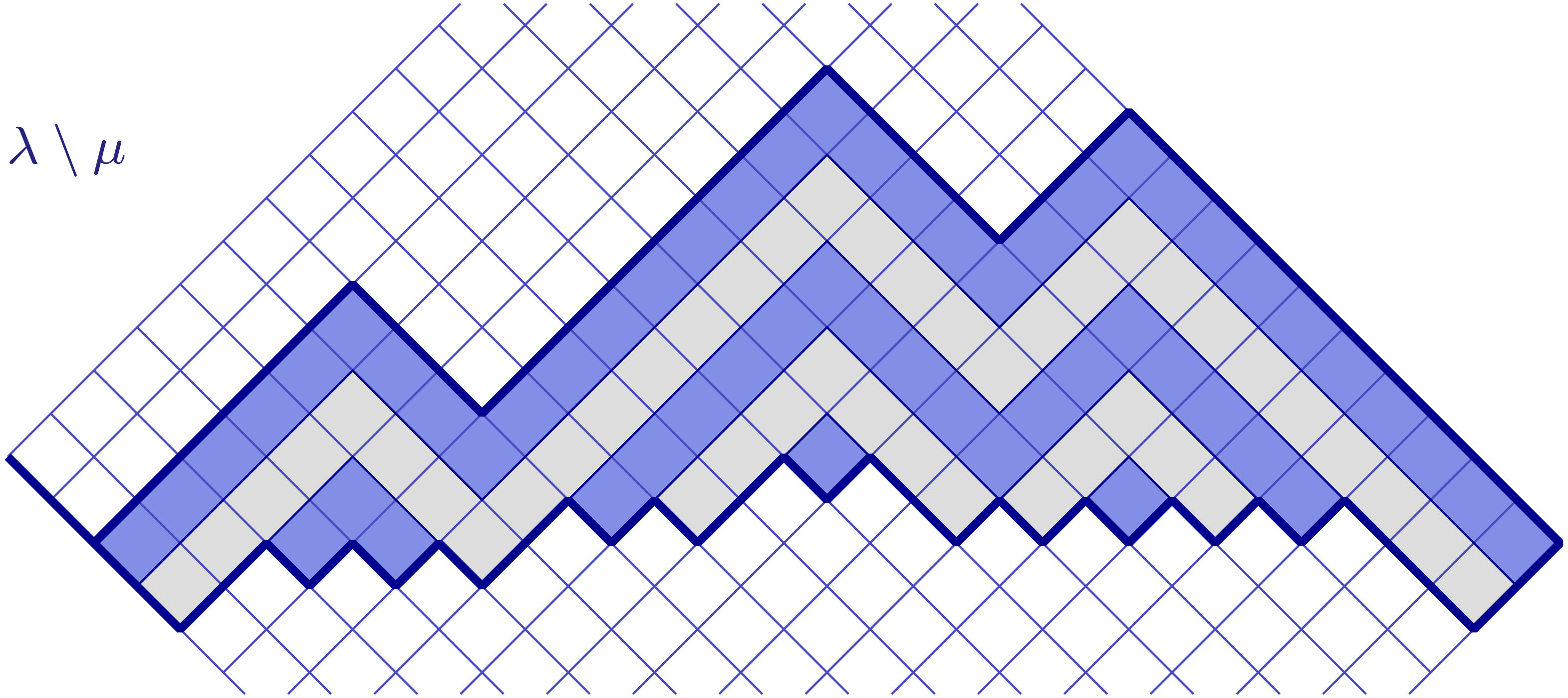
The *depth* of $\lambda \setminus \mu$ is defined inductively by

$$\mathrm{dp}(\lambda \setminus \mu) = \begin{cases} 0, & \text{if } \lambda = \mu, \\ c(\theta) + \mathrm{dp}((\lambda \setminus \mu) \setminus \theta), & \text{otherwise,} \end{cases}$$

where θ is the outer border strip of $\lambda \setminus \mu$ and

$$c(\theta) = \#\ \text{connected components of } \theta.$$

Example. Dyck skew partition:



$$\mathrm{dp}(\lambda \setminus \mu) = 8.$$

2. Parabolic Kazhdan-Lusztig polynomials

Theorem. (Deodhar, 1987) Let (W, S) be any Coxeter system and let $J \subseteq S$. Then, there is a unique family of polynomials

$$\{P^J_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbf{Z}[q]$$

such that, for all $u, v \in W^J$, with $u \leqslant v$, and fixed $s \in D(v)$, one has

$$P^J_{u,v}(q) = \widetilde{P}(q) - \sum_{\{u \leqslant w \leqslant vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w,v)}{2}} P^J_{u,w}(q),$$

where

$$\widetilde{P}(q) = \begin{cases} P^J_{us,vs}(q) + qP^J_{u,vs}(q), & \text{if } us < u, \\ qP^J_{us,vs}(q) + P^J_{u,vs}(q)\ , & \text{if } u < us \in W^J, \\ 0, & \text{if } u < us \notin W^J. \end{cases}$$

and

$$\mu(u, v) = \left[q^{\frac{\ell(u,v)-1}{2}} \right] (P^J_{u,v}).$$

The $P^J_{u,v}(q)$ are the *parabolic Kazhdan-Lusztig polynomials* of W^J.

For $J = \emptyset$, we get the (*ordinary*) *Kazhdan-Lusztig polynomials* of W:

$$P_{u,v}(q) = P^{\emptyset}_{u,v}(q).$$

Conversely, parabolic Kazhdan-Lusztig polynomials can be expressed in terms their ordinary counterparts.

Proposition. Let $J \subseteq S$, and $u, v \in W^J$. Then

$$P^J_{u,v}(q) = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q).$$

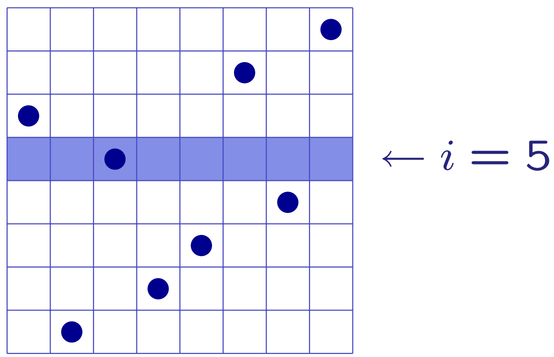
3. Quasi-minuscule quotiens

We will now give a combinatorial description of the quasi-minuscule quotients in S_n. We start with the following simple observation.

A permutation $v \in S_n$ belongs to $S_n^{[n-1]\setminus\{i-1,i\}}$ if and only if

$$v^{-1}(1) < \cdots < v^{-1}(i-1) \quad \text{and} \quad v^{-1}(i) < \cdots < v^{-1}(n).$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$



Let $\lambda \subseteq (n^m)$ be a partition and let

$$\mathsf{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{\mathsf{U}, \mathsf{D}\}.$$

We say that an index $k \in [n+m-1]$ is a

$$\begin{cases} \textit{valley} \text{ of } \lambda, & \text{if } (x_k, x_{k+1}) = (\mathsf{D}, \mathsf{U}), \\ \textit{peak} \text{ of } \lambda, & \text{if } (x_k, x_{k+1}) = (\mathsf{U}, \mathsf{D}). \end{cases}$$

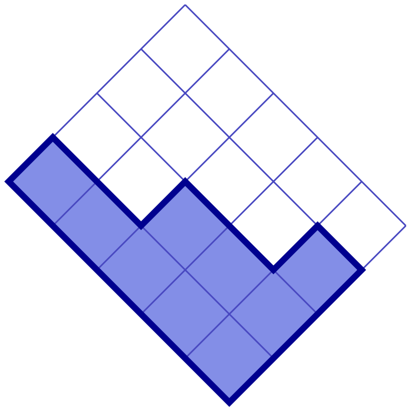
Definition. A *rooted partition* is a pair (λ, r), where λ is a partition with at least one valley and r is one of its valleys.

We think of a rooted partition as a lattice path with a ball in one of its valleys. If $\lambda \subseteq (n^m)$ and $\mathsf{path}(\lambda) = x_1 \ldots, x_{n+m}$, then

$$\mathsf{path}(\lambda, r) = x_1 \ldots x_r \bullet x_{r+1} \ldots x_{n+m}$$

Let $v \in S_n^{[n-1]\backslash\{i-1,i\}}$. The *partition* associated with v, denoted by $\Lambda(v)$, is the non-increasing rearrangement of the inversion table of v.

Example. $v = 61523748 \in S_8^{[7]\backslash\{4,5\}}$. Then

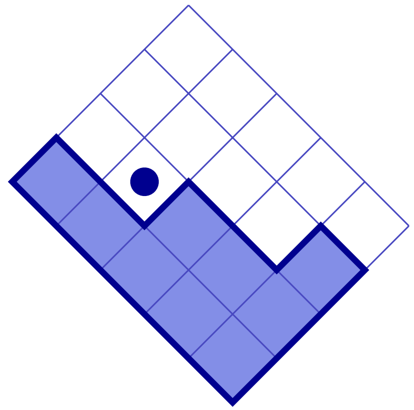
$\Lambda(v) = (3, 2, 2, 1, 1) =$ 

Remark. $\Lambda(v) \subseteq ((n-i+1)^i)$ and $v^{-1}(i)$ is a valley of $\Lambda(v)$.

The *rooted partition* associated with v is

$$\Lambda^\bullet(v) = (\Lambda(v), v^{-1}(i)).$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$. Then

$$\Lambda^\bullet(v) = ((3,2,2,1,1),3) =$$ 

Proposition. The map $v \mapsto \Lambda^\bullet(v)$ is a bijection

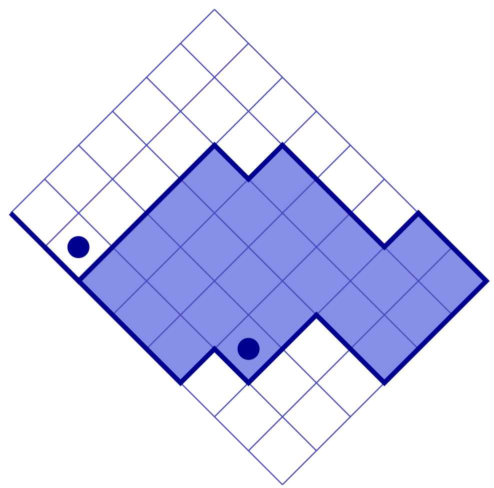
$$S_n^{[n-1]\setminus\{i-1,i\}} \longleftrightarrow \{\text{rooted partitions } \subseteq ((n-i+1)^i)\}.$$

Furthermore, $\ell(v) = |\Lambda(v)|$.

4. $\bullet$-Dyck partitions

This is the main new combinatorial concept arising from this work.

If (λ, r) and (μ, t) are two rooted partitions such that $\mu \subseteq \lambda$, then we call $(\lambda, r) \setminus (\mu, t)$ a *skew rooted partition*.



Definition. A skew rooted partition $(\lambda, r) \setminus (\mu, t)$ is $\bullet$*-Dyck* if

(1) there are no peaks of λ strictly between the two roots,

(2) at least one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck.

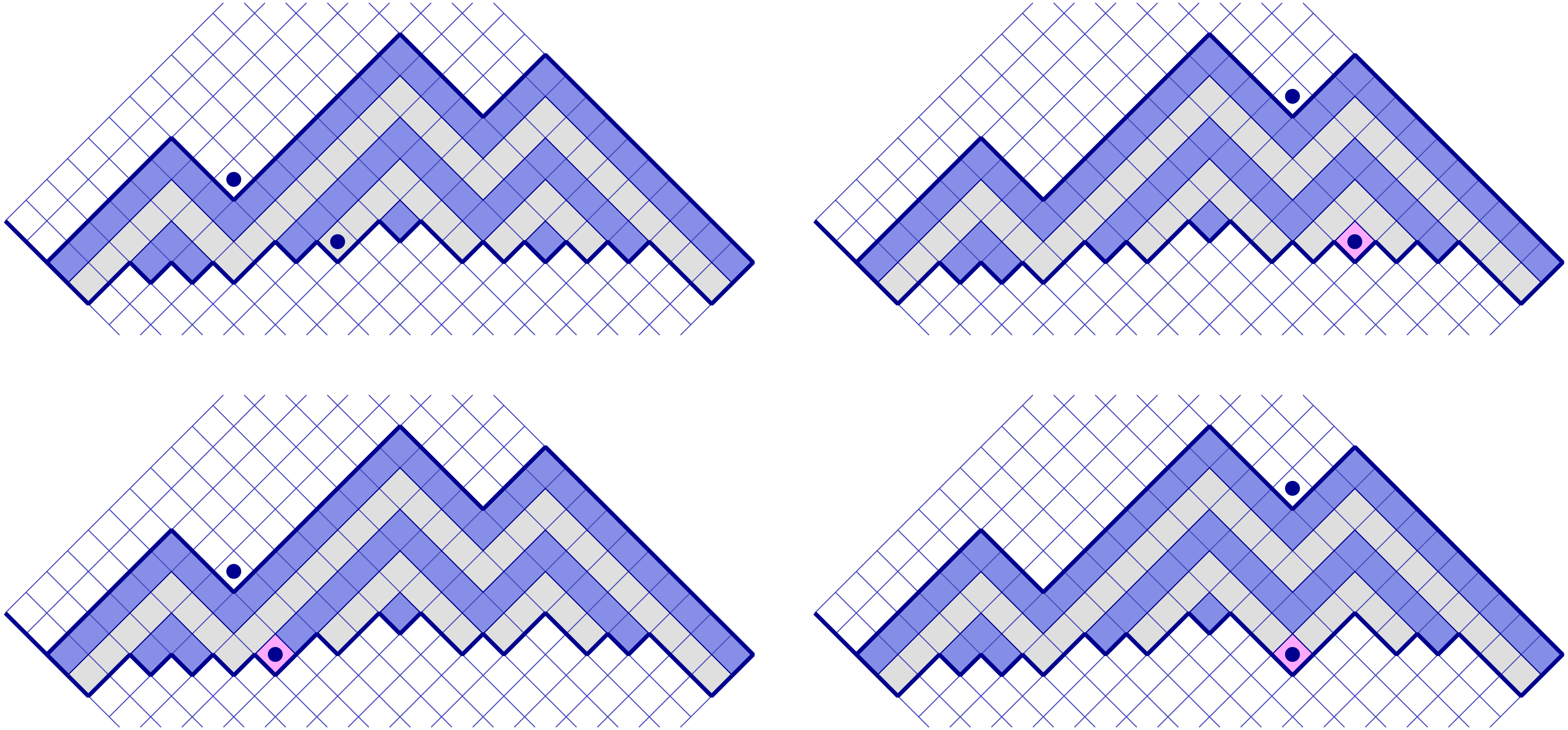
Let $(\lambda, r) \setminus (\mu, t)$ be $\bullet$-Dyck. The *depth* of $(\lambda, r) \setminus (\mu, t)$ is

$$\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \begin{cases} \mathrm{dp}(\lambda \setminus \mu), & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}(\lambda \setminus \mu^t) + 1, & \text{if } \lambda \setminus \mu^t \text{ is Dyck,} \end{cases}$$

Proposition. Let $\lambda \setminus \mu$ be skew partition and let t be a valley of μ. Suppose that at least one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck. Then $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are both Dyck if and only if t is a peak of λ. In this case,

$$\mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}(\lambda \setminus \mu^t) + 1.$$

Four $\bullet$-Dyck skew rooted partitions:



For all of them,

$$|\lambda \setminus \mu| = 98 \quad \text{and} \quad \mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 8.$$

5. Main result

Theorem. (Brenti, I., Marietti, 2008) Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with

$$\Lambda^{\bullet}(v) = (\lambda, r) \quad \text{and} \quad \Lambda^{\bullet}(u) = (\mu, t).$$

Then

$$P^J_{u,v}(q) = \begin{cases} q^{\frac{|\lambda\setminus\mu| - \mathrm{dp}((\lambda,r)\setminus(\mu,t))}{2}}, & \text{if } (\lambda, r) \setminus (\mu, t) \text{ is } \bullet\text{-Dyck}, \\ 0, & \text{otherwise.} \end{cases}$$

Example. If $(\lambda, r) \setminus (\mu, t)$ is one of the previous four, then

$$P^J_{u,v}(q) = q^{\frac{98-8}{2}} = q^{45}.$$

Our main result implies the analog result for *maximal quotients*, found by Brenti in [*Pacific Journal of Mathematics* **207** (2002), 257–286].

Corollary. (Brenti, 2002) Let $u, v \in S_n^{[n-1]\setminus\{i\}}$, with

$$\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.$$

Then

$$P^J_{u,v}(q) = \begin{cases} q^{\frac{|\lambda\setminus\mu| - \mathrm{dp}(\lambda\setminus\mu)}{2}}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases}$$

6. Enumerative results

6.1 Enumeration of Dyck partitions

Let $\lambda \subseteq (n^m)$ be a partition and consider the associated path

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{\mathsf{U}, \mathsf{D}\}.$$

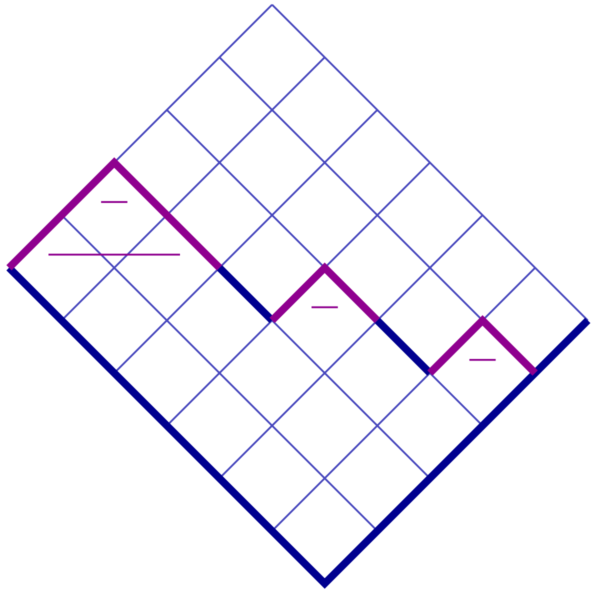
We make the substitution $\quad \mathsf{U} \longleftrightarrow (\qquad \mathsf{D} \longleftrightarrow)$.

We define the *matching set* and the *matching number* of λ by

$$M(\lambda) = \{k \in [n+m] : \text{parenthesis } x_k \text{ is matched}\},$$

$$\text{mtc}(\lambda) = \frac{|M(\lambda)|}{2} = \#\text{ pairs of matched parentheses in path}(\lambda).$$

Example. $\lambda = (4,3,3,2,2,2) \subseteq (5^6)$.

$\lambda =$ 

$\text{path}(\lambda) = \underline{(}\ \underline{(}\ \underline{)}\ \underline{)}\ )\ \underline{(}\ \underline{)}\ )\ \underline{(}\ \underline{)}\ ($

$M(\lambda) = \{1,2,3,4,6,7,10,11\}$

$\text{mtc}(\lambda) = 4$

In 2002, Brenti enumerated the partitions μ contained in a given partition λ such that $\lambda \setminus \mu$ is Dyck and found a q-analog formula.

This is a reformulation of his result.

Theorem. (Brenti, 2002) Let $\lambda \subseteq (n^m)$. Then

$$|\{\mu \subseteq \lambda : \lambda \setminus \mu \text{ is Dyck}\}| = 2^{\mathrm{mtc}(\lambda)}.$$

More generally, the following q-analog holds:

$$\sum_{\substack{\mu \subseteq \lambda \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\mathrm{dp}(\lambda \setminus \mu)} = (q+1)^{\mathrm{mtc}(\lambda)}.$$

Recently, *all* the Dyck skew partition contained in a given rectangle have been enumerated and a q-analog has been found.

Theorem. (I., August 2008)

$$|\{\lambda \setminus \mu \subseteq (n^m) \text{ Dyck}\}| = \sum_{k=0}^{\min\{n,m\}} \frac{n+m-2k+1}{n+m-k+1} \binom{n+m}{k} 2^k.$$

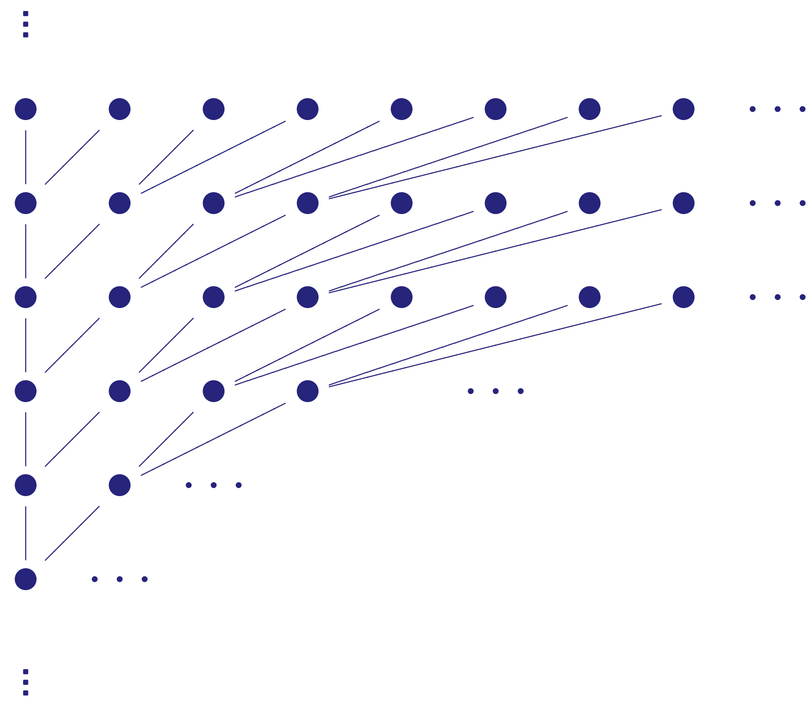
More generally, the following q-analog holds:

$$\sum_{\substack{\lambda \setminus \mu \subseteq (n^m) \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\mathrm{dp}(\lambda \setminus \mu)} = \sum_{k=0}^{\min\{n,m\}} \frac{n+m-2k+1}{n+m-k+1} \binom{n+m}{k} (q+1)^k.$$

6.2 Connection with paths on regular trees

For any integer $d \geqslant 2$, we denote by T_d the *d-regular tree*, that is the (infinite) tree where all the vertices have degree d.

T_3



Given two vertices x and y in a graph G, we denote by $\mathsf{Paths}_{G,\ell}(x, y)$ the set of all paths in G of length ℓ from x to y.

Theorem. (I., August 2008) Let $n, m \in \mathbf{P}$.

Let x, y be two vertices of T_3 at distance $|n - m|$. Then

$$|\{\lambda \setminus \mu \subseteq (n^m) : \lambda \setminus \mu \text{ is Dyck}\}| = |\mathsf{Paths}_{T_3,n+m}(x, y)|.$$

More generally, we have the following q-analog.

Let $q \in \mathbf{Z}_{\geqslant 0}$ and x, y be two vertices of T_{q+2} at distance $|n - m|$. Then

$$\sum_{\substack{\lambda\setminus\mu\subseteq(n^m) \\ \lambda\setminus\mu \text{ is Dyck}}} q^{\mathsf{dp}(\lambda\setminus\mu)} = |\mathsf{Paths}_{T_{q+2},n+m}(x, y)|.$$