Random walks on complex hyperplane arrangements

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Introduction: Tsetlin's library

- Shelf with n numbered books

- Choose book i with probability w_i, move it to front

Markov chain: States = permutations = S_n

Transition probabilities = $\begin{cases} w_i, & for''book-move'' \\ 0, & otherwise \end{cases}$

Studied also in CS: "dynamic file management", "cache management", ...

Much known: stationary distribution, eigenvalues of transition matrix P_w, ...

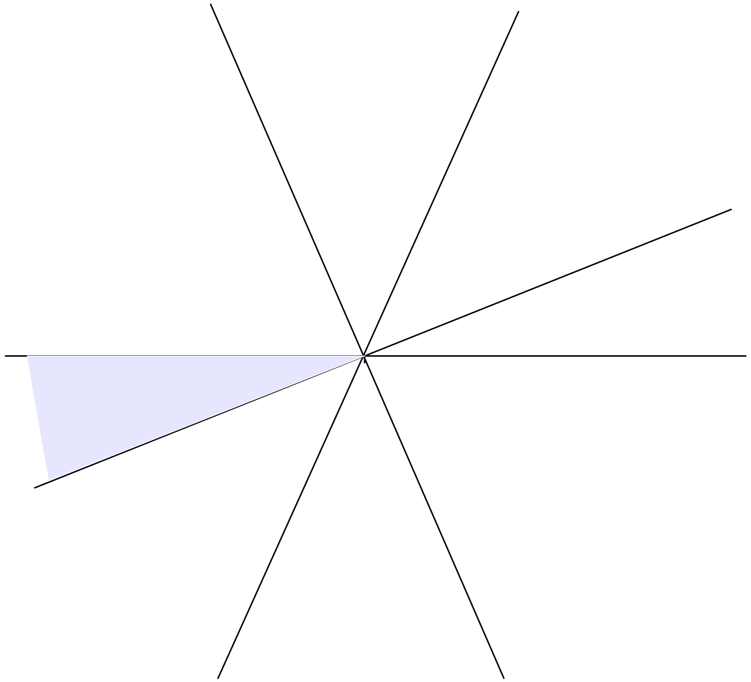
Theorem. *(Donnelly, Kapoor-Reingold, Phatarfod,* 1991*)*
For Tsetlin's library:

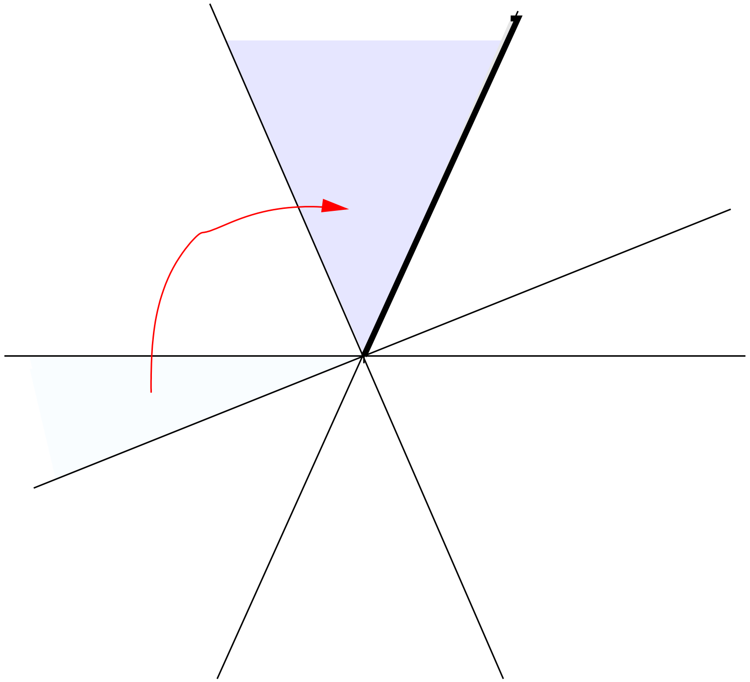
- *The eigenvalues* λ_E *of* P_w *are indexed by subsets* $E \subseteq [1, \ldots, n]$, *and* $\lambda_E = \sum_{i \in E} w_i$

- *The multiplicity of* λ_E *is the number of derangements of* $n - |E|$ *elements.*

Introduction

Libraries with one shelf (∼ Tsetlin), "random-to-front"	⟶	Random walks on complement of real hyperplane arrangements (Bidigare-Hanlon-Rockmore, 1998)



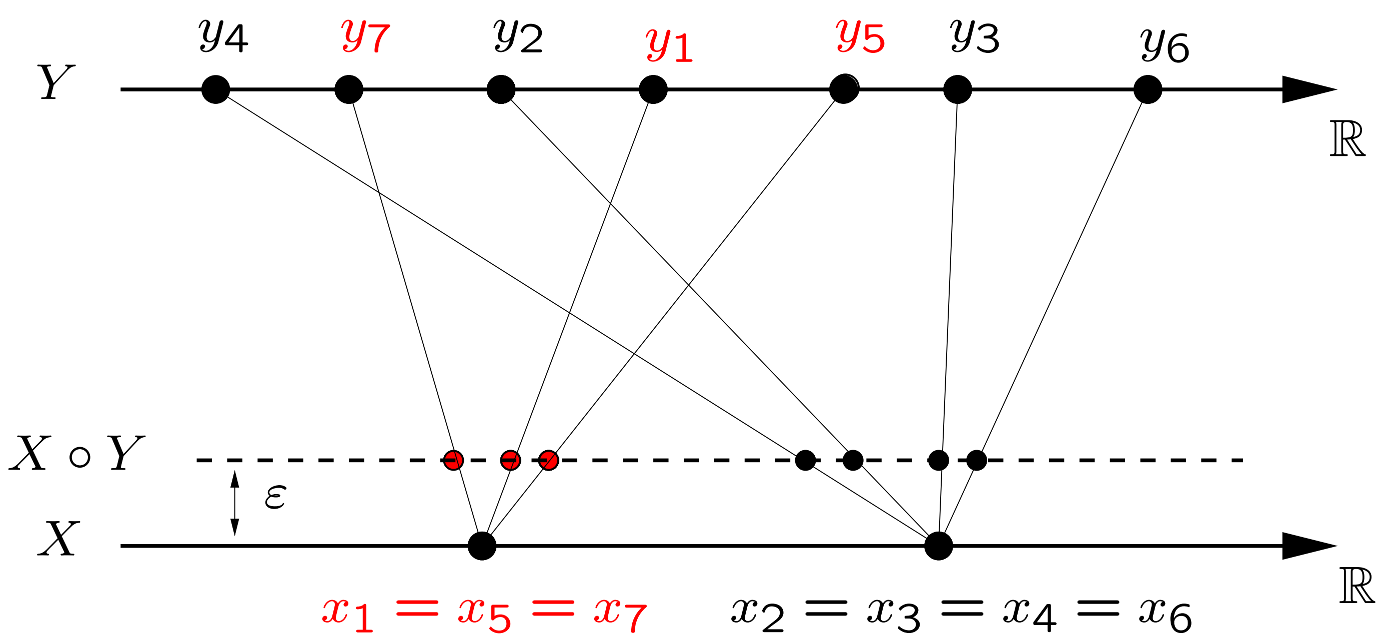


Introduction

Random-to-front shuffle:

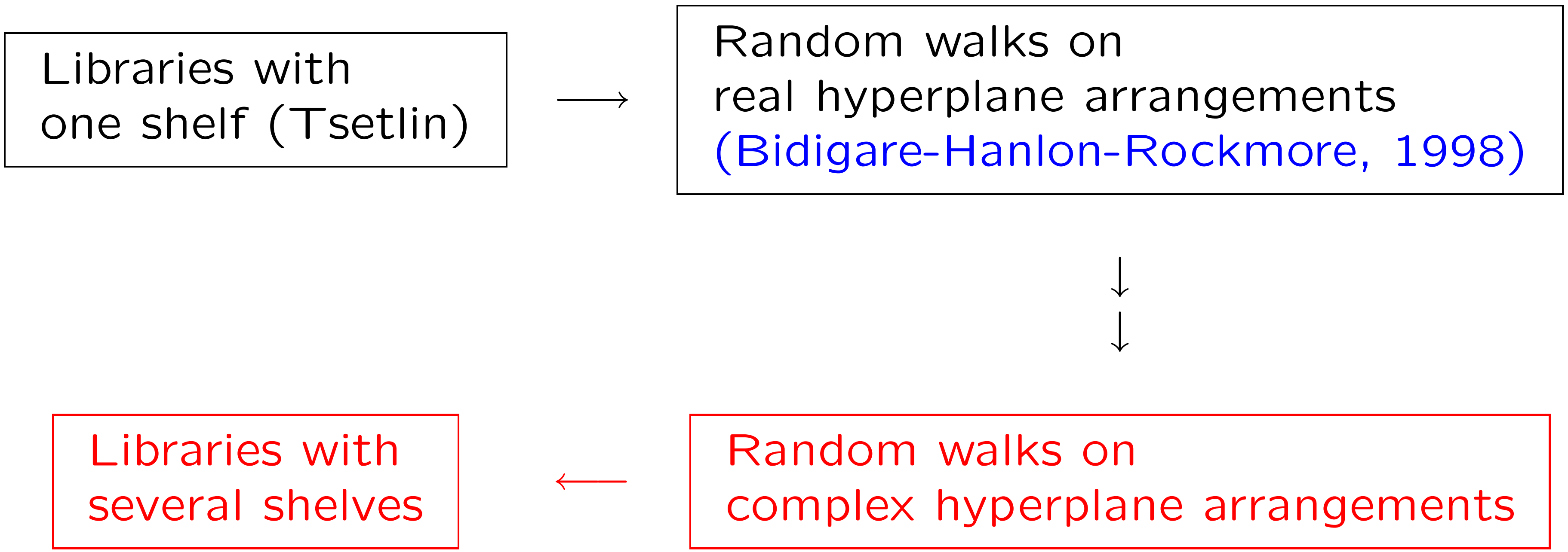
$$4\ 7\ 2\ 1\ 5\ 3\ 6 \quad \Longrightarrow \quad 7\ 1\ 5\ 4\ 2\ 3\ 6$$

Geometric version:



What about complex hyperplane arrangements?

Introduction



Introduction: Overview

Random walks on semigroups (Ken Brown, 2000)

↓ ↓ ↓

Random walks on $\mathbb{R}$-arrangements *	Random walks on $\mathbb{C}$-arrangements	Random walks on greedoids
↓	↓	
Libraries with one shelf (Tsetlin)	Libraries with several shelves	

* Bidigare-Hanlon-Rockmore (1998), Brown-Diaconis (1999)

Introduction: Library with several shelves

- k shelves with n numbered books, n_j books on shelf j
- Choose set of books $E \subseteq [n]$ with probability w_E
- Move chosen books to front of resp. shelf, in induced order
  AND
  Move affected shelves to top, in induced order

Markov chain: States = permutations = $S_{n_1} \times \ldots \times S_{n_k} \times S_k$

Transition probabilities = $\begin{cases} w_E, & for''book-borrow''move \\ 0, & otherwise \end{cases}$

Introduction

Example. Let $n = 3$ and $\pi = (1, 2 \mid 3)$.

Four library configurations. Transition matrix P_w:

	1 2 3	2 1 3	3 1 2	3 2 1
1 2 3	$w_1 + w_{1,2} + w_{1,3}$	$w_1 + w_{1,3}$	$w_1 + w_{1,2}$	w_1
2 1 3	$w_2 + w_{2,3}$	$w_2 + w_{1,2} + w_{2,3}$	w_2	$w_2 + w_{1,2}$
3 1 2	w_3	0	$w_3 + w_{1,3}$	$w_{1,3}$
3 2 1	0	w_3	$w_{2,3}$	$w_3 + w_{2,3}$

Example (cont'd). Let $n = 3$ and $\pi = (1, 2 \mid 3)$. Transition matrix P_w (previous slide) has four eigenvalues, all of multiplicity one:

$$\begin{cases} \varepsilon_1 = 0 \\ \varepsilon_2 = w_{1,3} + w_{2,3} \\ \varepsilon_3 = w_3 + w_{1,2} \\ \varepsilon_4 = 1 \end{cases}$$

Introduction

Theorem. *(Eigenvalues for the k-shelf library walk.)*
Let π be the partition of $\{1, \ldots, n\}$ into k blocks according to placement on shelves.

1. *For each pair of unordered partitions (α, β)such that $\alpha \leq \pi \leq \beta$ (i.e., β refines π and π refines α) there is an eigenvalue*

$$\varepsilon_{(\alpha,\beta)} = \sum w_E,$$

   the sum extending over all $E \subseteq [n]$ such that E is a union of blocks from β and the union of shelves containing some element of E is a union of blocks from α.

2. *The multiplicity of $\varepsilon_{(\alpha,\beta)}$ is*

$$\prod (p_i - 1)! \prod (q_j - 1)!$$

   where $(p_1, p_2, \ldots)$ are the block sizes of β and $(q_1, q_2, \ldots)$ the block sizes of α modulo π.

3. *These are all the eigenvalues.*

Questions

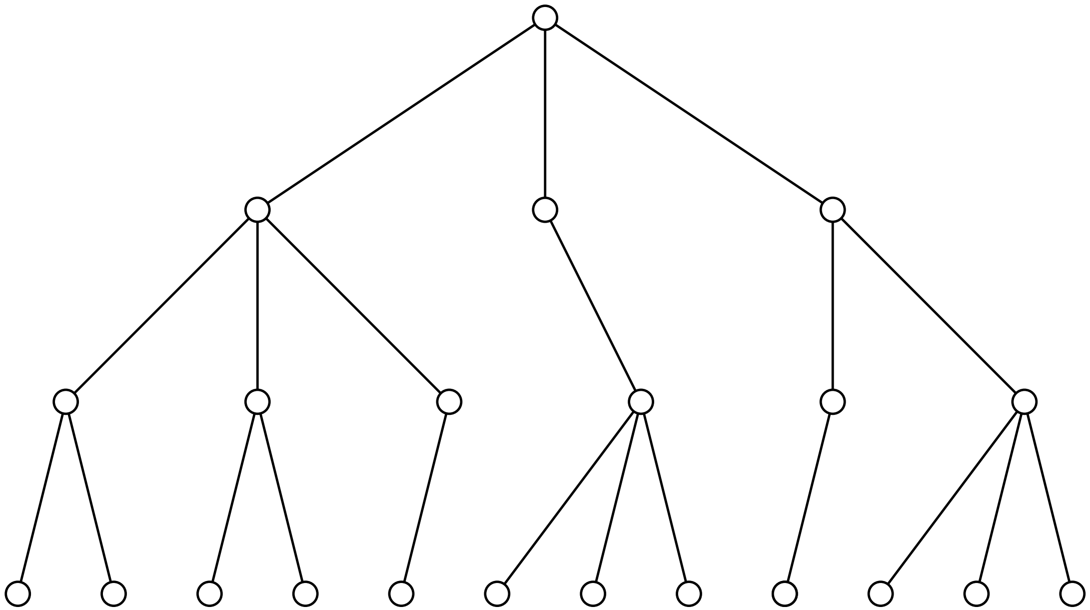
1. Is the detour via complex geometry really needed?

– No, not if one wants only the k-shelf library application, which needs no geometry at all.

2. Why stop at k shelves?

– No need to do that. One can have several library rooms, each with a certain number of shelves each carrying books, such that rooms, shelves and books are permuted at each step.
– Or, several library buildings, . . . several planets, and so on ....

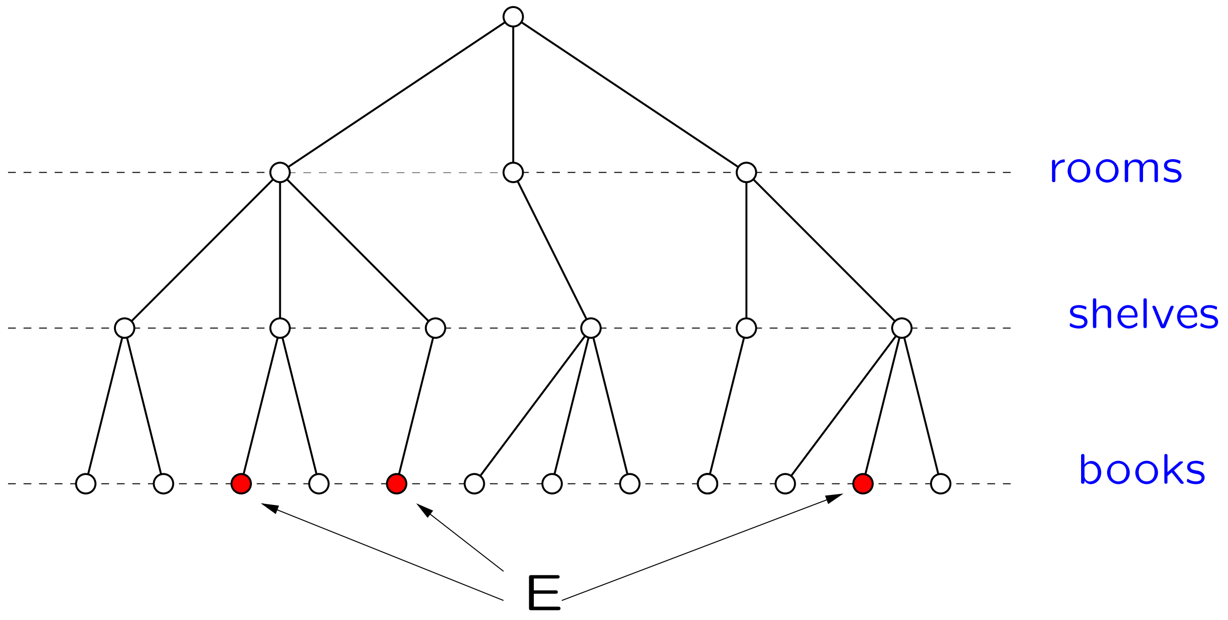
General picture: Random-to-front shuffle on tree:



Probability distribution on subsets $E \subseteq \{\text{leaves}\}$

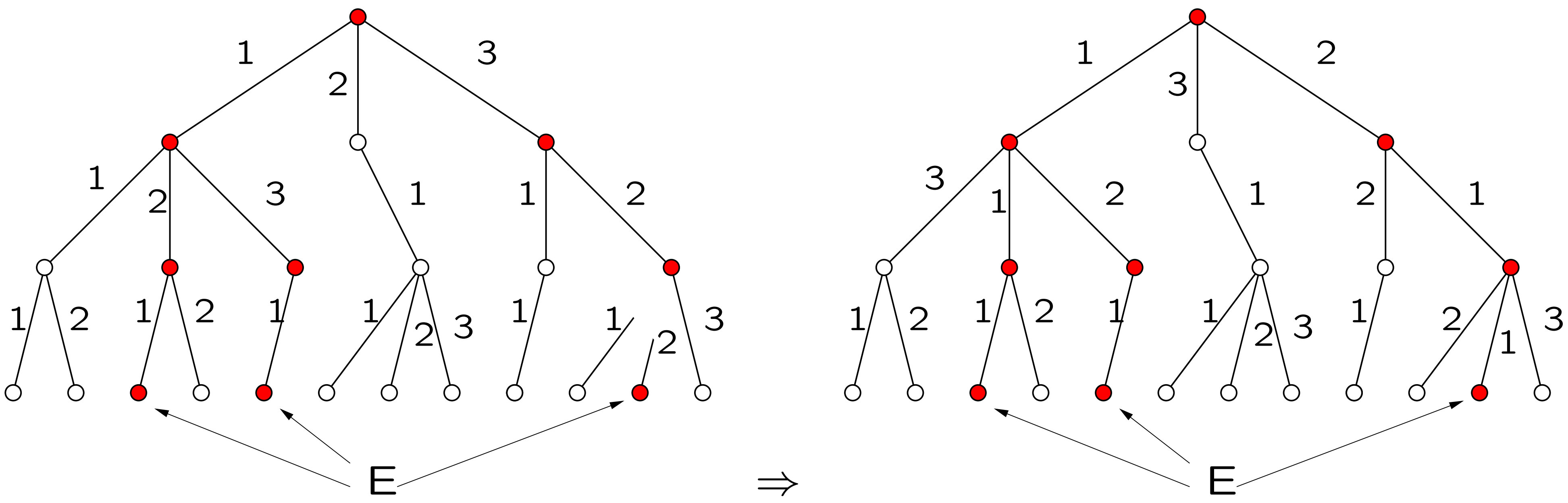
Introduction

Random-to-front shuffle on tree:



Introduction

Random-to-front shuffle (relabeling) on tree:



Theorem. *Explicit description of eigenvalues of transition matrix. They are all partial sums of the probabilities* w_E.

1. Random walks on semigroups

Def: An LRB (left regular band): semigroup Σ satisfying

$$\begin{cases} x^2 = x & \text{for all } x \in \Sigma \\ xyx = xy & \text{for all } x, y \in \Sigma \end{cases}$$

There are *two posets* related to an LRB semigroup Σ.

Proposition 1. *Define a relation "$\leq$" on Σ by*

$$x \leq y \quad \Leftrightarrow \quad xy = y \tag{1}$$

This is a partial order relation.

So, an LRB semigroup is also a poset.
The identity element e is the unique minimal element.
The set $\max(\Sigma)$ of maximal elements satisfies

$$x \in \Sigma, \;\; y \in \max(\Sigma) \; \Rightarrow \; xy \in \max(\Sigma)$$

Proposition 2. *Let Σ be an LRB semigroup. Then there exists a unique finite lattice Λ and an order-preserving and surjective map*

$$\text{supp} : \Sigma \to \Lambda \tag{2}$$

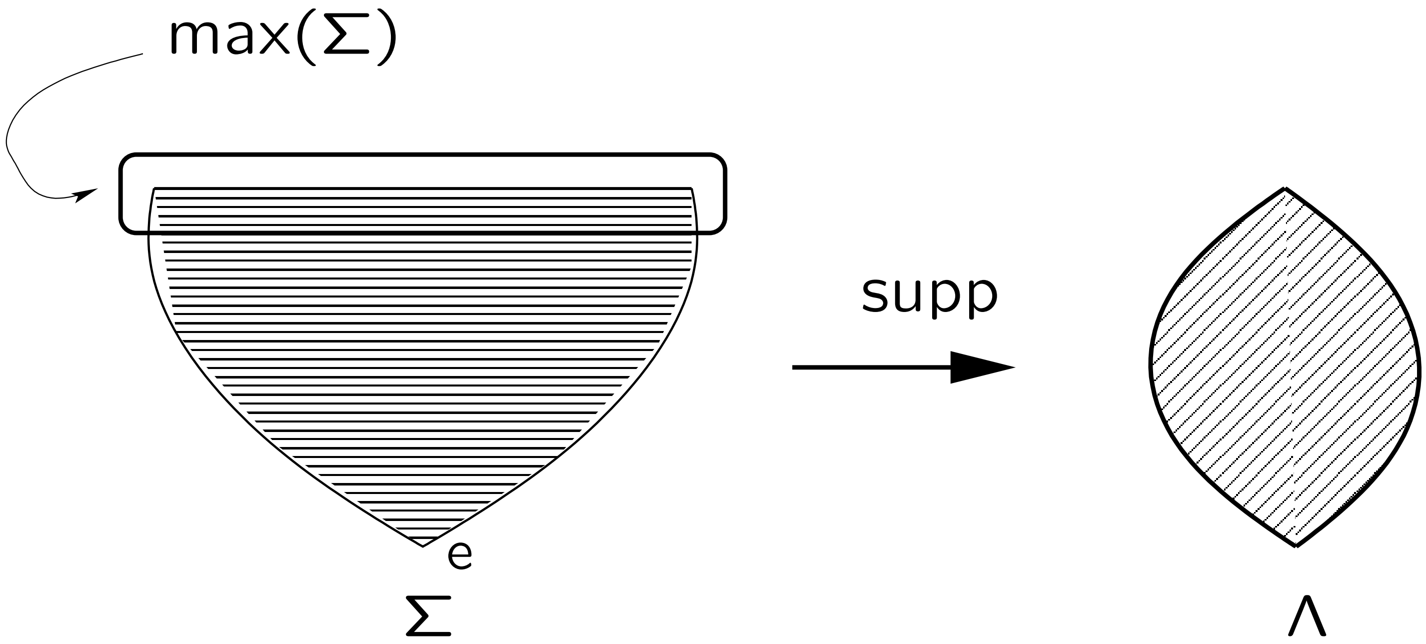
such that for all $x, y \in \Sigma$:

1. $\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y)$

2. $\text{supp}(x) \leq \text{supp}(y) \quad \Leftrightarrow \quad yx = y$

We call Λ the *support lattice* and supp the *support map*.

"The picture"



semigroup

support lattice

1. Random walks on semigroups

Random walk on max(Σ): Probability distribution $\{w_x\}$ on Σ.

STEP: $y \mapsto xy$, where $y \in$ max(Σ) and

$x \in \Sigma$ is chosen according to w.

Let P_w be the transition matrix of the random walk on max(Σ):

$$P_w(c,d) = \sum_{x\,:\,xc=d} w_x$$

for $c, d \in$ max(Σ).

1. Random walks on semigroups

Two fundamental theorems of Brown (2000),

on *eigenvalues of P_w* resp. *stationarity*,

generalizing work of Bidigare 97, Bidigare-Hanlon-Rockmore 99, and Brown-Diaconis 98.

Theorem 1. *(Eigenvalues)*

1. *The matrix P_w is diagonalizable.*

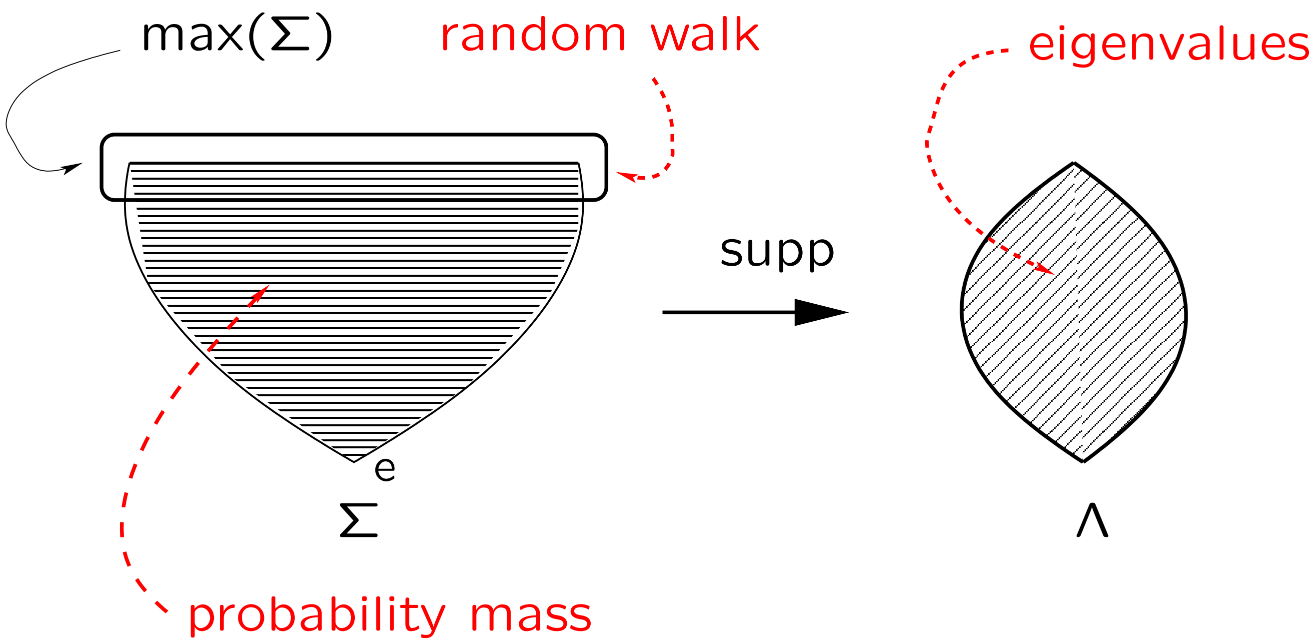
2. *For each $X \in \Lambda$ there is an eigenvalue $\varepsilon_X = \sum_{y\,:\,\mathrm{supp}(y) \le X} w_y$.*

3. *The multiplicity of the eigenvalue ε_X is $m_X = \sum_{Y\,:\,Y \ge X} \mu_\Lambda(X, Y) c_Y$, where $c_Y \stackrel{\mathrm{def}}{=} |\max(\Sigma_{\ge y})|$, for any $y \in \mathrm{supp}^{-1}(Y)$.*

4. *These are all the eigenvalues of P.*

1. Random walks on semigroups

"The picture"



Theorem 2. *(Stationarity)*
Suppose that Σ *is generated by* $\{x \in \Sigma : w_x > 0\}$*. Then the random walk on* $\max(\Sigma)$ *has a unique stationary distribution* π*.*

Also provided:

- Algorithm how to sample an element distributed from π.

- Measure of convergence to π.

— stationarity will not be further discussed in this talk

2. Real hyperplane arrangements

$\mathcal{A} = \{H_1, \ldots, H_t\}$ arrangement

$\ell_1, \ldots, \ell_t$ linear forms on $\mathbb{R}^d$

$H_i = \{x : \ell_i(x) = 0\} \subseteq \mathbb{R}^d$ hyperplane

$L_{\mathcal{A}} = \{$intersections of H_i's$\}$ ordered by reverse inclusion

— intersection lattice

Complement — convex cones "regions" or "chambers"

Theorem. *(Zaslavsky 1975)*

$$\#\ \textit{regions} = \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)|$$

— Where is the semigroup?

Encode position of point $x \in \mathbb{R}^d$ with respect to $\mathcal{A}$.

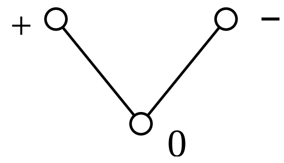
Sign vector (position vector): $\sigma(x) = \{\sigma_1, \ldots, \sigma_t\} \in \{+, -, 0\}^t$

$$\sigma_i = \begin{cases} 0, & \text{if } \ell_i(x) = 0 \\ +, & \text{if } \ell_i(x) > 0 \\ -, & \text{if } \ell_i(x) < 0 \end{cases}$$

Combinatorics of sign vectors $\longrightarrow$ oriented matroid theory

2. Real hyperplane arrangements

Face semilattice:
$F_{\mathcal{A}} = \sigma(\mathbb{R}^d) \subseteq \{+,-,0\}^t$ — ordered componentwise by



Note: maximal el'ts of $F_{\mathcal{A}} \leftrightarrow$ regions

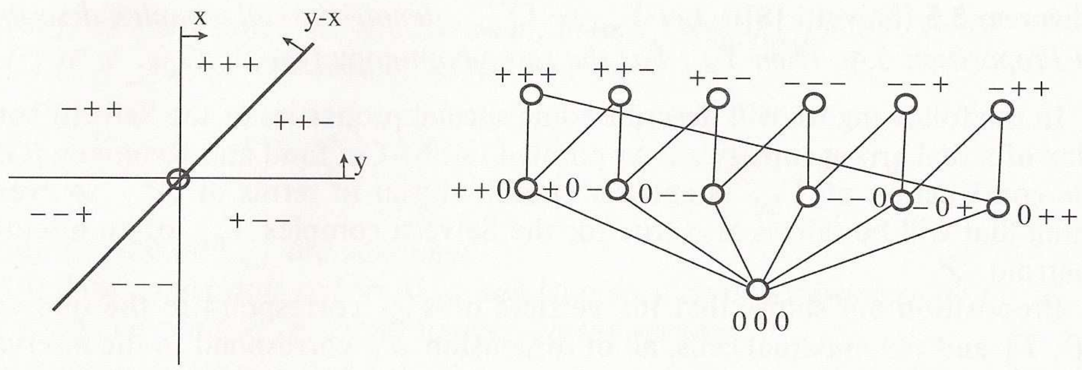


FIGURE 5.1. Face poset $\mathscr{L}$ of an arrangement of three lines in $\mathbb{R}^2$.

2. Real hyperplane arrangements

Fact: $F_{\mathcal{A}}$ describes cell structure of regular CW-decomposiotion of the unit sphere in $\mathbb{R}^d$ (the cell decomposition induced by the hyperplanes)

Composition: $X \circ Y \in \{+,-,0\}^t$ defined by

$$(X \circ Y)_i = \begin{cases} X_i, & \text{if } X_i \neq 0 \\ Y_i, & \text{if } X_i = 0 \end{cases}$$

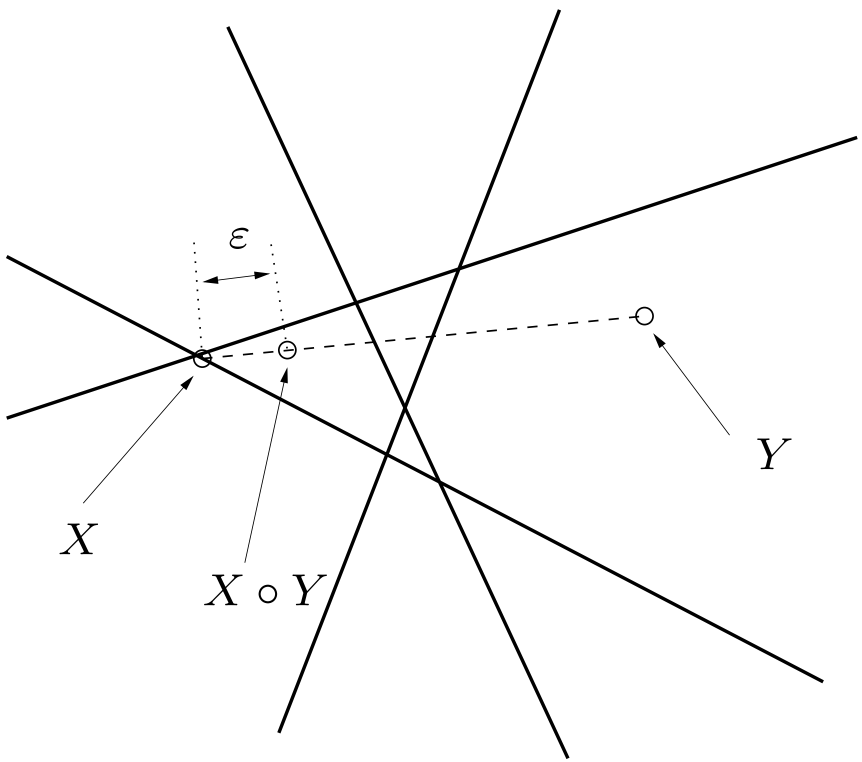
- associative, idempotent, unit element $= (0, \ldots, 0)$

- $X, Y \in F_{\mathcal{A}} \Rightarrow X \circ Y \in F_{\mathcal{A}}$
  (geometric reason: move ε distance from X toward Y)

2. Real hyperplane arrangements

$X, Y \in F_{\mathcal{A}} \Rightarrow X \circ Y \in F_{\mathcal{A}}$

(geometric reason: move ε distance from X toward Y)



Proposition 3. $(F_{\mathcal{A}}, \circ)$ *is LRB semigroup with support lattice* $L_{\mathcal{A}}$*. The support map*

$$\textit{supp} : F_{\mathcal{A}} \to L_{\mathcal{A}}$$

sends cell σ *to linear span* $\overline{\sigma}$*. (Equivalently, sends sign-vector to the set of positions of its zeroes.)*

Consequence: Theory of random walks on $\mathbb{R}$-arrangements

Probability distribution w on $F_{\mathcal{A}}$ $\Rightarrow$ Random walk on $C_{\mathcal{A}}$

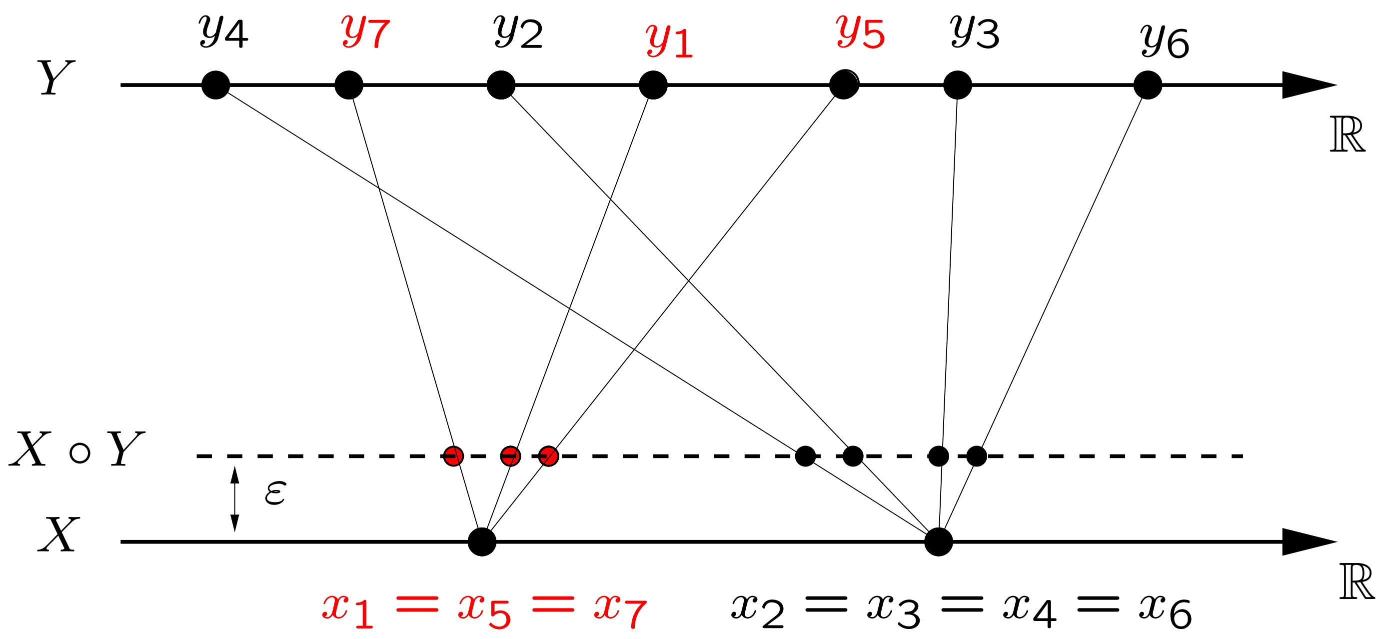
STEP: Choose $X \in F_{\mathcal{A}}$ according to measure w. Then, from current region $C \in C_{\mathcal{A}}$ move to $X \circ C$.

Special case (real braid arrangement): Tsetlin's library

RECALL Random-to-front shuffle:

4 7 2 1 5 3 6 $\Longrightarrow$ 7 1 5 4 2 3 6

Geometric version:



2. Real hyperplane arrangements

As before: $\mathcal{A}$, $L_{\mathcal{A}}$, $F_{\mathcal{A}}$
$C_{\mathcal{A}} = \{\text{regions}\} \leftrightarrow \max F_{\mathcal{A}}$

Probability distribution w on $F_{\mathcal{A}}$ $\Rightarrow$ Random walk on $C_{\mathcal{A}}$

STEP: Choose $X \in F_{\mathcal{A}}$ according to measure w. Then, from current region $C \in C_{\mathcal{A}}$ move to $X \circ C$.

Transition matrix: $P_w = (p_{C,D})_{C,D \in C_{\mathcal{A}}}$

$$p_{C,D} = \sum_{X \circ C = D} w(X)$$

Theorem 3. *(Bidigare-Hanlon-Rockmore, Brown-Diaconis)*
(a) P_w *is diagonalizable.*
(b) For each $F \in L_{\mathcal{A}}$ *there is an eigenvalue*

$$\lambda_F = \sum_{X:\mathit{supp}X \subseteq F} w(X)$$

.
(c) The multiplicity of λ_F *is* $|\mu(\hat{0}, F)|$.
(d) These are all the eigenvalues.

Corollary 1. *(Zaslavsky's formula)*

$$\#\ \text{regions} = \text{size of matrix} = \sum_{F \in L_{\mathcal{A}}} |\mu(\hat{0}, F)|$$

3. Complex hyperplane arrangements

Arrangement $\mathcal{A} = \{H_1, \ldots, H_t\}$
$\ell_1, \ldots, \ell_t$ linear forms on $\mathbb{C}^d$
$H_i = \{x : \ell_i(x) = 0\} \subseteq \mathbb{C}^d$ hyperplane

$M_{\mathcal{A}} = \mathbb{C}^d \setminus \cup\mathcal{A}$ — complement ($2d$-dimensional manifold))

How define random walk? Where is semigroup?

Encode position of point $x \in \mathbb{C}^d$ with respect to $\mathcal{A}$.

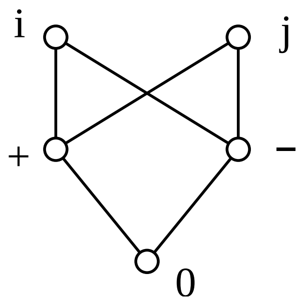
Sign vector (position vector): $\sigma(x) = \{\sigma_1, \ldots, \sigma_t\} \in \{0, +, -, i, j\}^t$

$$\sigma_i = \begin{cases} 0, & \text{if } \ell_i(x) = 0 \\ +, & \text{if } \Im(\ell_i(x)) = 0,\ \Re(\ell_i(x) > 0 \\ -, & \text{if } \Im(\ell_i(x)) = 0,\ \Re(\ell_i(x) < 0 \\ i, & \text{if } \Im(\ell_i(x)) > 0 \\ j, & \text{if } \Im(\ell_i(x)) < 0 \end{cases}$$

3. Complex hyperplane arrangements

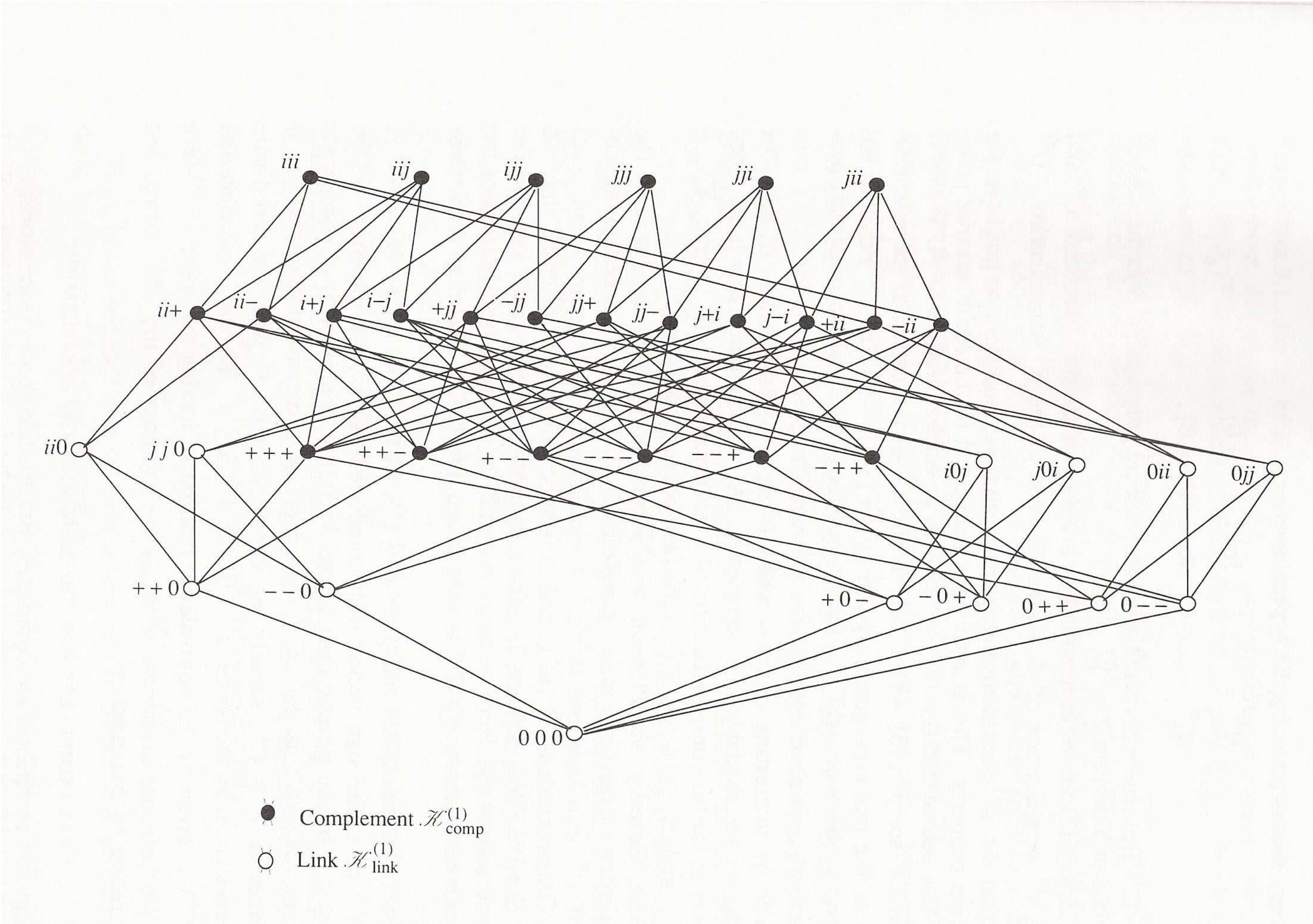
Face poset: $F_{\mathcal{A}} = \sigma(\mathbb{C}^d) \subseteq \{0, +, -, i, j\}^t$

— ordered componentwise by



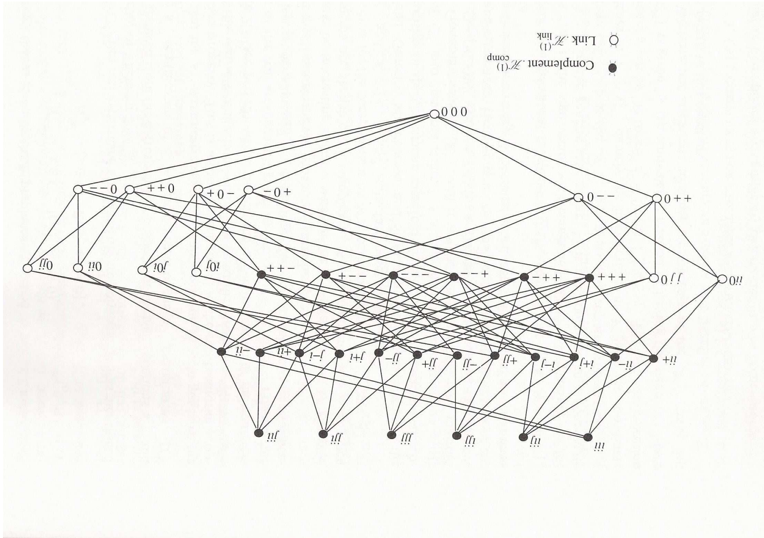
Proposition 4. *$F_{\mathcal{A}}$ is a ranked poset of length $2d$*

3. Complex hyperplane arrangements



Face poset of arrangement of 3 lines in $\mathbb{C}^2$

3. Complex hyperplane arrangements



Theorem. *(Bj-Ziegler 1992)*
(a) $F_{\mathcal{A}}$ *determines cell structure of regular CW-decomposition of the unit sphere in* $\mathbb{R}^{2d} \cong \mathbb{C}^d$
(b) $C_{\mathcal{A}} \stackrel{\text{def}}{=} F_{\mathcal{A}} \cap \{+,-,i,j\}^t$ *with opposite order determines cell structure of a regular CW complex having the homotopy type of the complement* $M_{\mathcal{A}}$*.*

Composition: $Z \circ W \in \{0, +, -, i, j\}^t$ defined by

$$(Z \circ W)_i = \begin{cases} Z_i, & \text{if } W_i \not> Z_i \\ W_i, & \text{if } W_i > Z_i \end{cases}$$

- associative, idempotent, unit element $= (0, \dots, 0)$
- $X, Y \in F_{\mathcal{A}} \;\Rightarrow\; X \circ Y \in F_{\mathcal{A}}$
  (geometric reason: move ε distance from X toward Y)

Proposition 5. $(F_{\mathcal{A}}, \circ)$ *is LRB semigroup.*

What is its support lattice?

3. Complex hyperplane arrangements

For $\mathbb{C}$-arrangements, notion of intersection lattice splits into two.

1. The *intersection lattice* $L_{\mathcal{A}}$: all intersections of subfamilies of hyperplanes H_i ordered by set inclusion.

2. The *augmented intersection lattice* $L_{\mathcal{A},\,\mathrm{aug}}$: all intersections of subfamilies of the *augmented arrangement*

$$\mathcal{A}_{\mathrm{aug}} = \{H_1, \ldots, H_t, H_1^{\mathbb{R}}, \ldots, H_t^{\mathbb{R}}\}$$

ordered by set inclusion.

Here, $H_i^{\mathbb{R}} \overset{\mathrm{def}}{=} \{z \in \mathbb{C}^d : \Im(\ell_i(z)) = 0\}$ is a $(2d-1)$-dimensional real hyperplane in $\mathbb{C}^d \cong \mathbb{R}^{2d}$ containing H_i.

Proposition 6. *Reversing the partial order:*

1. $L_{\mathcal{A}}^{\text{op}}$ *is a geometric lattice of length* d.
2. $L_{\mathcal{A},\text{aug}}^{\text{op}}$ *is a semimodular lattice of length* $2d$.

Proposition 7. *$(F_{\mathcal{A}}, \circ)$ is an LRB semigroup with support lattice $L_{\mathcal{A},\text{aug}}$. The support map*

$$\text{supp} : F_{\mathcal{A}} \to L_{\mathcal{A},\text{aug}}$$

sends the convex cone $\sigma^{-1}(Z)$, for $Z \in F_{\mathcal{A}}$, to the intersection of all subspaces in $\mathcal{A}_{\text{aug}}$ that contain $\sigma^{-1}(Z)$.

4. Walks on complex hyperplane arrangements

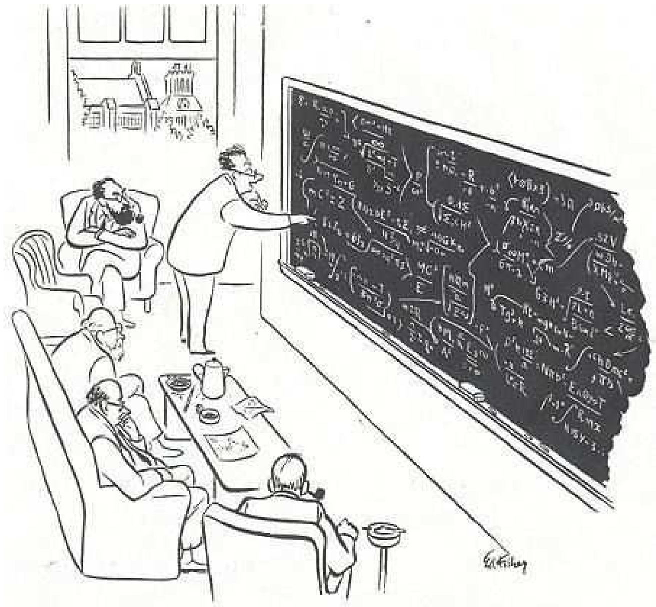
Consequence: $\exists$ theory of random walks on $\mathbb{C}$-arrangements

Different versions exist

obtained by choosing various sub-LRB-semigroups of the complex sign vector semigroup $(F_{\mathcal{A}}, \circ)$

4. Walks on complex hyperplane arrangements

Working out the combinatorial details for one version of random walk on the complex braid arrangement ....



"Say, I think I see where we went off. Isn't eight times seven fifty-six?"

...... we arrive at the description of eigenvalues of the transition matrix of the k-shelf library walk, as stated earlier.

5. Complexified $\mathbb{R}$-arrangements

All forms $\ell_i(z)$ have $\mathbb{R}$-coefficients $\rightarrow \begin{cases} \text{real arrangement } \mathcal{A}^{\mathbb{R}} \\ \text{complex arrangement } \mathcal{A}^{\mathbb{C}} \end{cases}$

Fact: $F_{\mathcal{A}^{\mathbb{C}}}$ is determined by $F_{\mathcal{A}^{\mathbb{R}}}$, namely,

$$\begin{array}{rcl} \phi : \mathrm{Int}(F_{\mathcal{A}^{\mathbb{R}}}) & \longrightarrow & F_{\mathcal{A}^{\mathbb{C}}} \\ [Y, X] & \longmapsto & X \circ iY \end{array}$$

is a poset isomorphism.

Here $\mathrm{Int}(F_{\mathcal{A}^{\mathbb{R}}}) \stackrel{\text{def}}{=}$ set of intervals in the real face poset $F_{\mathcal{A}^{\mathbb{R}}}$

5. Complexified $\mathbb{R}$-arrangements

Structure of $F_{\mathcal{A}^{\mathbb{C}}}$ in terms of intervals $\mathrm{Int}(F_{\mathcal{A}^{\mathbb{R}}})$

Order:

$$[Y, X] \leq [R, S] \quad \Leftrightarrow \quad \begin{cases} Y \leq R \\ R \circ X \leq S \end{cases}$$

Composition:

$$[Y, X] \circ [R, S] = [Y \circ R, Y \circ R \circ X \circ S]$$

6. The real braid arrangement

$\mathcal{A} = \{x_i - x_j \mid 1 \le i < j \le n\}$ in $\mathbb{R}^n$.

Intersection lattice $L_{\mathcal{A}} \cong \Pi_n$ (set partitions, refinement)

Ex: $(134 \mid 27 \mid 5 \mid 6) \leftrightarrow \begin{cases} x_1 = x_3 = x_4 \\ x_2 = x_7 \end{cases}$

Face semilattice $F_{\mathcal{A}} \cong \Pi_n^{\text{ord}}$ (ordered set partitions, refinement)

Ex: $\langle\, 134 \mid 6 \mid 27 \mid 5 \,\rangle \leftrightarrow \begin{cases} x_1 = x_3 = x_4 \\ < x_6 \\ < x_2 = x_7 \\ < x_5 \end{cases}$

Complement: Regions $C_{\mathcal{A}} \cong S_n$ (permutations of $[n]$)

Composition in $F_{\mathcal{A}}$:

If $X = \langle X_1, \ldots, X_p \rangle$ and $Y = \langle Y_1, \ldots, Y_q \rangle$, $X_i, Y_j \subseteq [n]$, then

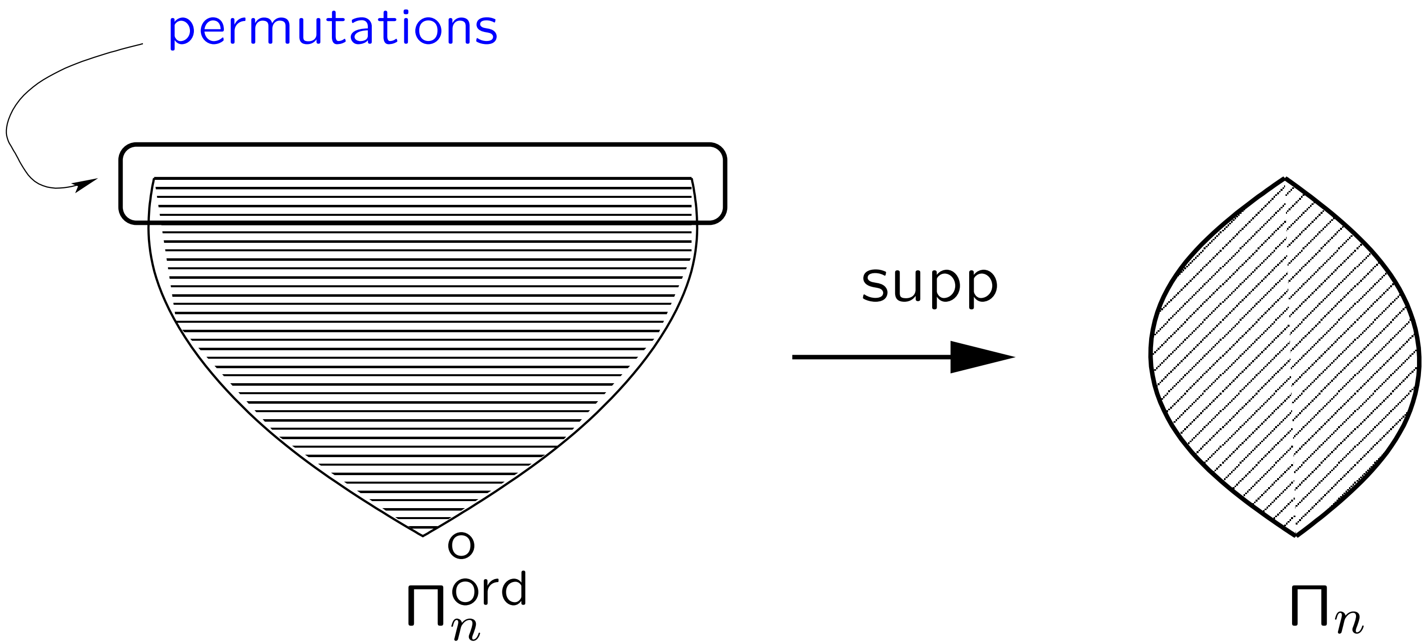
$$X \circ Y = \left\langle X_i \cap Y_j \right\rangle,$$

blocks ordered lexicographically by indices (i, j)

Ex: $\langle\ 257 \mid 3 \mid 146\ \rangle \circ \langle\ 17 \mid 25 \mid 346\ \rangle = \langle\ 7 \mid 25 \mid 3 \mid 1 \mid 46\ \rangle$

6. The real braid arrangement

"The picture"



ordered set partitions

set partitions

6. The complex braid arrangement

$\mathcal{A} = \{x_i - x_j \mid 1 \leq i < j \leq n\}$ in $\mathbb{C}^n$.

Intersection lattice $L_{\mathcal{A}} \cong \Pi_n$
(set partitions)

Augmented intersection lattice $L_{\mathcal{A},\,\mathrm{aug}} \cong \mathrm{Int}(\Pi_n)$
(intervals of set partitions)

Face semilattice $F_{\mathcal{A}} \cong \mathrm{Int}(\Pi_n^{\mathrm{ord}})$
(intervals $[Y, X]$ in semilattice of ordered set partitions)

Complement $C_{\mathcal{A}} \cong$ intervals $[Y, X]$, X maximal
$\leftrightarrow$ permutation X divided into blocks Y

6. The complex braid arrangement

Hence, we get random walk on block-divided permutations

Block-divided permutations $\leftrightarrow$ library placements of books

Other specializations possible ....

Further developments

Libraries with one shelf (Tsetlin) $\longrightarrow$ Random walks on real hyperplane arrangements (Bidigare-Hanlon-Rockmore, 1998)

$\downarrow$

$\downarrow$

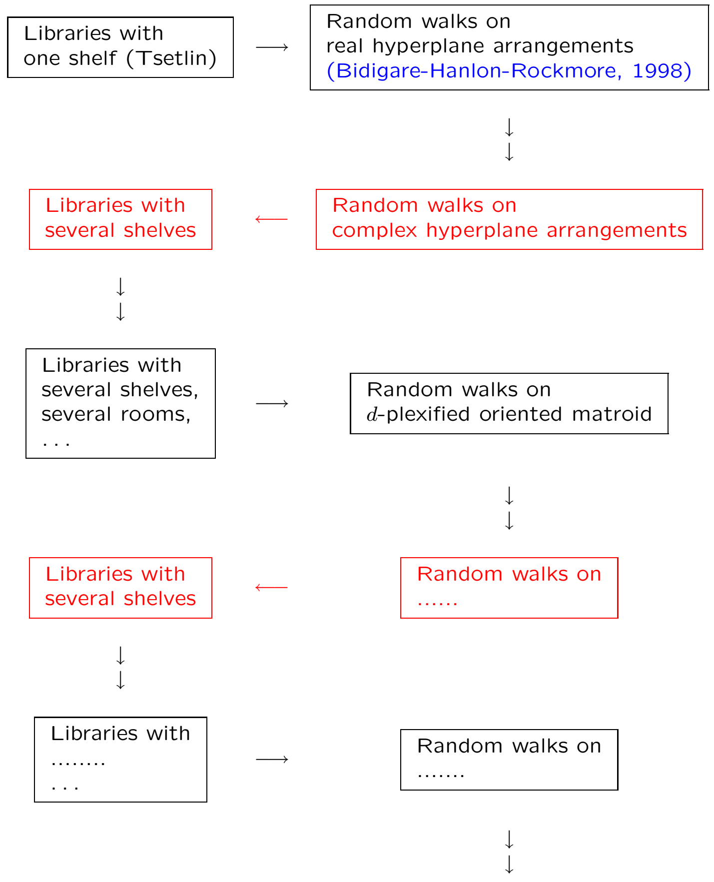
Libraries with several shelves $\longleftarrow$ Random walks on complex hyperplane arrangements

$\downarrow$

$\downarrow$

Libraries with several shelves, several rooms, . . . $\longrightarrow$ Random walks on d-plexified oriented matroid

Even further developments



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