Mixed connectivity

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A motivating application

(W, S) Coxeter group

Operations on words in the Coxeter generators $S = \{a, b, c, \ldots\}$

nil-move: deleting or adding a factor of the form aa
braid-move: replacement of a factor $ababa...$ by $babab....$

Ex: a sequence of two nil-moves and two braid-moves in H_3:

$$cb\underline{c}\,\underline{a}cbabac \sim cba\underline{c}\,\underline{c}babac \sim c\underline{b}\,\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}ac \sim cabab\underline{a}\,\underline{a}c \sim cababc$$

A motivating application

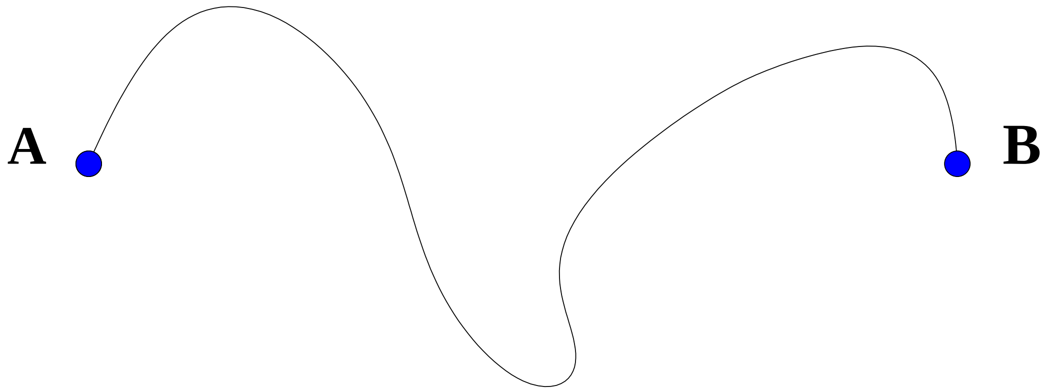
Let $A \subseteq W \setminus \{e\}$ and $w \in W \setminus A$. An expression $w = s_1 \cdots s_k$, $s_i \in S$, is *A-avoiding* if $s_1 \cdots s_i \notin A$ for $i = 1, 2, \ldots, k$.

Theorem 1. *Suppose that $|S| = r$, and let $w \in W \setminus A$.*
(1) If $|A| \leq r - 1$ there exists an A-avoiding expression

$$w = s_1 \cdots s_k$$

(2) If $|A| \leq r - 2$ then every pair of such A-avoiding expressions for w are connected by a sequence of A-avoiding nil-moves and braid moves.

Intuitive idea of **connected**



Higher connectivity:

- Graph theory: "k-connected"
- Topology: "k-connected"

Graph-theoretic k-connectivity

Def: Graph is k-connected if $|V(G)| \geq k+1$ and removal of any $\leq k-1$ vertices leaves connected induced subgraph.

Theorem. *(Menger, 1927): Graph G is k-connected $\Longleftrightarrow$ any pair of vertices is connected by k vertex-disjoint paths.*

Theorem. *(Balinski, 1961) The graph (1-skeleton) of a convex d-polytope is d-connected.*

Theorem. *(Barnette, 1973) The graph (1-skeleton) of a $(d-1)$-dimensional "graph-manifold" is d-connected.*

Theorem. *(Steinitz 1922) A graph G is the 1-skeleton of a convex 3-polytope*
$\Longleftrightarrow$ G is planar and 3-connected.

Topological k-connectivity

Def: Topological space $\mathbf{X}$ is k-connected if for all $j \leq k$ every mapping $S^j \to \mathbf{X}$ extends from the j-sphere $S^j = \partial B^{j+1}$ to the entire $(j+1)$-ball B^{j+1}.

Ex: $j = 0 \iff$ connected.

Ex: $j = 1 \iff$ simply-connected (i.e., fundamental group =0).

Concept fundamental in topological combinatorics,
e..g. in applications of Borsuk-Ulam

We now define "cell complex"

Not used to this stuff? Don't worry, just let

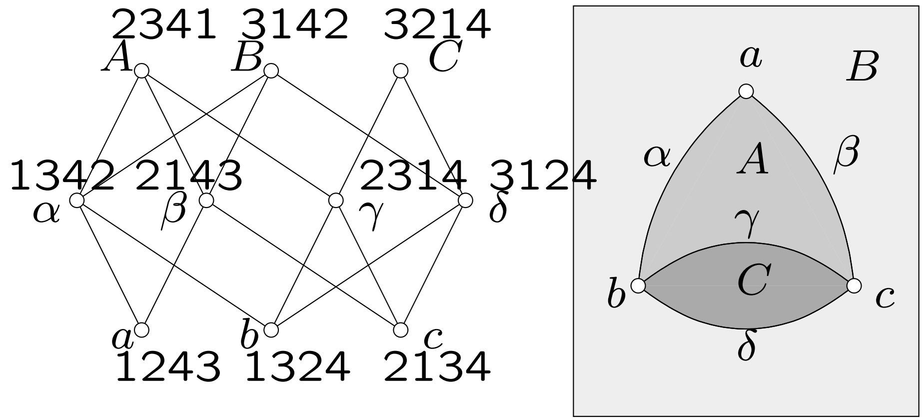
$$\text{cell complex} = \text{simplicial complex}$$

Topological k-connectivity

In this talk:

Cell complex $\overset{\text{def}}{=}$ regular CW complex
"with the intersection property" ($\cap$ cells = cell),
e.g. simplicial complex, polyhedral complex, ...

Example:



Regular CW complex (from the Bruhat interval $[1234, 3241]$ in S_4).
Does not have the intersection property.

Lemma 1. *Cell complex* Γ *is* k*-connected*

$\iff$ *its* $(k+1)$*-skeleton* $\Gamma^{\leq k+1}$ *is* k*-connected.*

Definition 1. *A cell complex is*

(k,t)*-biconnected*

if removal of any set of $\leq k-1$ *open cells (and the cells that contain any one of them in their closure) leaves a topologically* t*-connected induced subcomplex of the same dimension.*

Remark 1. Just removing vertices gives a weaker concept.
Counterex: 2 tetrahedron boundaries glued along edge.

Remark 2. Baclawski's concept of k-CM-connectivity earlier idea in this direction.

Theorem. *The boundary complex of a convex d-polytope is $(d-j, j)$-connected, for $j = 0, 1, \ldots, d-2$.*

- $j = 0$ case $\iff$ Balinski's theorem

- Homology version due to Fløystad (2005), using "enriched homology" and ring theory

- Proof method here based on poset homotopy tools applied to the face lattice

- Important point: method works also for other lattices, e.g. geometric lattices.

PLAN

First: A general theorem for posets

Then: three applications

- Cell complexes, polytopes and manifolds
- Coxeter groups
- Matroid basis graphs

Poset notions:

Order complex $\Delta(P)$ — simplicial complex of chains

$$x_0 < x_1 < \cdots < x_p$$

Length — $\ell(P)$, length (card -1) of longest chain

Length of interval — $\ell(x, y)$, length of $(x, y) = \{z : x < z < y\}$

Filter — up-directed subposet

Width of filter — number of its minimal elements

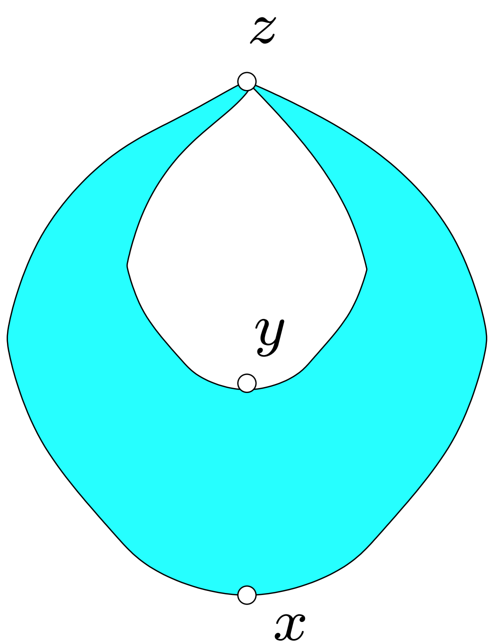
Fact for d-dimensional simplicial complex Δ:

- $(d-1)$-connected $\iff$ homotopy equiv to wedge of d-spheres

Def: Δ is d-spherical if $\dim \Delta = d$ and Δ is $(d-1)$-connected

Δ is spherical if it is $(\dim \Delta)$-spherical.

Definition 2. *A pure poset P is* locally rigid $\iff$ *for all $x < y \leq z$ in P the order complex of $(x, z) \setminus [y, z)$ is $(\ell(x, z) - 2)$-spherical.*



Examples of locally rigid posets:

- face posets of simplicial and polyhedral complexes (via Alexander duality ...)
- geometric (semi)lattices (via lex. shellability ...)
- the order duals of these

Definition 3. *A pure poset* P *is* (k,t)-rigid *if* $P \setminus F$ *is topologically* t*-connected, pure and of the same length as* P*, for every filter* $F \subset P$ *of width at most* $k-1$ *elements.*

Theorem 2. *Let P be a pure poset of length r, and let* $0 \leq t \leq r-1$*. Assume that*

(i) $P \cup \{\hat{0}\}$ *is a semilattice,*

(ii) $P \cup \{\hat{0}\}$ *is locally rigid,*

(iii) the upper interval $P_{>x}$ *is* $\min\{t, r-2-\mathrm{rk}(x)\}$*-connected, for all* $x \in P \cup \{\hat{0}\}$*.*

Then, the truncated poset $P^{\leq(s+1)}$ *is* $(r-s, s)$*-rigid,* $\forall s \leq t$*.*

Poset rigidity, main theorem

Proof. A bit technical ....

□

Comment "for the experts":

Comparison of conditions on intervals (x, y) in $\hat{P}$ in Thm 1 with the Cohen-Macaulay case:

	$y < \hat{1}$	$y = \hat{1}$
Thm 1	(x, y) locally rigid	$(x, \hat{1})$ . . . -connected
CM	(x, y) spherical	$(x, \hat{1})$ spherical

Stronger condition in red

Application 1: Polytopes and manifolds

Recall:

$$\textit{Cell complex} = \begin{cases} \bullet\, \text{regular CW complex} \\ \bullet\, \text{intersection property:} \bigcap \text{cells} = \text{cell} \\ \quad \Longleftrightarrow \text{ face poset is meet-semilattice} \end{cases}$$

Examples:

- simplicial complexes
- polyhedral complexes

Theorem. *The face semilattice of a cell complex is locally rigid*

Method yields

Theorem 3. *Let Γ be a pure cell complex.*
(1) If Γ is a d-dimensional compact manifold, then its 1-skeleton is graph-theoretically $(d+1)$-connected.
(2) If Γ is a d-dimensional compact manifold with boundary, then its 1-skeleton is graph-theoretically d-connected.

If Γ is polytope boundary: (1) $\Longrightarrow$ Balinski's theorem
If Γ is "graph-manifold": (1) $\Longrightarrow$ Barnette's theorem

— All that's needed: graph-theor. connectivity of links

Application 2: Coxeter groups

Consider a group W, and subset $S \subseteq W$ such that

$$s^2 = e, \quad \text{for all } s \in S,$$

(W, S) is Coxeter group $\Longleftrightarrow$ has Coxeter presentation:

Generators: S

Braid relations: for $s \neq s' \in S$

$$\underbrace{s\, s'\, s\, s'\, s \ldots}_{m(s,s')} = \underbrace{s'\, s\, s'\, s\, s' \ldots}_{m(s,s')}$$

Application 2: Coxeter groups

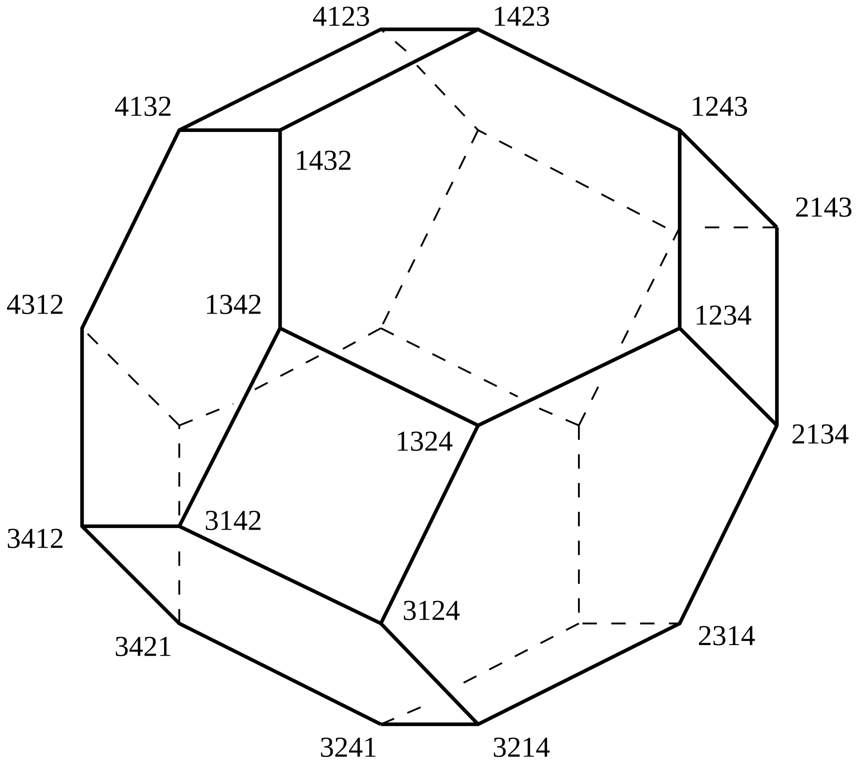
Build 2-dim'l cell complex $\Gamma_{(W,S)}$:

{vertices} = W
{edges} = pairs $w - ws$, $s \in S$
{2-cells} = braid relations

So, we glue some membranes (2-cells) into the Cayley graph.

FACT: (W, S) has the Coxeter presentation
$\iff$ the complex $\Gamma_{(W,S)}$ is 1-connected.

Application 2: Coxeter groups



The complex $\Gamma_{(W,S)}$ for S_4

4-gons $\leftrightarrow s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ 6-gons $\leftrightarrow s_i s_j = s_j s_i$.

Application 2: Coxeter groups

Operations on words in the Coxeter generators $S = \{a, b, c, \ldots\}$

nil-move: deleting or adding a factor of the form aa
braid-move: replacement of a factor $ababa...$ by $babab....$

Basic facts:

• Closed paths in the 1-skeleton of $\Gamma_{(W,S)}$ (the Cayley graph) correspond to relations in W (i.e., words evaluating to the identity)

• A homotopy in $\Gamma_{(W,S)}$ from one such path to another corresponds to a sequence of nil-moves and braid-moves.

Application 2: Coxeter groups

Let $A \subseteq W \setminus \{e\}$ and $w \in W \setminus A$. An expression $w = s_1 \cdots s_k$, $s_i \in S$, is *A-avoiding* if $s_1 \cdots s_i \notin A$ for $i = 1, 2, \ldots, k$.

Theorem 4. *Suppose that W is finite and $|S| = r$.*
(1) If $|A| \leq r - 1$ there exists an A-avoiding expression

$$w = s_1 \cdots s_k$$

(2) If $|A| \leq r - 2$ then every pair of such A-avoiding expressions are connected by a sequence of A-avoiding nil-moves and braid moves.

Proof. $\Gamma_{(W,S)}$ is the 2-skeleton of a polytope boundary, namely the r-dimensional dual zonotope, which is $(r-j, j)$-biconnected.
For (1), use the $j = 0$ case (the 1-skeleton) (Balinski)
For (2), use the $j = 1$ case (the 2-skeleton) □

Comments on Theorem 5

• Part (1) holds for all (W, S) whose Coxeter diagram has no ∞-labeled edges.
Combinatorial (non-topological) proof.

• Part (2) does not hold for infinite groups, e.g. fails for $\tilde{A}_3$.
Reason: affine plane minus point is not simply-connected.

Comments on Theorem 5

• Does there exist an A-avoiding "Tits word theorem", i.e. demanding that all intermediate expressions are A-avoiding and reduced? If so, for what size of A?

• Observation: $|A| \leq r - 2$ will not work. For example, let $W = D_4$, $S = \{a, b, c, d\}$ with a, b and c commuting, $A = \{b, ac\}$. Then

$$abc \sim cba$$

but not via A-avoiding length-preserving braid moves.

Application 3: Matroid basis graphs

Motivating problem:

Say we have a finite set V of vectors spanning $\mathbb{R}^d$.

Want to define "orientation" $+1$ or -1 for ordered bases $(b_1, b_2, \ldots, b_d)$.

Using determinants not allowed.

How to do it?

Application 3: Matroid basis graphs

The *basis graph* $\Gamma^1(M)$ of a matroid M has

$$\begin{cases} \text{vertices} = \text{the bases of } M \\ \text{edges} = \text{pairs of bases } (B_1, B_2) \text{ such that } |B_1 \cap B_2| = \text{rk}(M) - 1 \end{cases}$$

Connectivity?

Theorem. *(G.Z. Liu, 1990)* $\Gamma^1(M)$ *is* δ*-connected, where* δ *is the minimal vertex-degree.*

Note: best possible

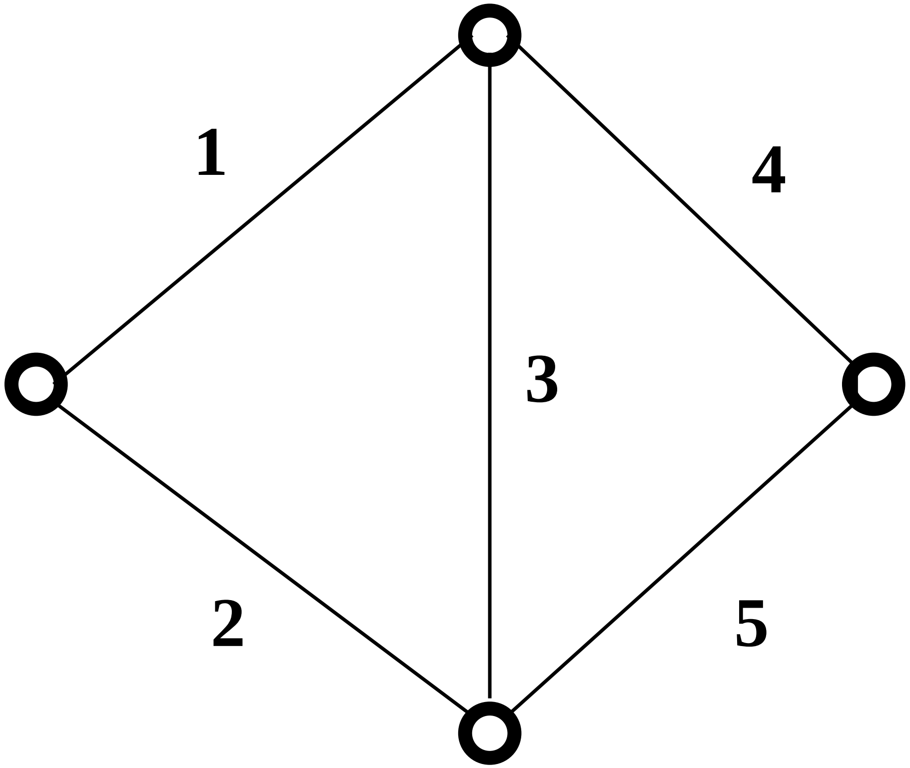
Application 3: Matroid basis graphs

Let M be a matroid of rank r on the ground set E.

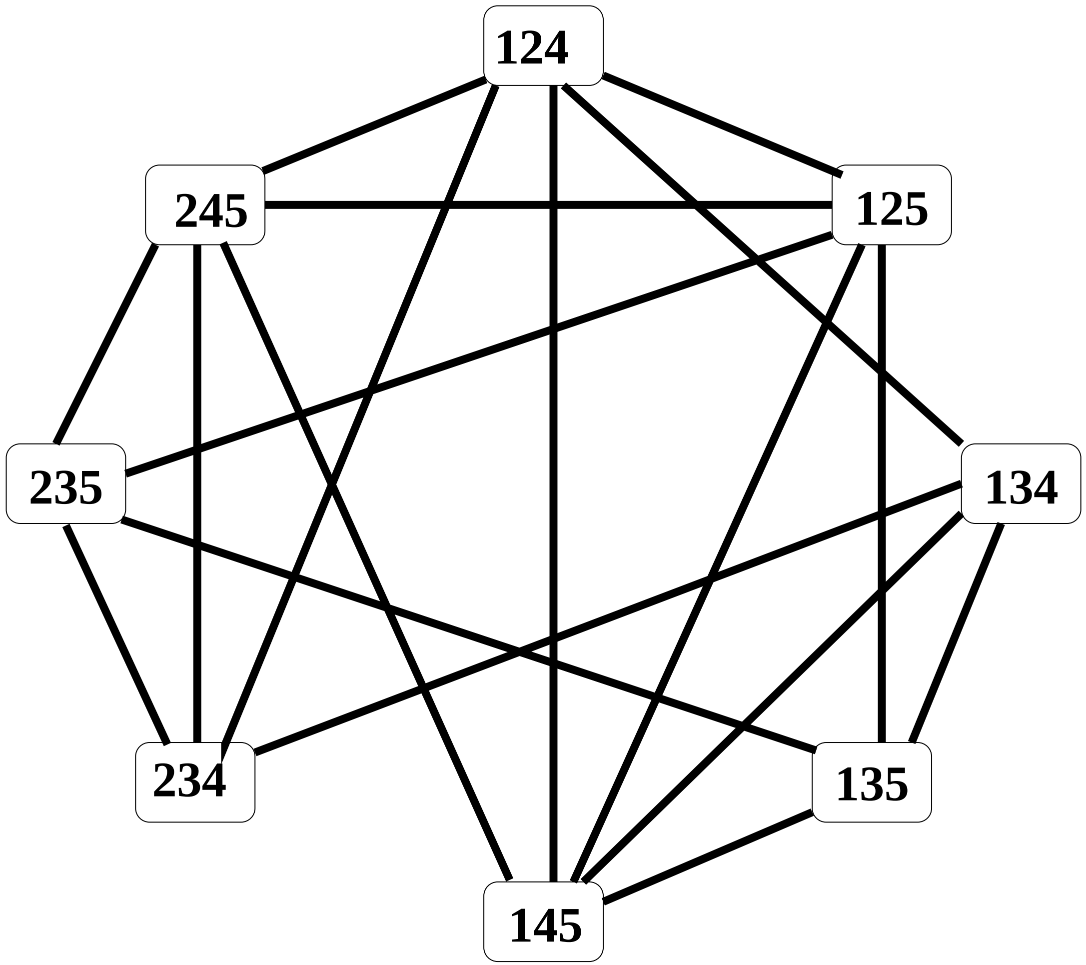
Def: M has the *disjoint basis property* if
for $\forall$ basis B $\exists$ a basis C such that $B \cap C = \emptyset$,
or else $E \setminus B$ is independent.

Def: For a basis B, an edge (B_1, B_2) is *B-related*
if $B_1 \cap B_2 \subset B$.

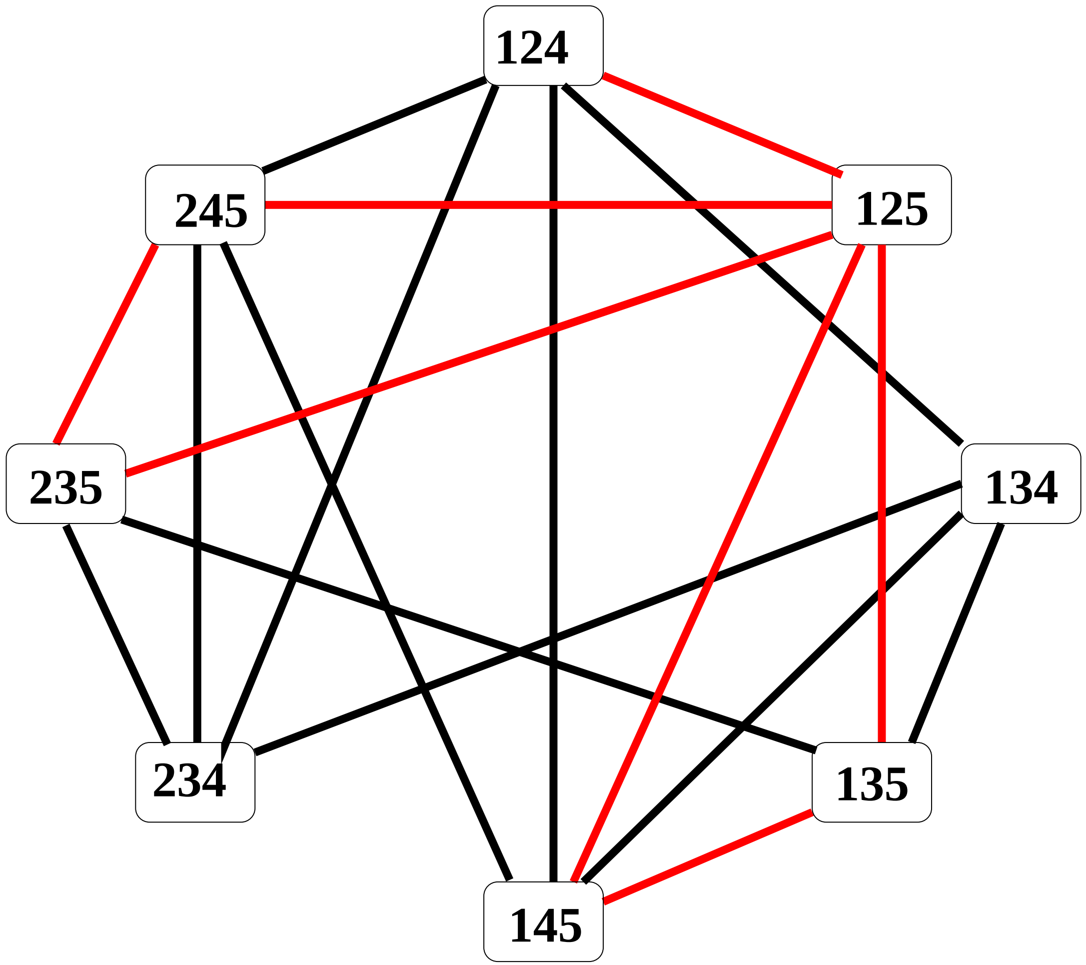
Example of basis graph: A graphic matroid



Example of basis graph



Example of basis graph: 125-related edges



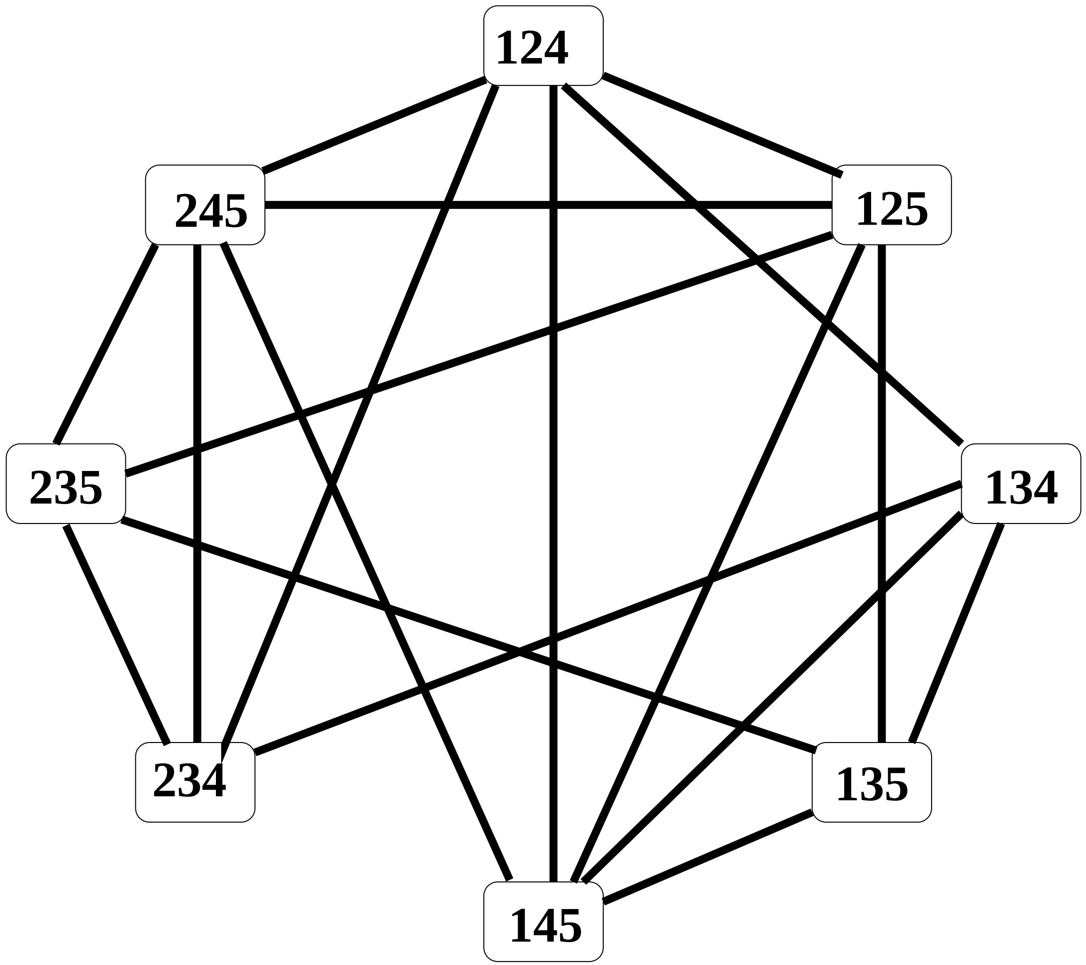
Theorem 5. *Let M be a matroid of rank r with the disjoint basis property. Then any collection of at most $r-1$ vertices and all related edges can be removed from its basis graph $\Gamma^1(M)$ without losing connectivity.*

Compare Liu's theorem: $\Gamma^1(M)$ is (graph-theoretically) δ-connected

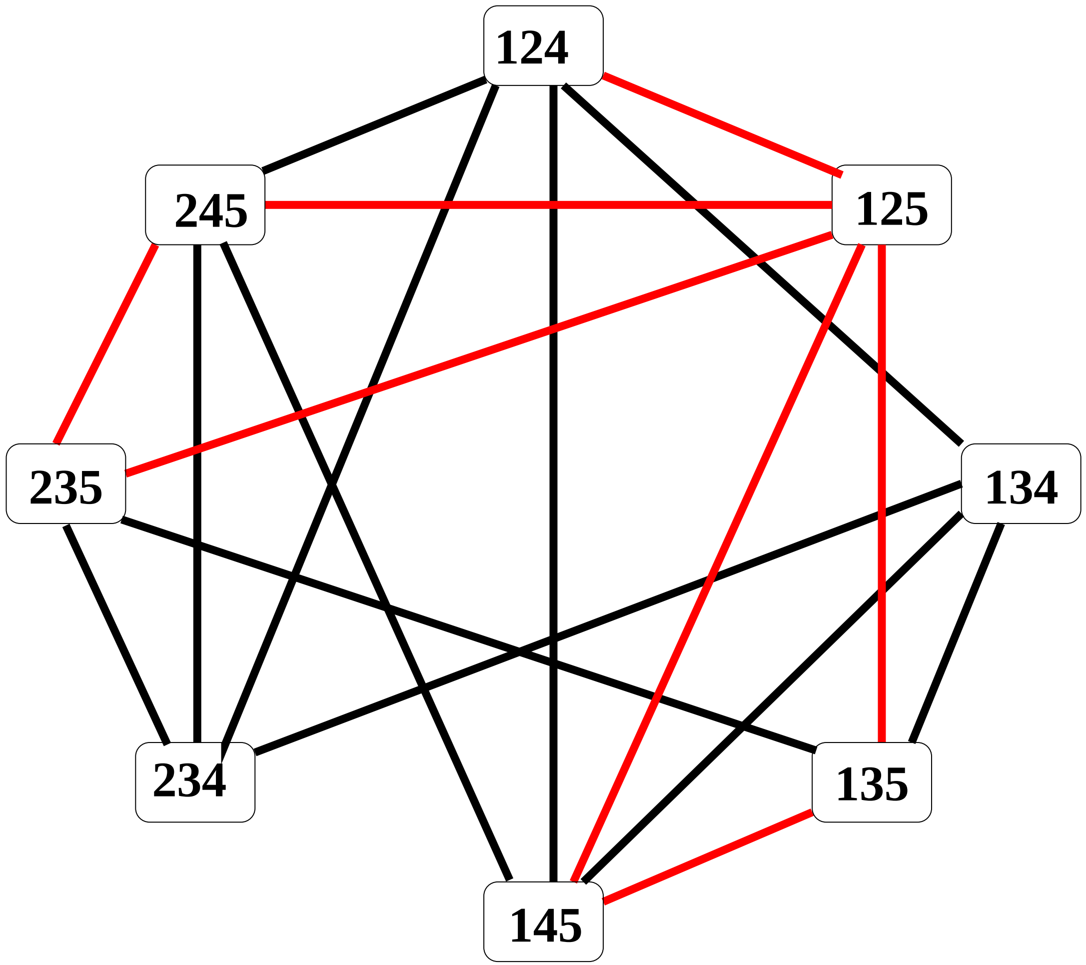
Comment: In Liu's theorem one removes all incident edges
— fewer edges, but more vertices $\delta \geq r, \ldots$

Neither result implies the other.

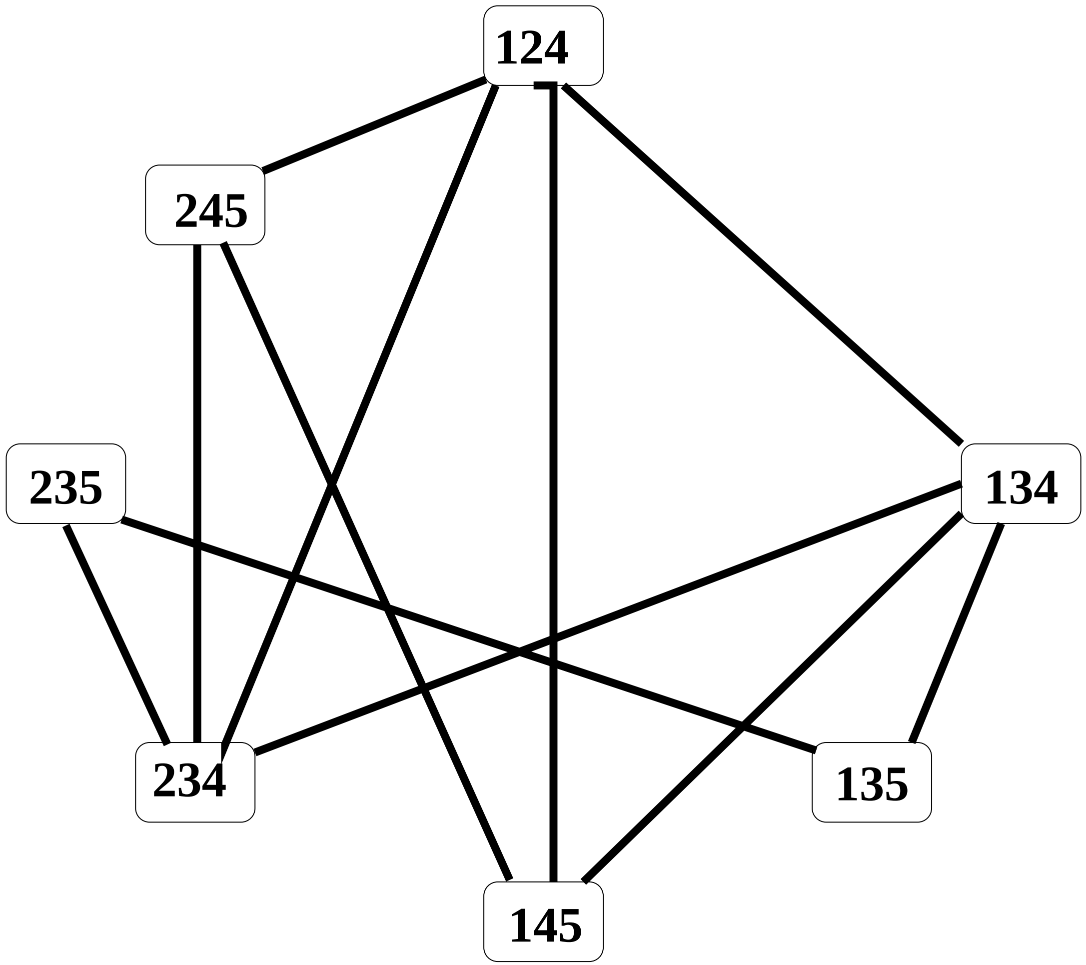
Example of basis graph



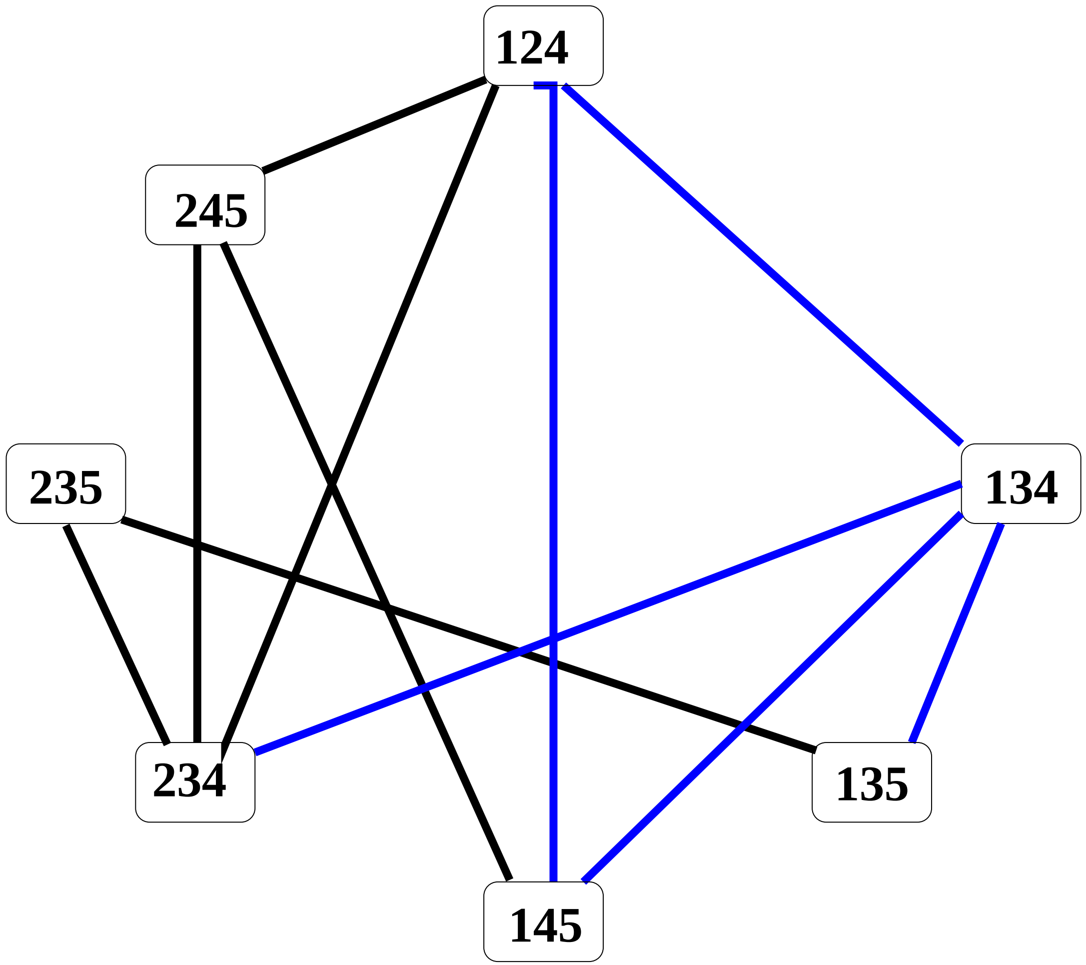
Example of basis graph: 125-related edges



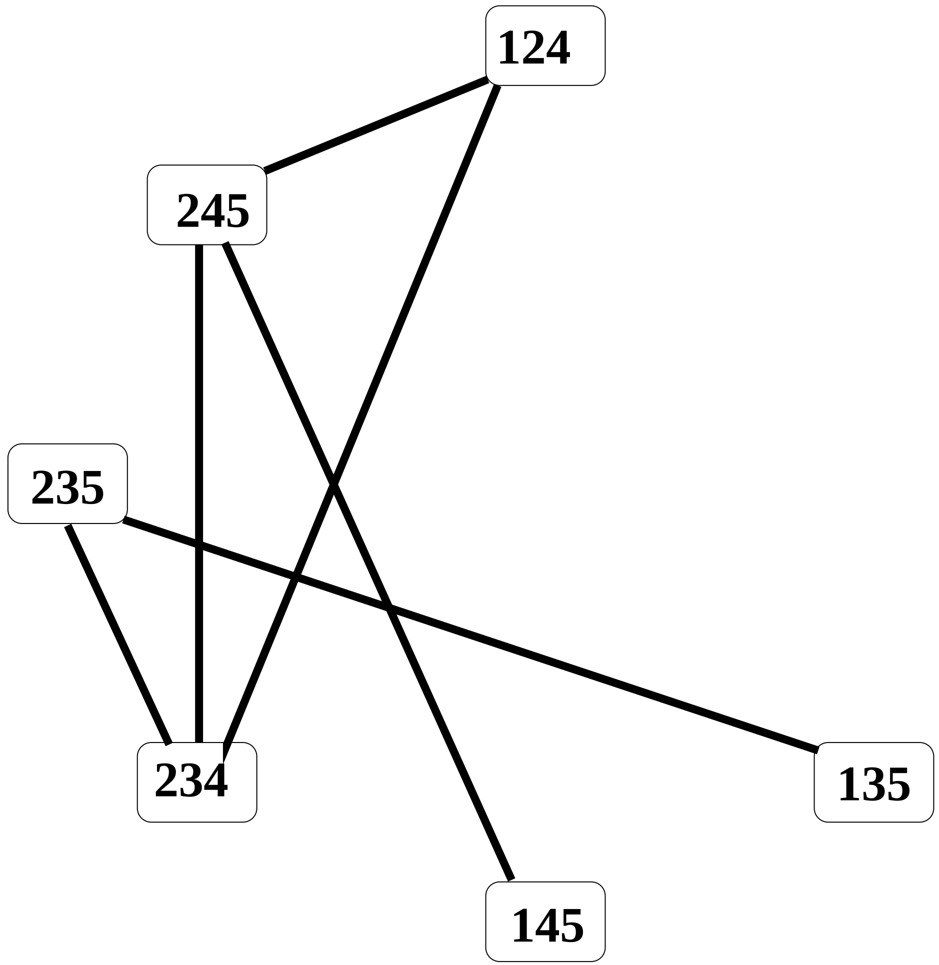
Example of basis graph: 125-related edges removed



Example of basis graph: 134-related edges



Example of basis graph: 125- and 134-related edges removed



Matroid basis complex

The *basis complex* $\Gamma^2(M)$ of a matroid M is the polyhedral complex obtained from the basis graph by gluing 2-cells (or "membranes") into all 3- and 4-cycles of the basis graph.

Theorem. *(Maurer, 1973)* $\Gamma^2(M)$ *is simply connected.*

Given a basis B, an 1-cell (edge) or a 2-cell is *B-related* if the intersection (as sets) of its vertices is a subset of B.

Theorem 6. *Let M be a matroid of rank r with the disjoint basis property. Then, if any collection of at most $r-2$ vertices and all related cells are removed from its basis complex $\Gamma^2(M)$, the remaining cell complex is 1-connected.*

Remark. These results can fail for matroids without the disjoint basis property.

Matroid basis graph — sketch of proof

Let M be a matroid of rank r with the disjoint basis property.
$P \stackrel{\text{def}}{=} (IN(M), \supseteq)$ — independent sets ordered by reverse inclusion (minimal elements = bases)

- $P \cup \{\hat{0}\}$ is locally rigid
- $P^{\leq 1}$ is $(r, 0)$-rigid, by main theorem
- $\exists$ order-pres map $f : \Gamma^1(M) \to P^{\leq 1}$
- fibers $f^{-1}((P^{\leq 1})_{\leq p})$ are sufficiently connected
- rigidity transfers back from $P^{\leq 1}$ to $\Gamma^1(M)$