On the shape of Bruhat intervals

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Anders Björner
Institut Mittag-Leffler

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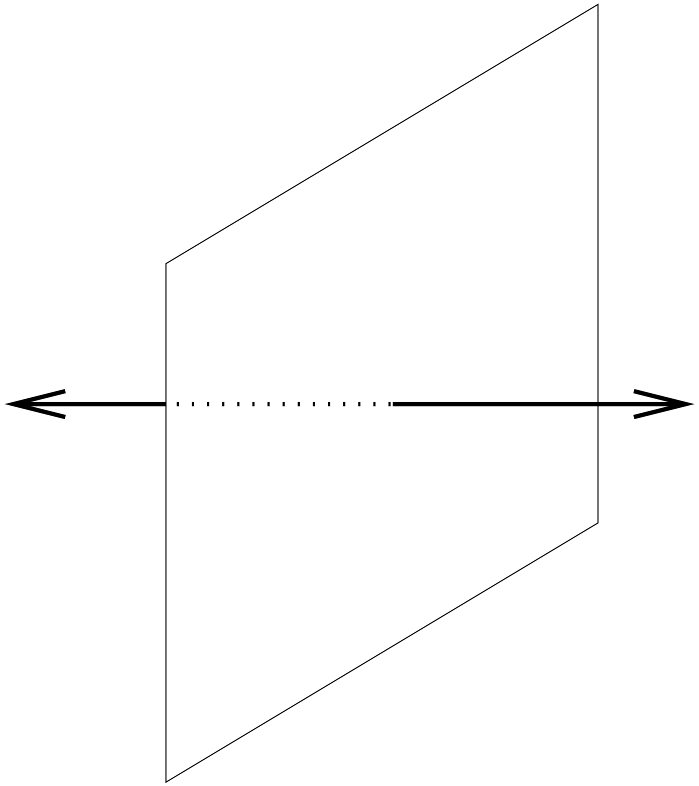
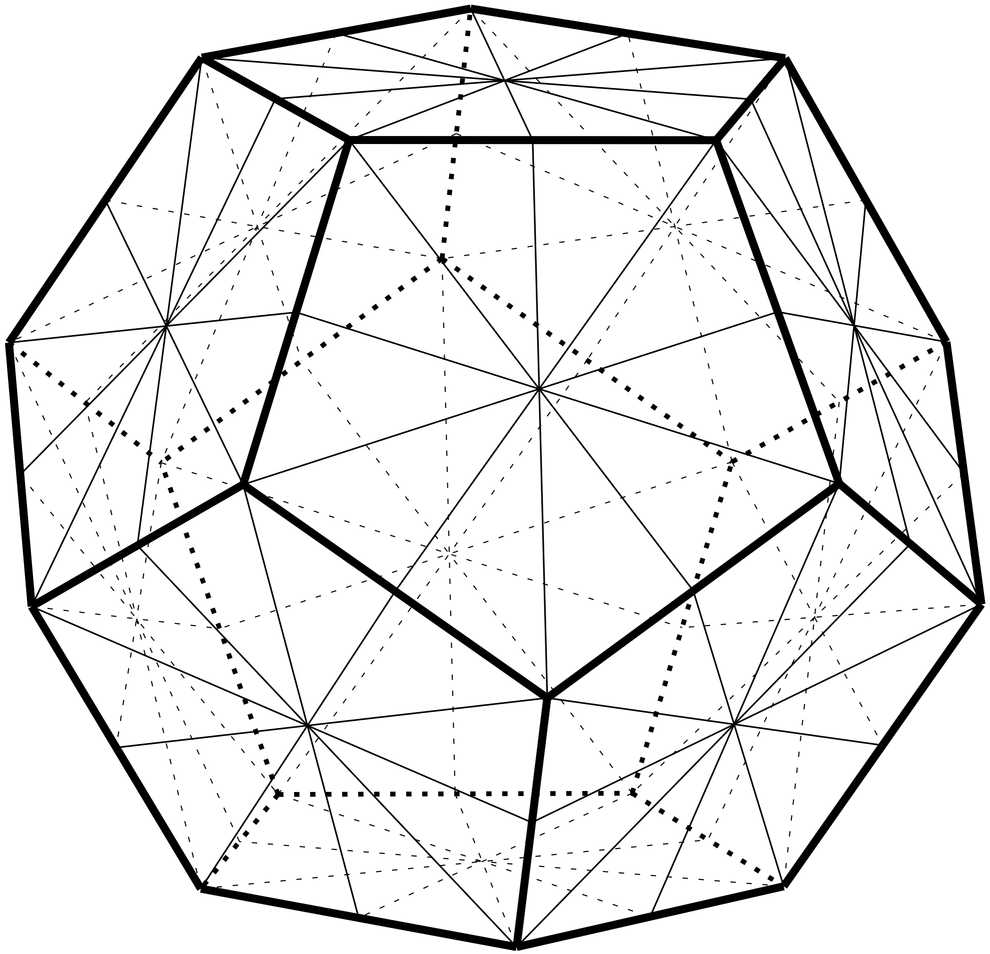
Coxeter groups



Finite Coxeter groups $\longleftrightarrow$ Finite reflection groups (i.e., groups generated by orthogonal reflections in hyperplanes)



The dodecahedron as a reflection group

The pair (W, S) is a *Coxeter group* (Coxeter system) if W is a group with presentation

Generators: S, such that

$$s^2 = e, \quad \text{for all } s \in S,$$

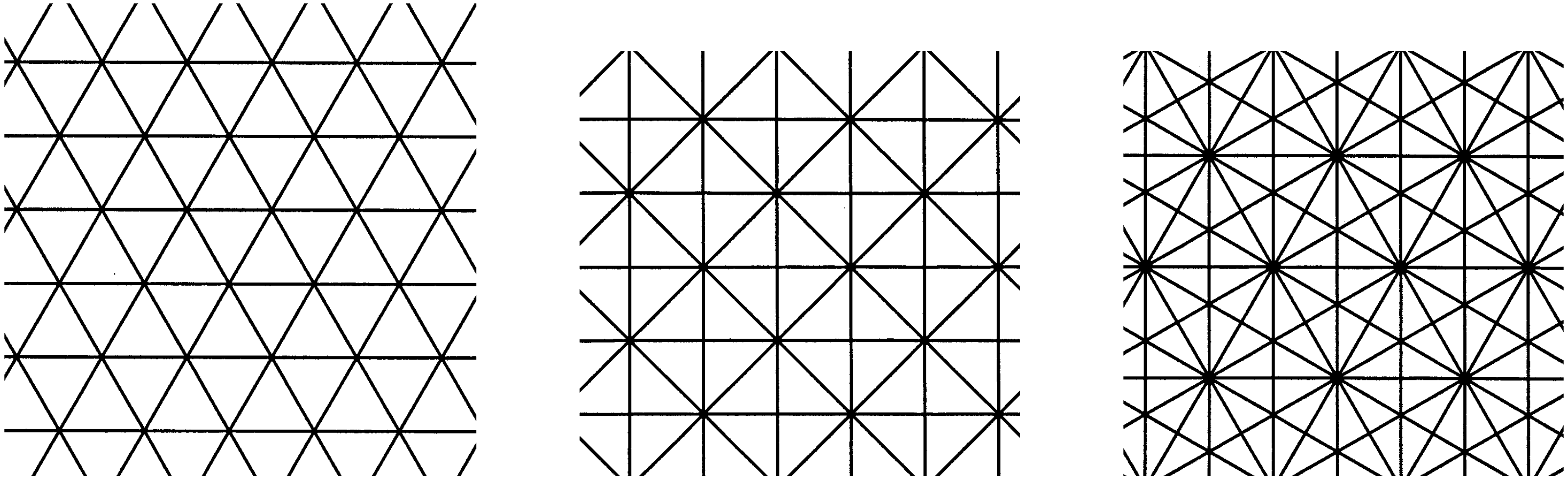
Relations: for $s, s' \in S$

$$\underbrace{s\, s'\, s\, s'\, s \ldots}_{m(s,s')} = \underbrace{s'\, s\, s'\, s\, s' \ldots}_{m(s,s')}$$

Examples

1. The symmetric group S_n.
Coxeter generators = Adjacent transpositions $(i, i+1)$

2. Affine reflection groups



The $\widetilde{A}_2$, $\widetilde{C}_2$ and $\widetilde{G}_2$ tesselations of the affine plane.

$\exists$ **classifications**

finite Coxeter groups: type A_n, B_n, ... etc.

affine Coxeter groups: type $\widetilde{A}_n$, $\widetilde{B}_n$, ... etc.

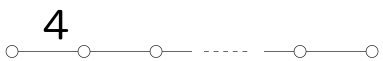
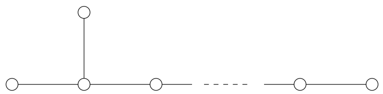
hyperbolic Coxeter groups

Definition: (W, S) is *crystallographic* if $m(s,t) \in \{2, 3, 4, 6, \infty\}$ for all distinct generators s and t.

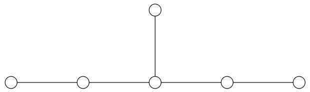
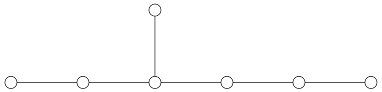
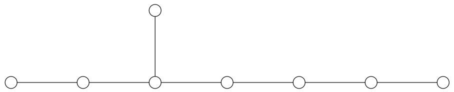
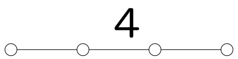
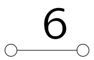
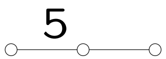
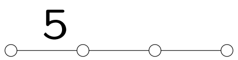
E.g., *finite and affine Weyl groups* are crystallographic.

Coxeter groups

The finite irreducible Coxeter systems

Name	Diagram	Order	$\lvert T \rvert$	Exponents
A_n $(n \geq 1)$		$(n+1)!$	$\binom{n+1}{2}$	$1, 2, \ldots, n$
B_n $(n \geq 2)$		$2^n n!$	n^2	$1, 3, \ldots, 2n-1$
D_n $(n \geq 4)$		$2^{n-1} n!$	$n^2 - n$	$1, 3, \ldots, 2n-3, n-1$

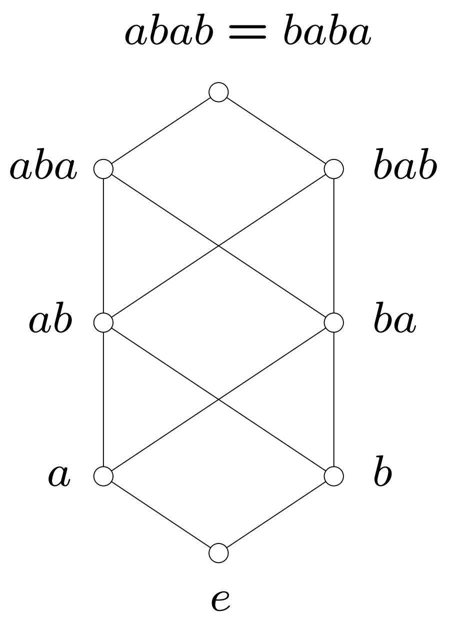
Coxeter groups

E_6		$2^7\,3^4\,5$	36	1, 4, 5, 7, 8, 11
E_7		$2^{10}\,3^4\,5\,7$	63	1, 5, 7, 9, 11, 13, 17
E_8		$2^{14}\,3^5\,5^2\,7$	120	1, 7, 11, 13, 17, 19, 23, 29
F_4		1152	24	1, 5, 7, 11
G_2		12	6	1, 5
H_3		120	15	1, 5, 9
H_4		14400	60	1, 11, 19, 29
$I_2(m)$ $(m \geq 3)$		1 $2m$	m	1, $m-1$

Bruhat order

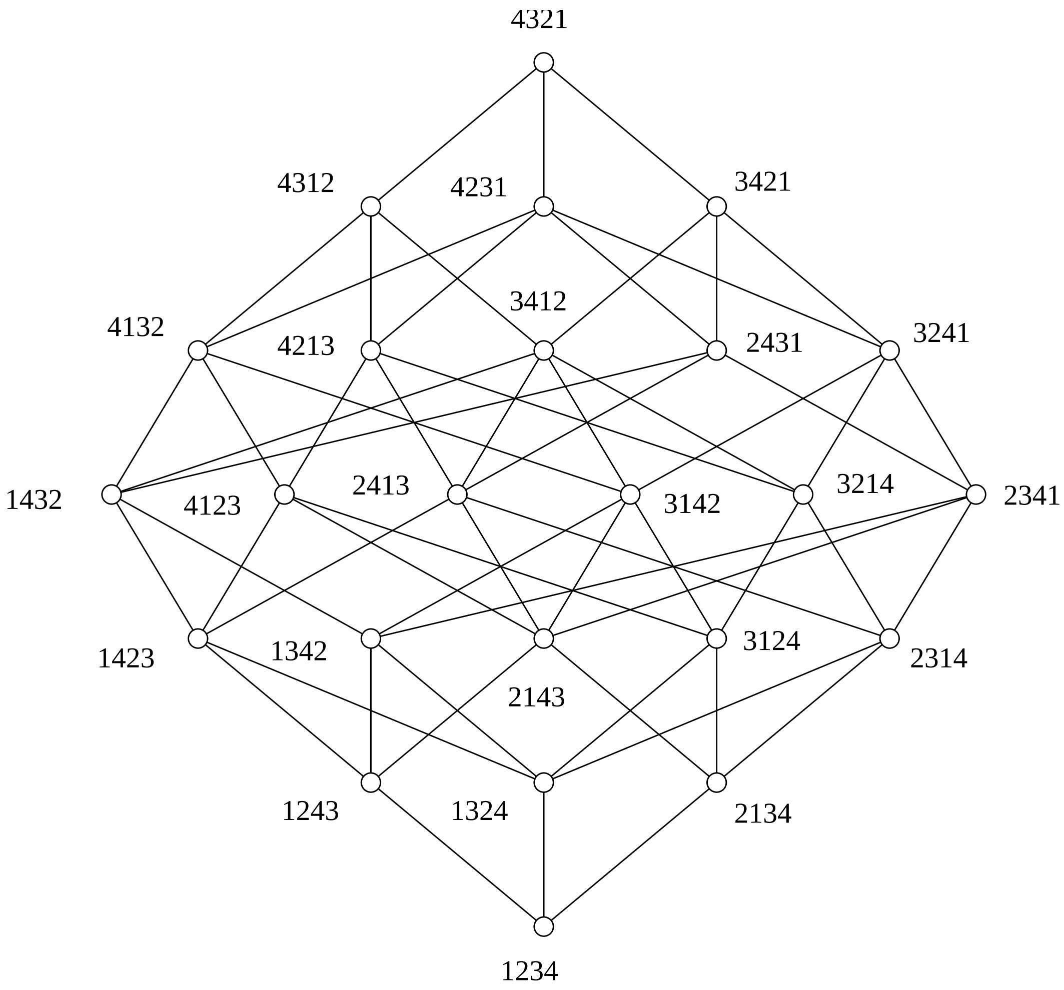
Bruhat order: For $u, w \in W$:

$$u \leq w \overset{\text{def}}{\iff} \begin{array}{l} \text{for } \forall \text{ reduced expression } w = s_1 s_2 \ldots s_q \\ \exists \text{ a reduced subexpression } u = s_{i_1} s_{i_2} \ldots s_{i_k}, \\ \; 1 \leq i_1 < \ldots < i_k \leq q. \end{array}$$



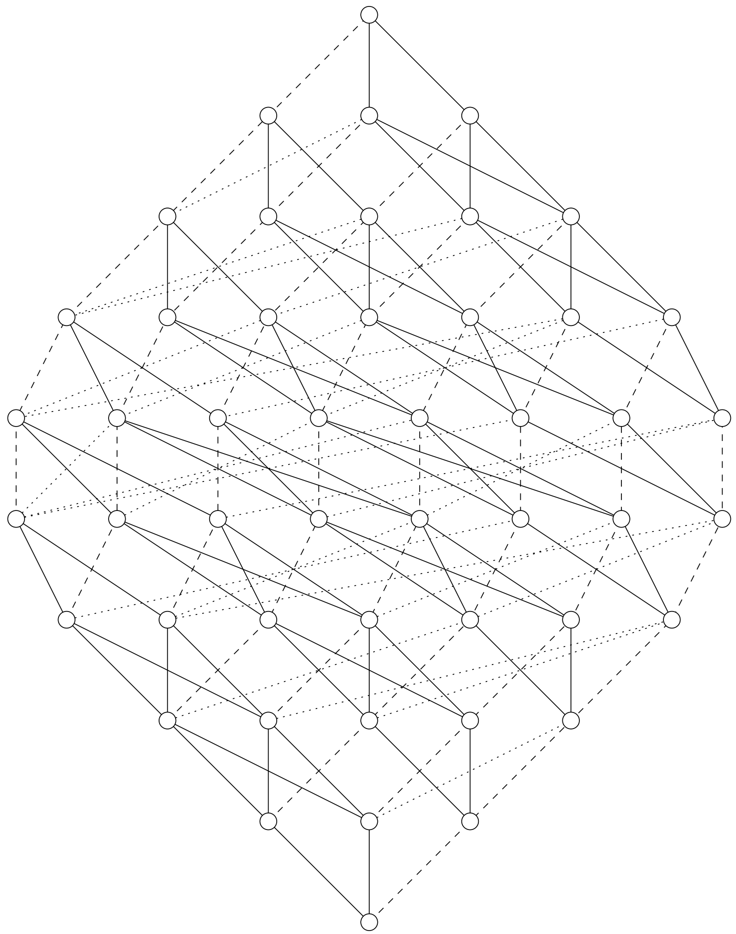
Bruhat order of B_2

Bruhat order



Bruhat order of S_4.

Bruhat order



Bruhat order of B_3.

Bruhat order

Some global properties of Bruhat order of a finite W, as a poset:

** Bottom element e, top element w_0

** Graded (all maximal chains of same size)

** Poset rank = Group-theoretic length $\ell(\,\cdot\,)$

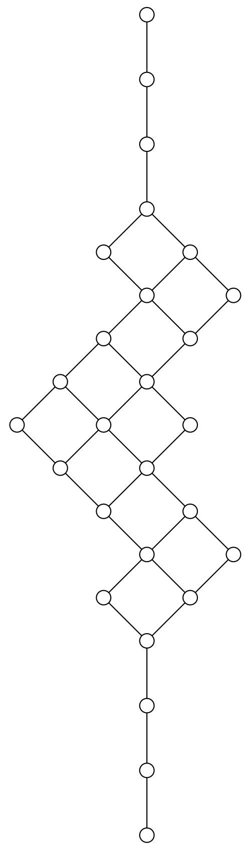
** Rank-generating function

$$\sum_{w \in W} q^{\ell(w)} = \prod_{1 \leq i \leq d} (1 + q + q^2 + \cdots q^{e_i})$$

** Anti-automorphic under map $w \mapsto ww_0$

Bruhat order

Quotients W^J: Minimal coset representatives modulo parabolic subgroups $W_J = < J >$, $J \subseteq S$, with induced order.



The Bruhat poset E_6 modulo D_5.

Bruhat order

Global poset properties of Bruhat order of finite quotients W^J:

** Graded

** Bottom element e, top element w_0^J

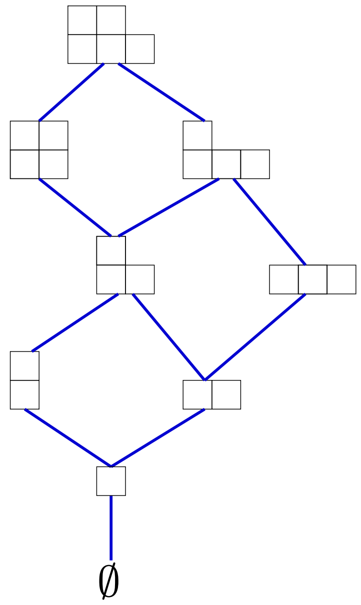
** Poset rank = Group-theoretic length $\ell(\,\cdot\,)$

** Rank-generating function $\sum_{w \in W^J} q^{\ell(w)} = \frac{\sum_{w \in W} q^{\ell(w)}}{\sum_{w \in W_J} q^{\ell(w)}}$

** Anti-automorphic under map $w \mapsto w_{J,0} w w_0$

Bruhat order

A special case of quotient W^J: Young's lattice



Lower intervals $[\emptyset, \lambda]$: Ferrers' diagrams contained in shape λ, and ordered by containment

maximal chains = # standard Young tableaux of shape λ

General Problem:

Study the combinatorial structure of intervals

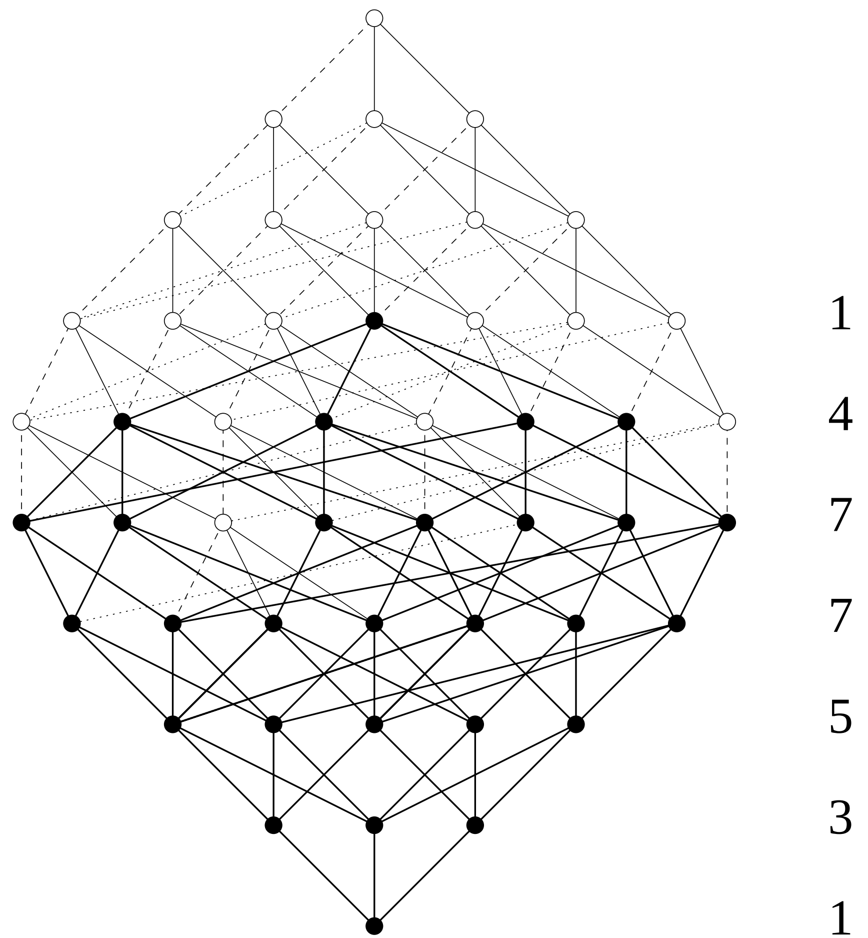
$$[u,w]^J \stackrel{\text{def}}{=} \{z : u \le z \le w\} \cap W^J$$

TOPIC 1: f-vectors of Bruhat intervals

– Joint work with T.Ekedahl

If asking for global interval structure is too hard, study the enumerative "shadow".

Bruhat interval f-vectors



f^w-vector of Bruhat interval $[e, w]$

Shape (or f-vector) of lower interval $[e, w]^J$:

$$f^{w,J} = \{f_0^{w,J}, f_1^{w,J}, \ldots, f_{\ell(w)}^{w,J}\},$$

$f_i^{w,J} \overset{\text{def}}{=}$ number of elements $x \le w$, $x \in W^J$, of length i.

Special case of full group:
$W = W^{\emptyset}$
$f^w \overset{\text{def}}{=} f^{w,\emptyset}$

Another example of f^w-vector of Bruhat interval $[e, w]$
Here $w \in C_4$, $\ell(w) = 13$:

$$f^w = (1, 4, 9, 16, 24, 32, 39, 44, 46, 42, 31, 17, 6, 1)$$

Another example of f^w-vector of Bruhat interval $[e, w]$
Here $w \in C_4$, $\ell(w) = 13$:

$$f^w = (1, 4, 9, 16, 24, 32, 39 \mid 44, 46, 42, 31, 17, 6, 1)$$

$$\uparrow$$

MID

Bruhat interval f-vectors

∃ analogy

Intervals $[e, w]$ in Bruhat order	$\leftrightarrow$	Face lattices of convex polytopes
Weyl group	$\leftrightarrow$	rational polytope
Schubert variety	$\leftrightarrow$	toric variety
Kazhdan-Lusztig polynomial	$\leftrightarrow$	g-polynomial

Also: Both determine regular CW decompositions of a sphere
Intersection cohomology lurks in the background

Remark:

For all polytopes: ∃ combinatorial intersection cohomology theory satisfying hard Lefschetz (recent work of K. Karu and others)

Question: ∃ ??? combinatorial intersection cohomology theory for all Coxeter groups ("virtual Schubert varieties")?

Bruhat interval f-vectors

Note: Analogy with *f-vector* of convex polytope Compare: Known for f-vector of simplicial $(d+1)$-dimensional polytope:

(1) $f_i \leq f_j$ if $i < j \leq d - i$. In particular,

- $f_0 \leq f_1 \leq \cdots \leq f_{d/2}$ and $f_i \leq f_{d-i}$

(2) $f_{3d/4} \geq f_{(3d/4)-1} \geq \cdots \geq f_d$

(3) The bounds $d/2$ and $3d/4$ are best possible.

Conjecture: (2) is true for all polytopes.

Does it make sense to ask such questions for f^w-vectors of Bruhat intervals $[e, w]$?

Perhaps . . . — consider this:

THM (Carrell-Peterson 1994)
The Schubert variety X_w is rationally smooth

$$\iff \; f_i^w = f_{\ell(w)-i}^w, \; \forall i$$

THM (Brion 2000)

$$\sum_{0 \leq i \leq k} f_i^w \leq \sum_{0 \leq i \leq k} f_{\ell(w)-i}^w$$

Theorem 1. *The $f^{w,J}$-vector $f^{w,J} = \{f_0, f_1, \ldots, f_{\ell(w)}\}$ of an interval $[e,w]^J$ in a crystallographic Coxeter group satisfies:*

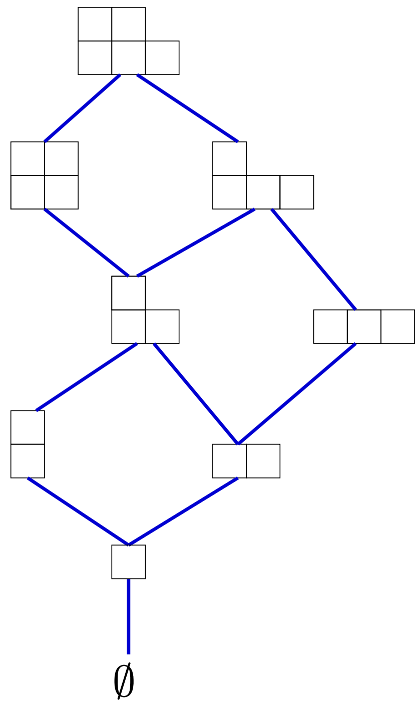
$$f_i \leq f_j \quad , \text{ if } 0 \leq i < j \leq \ell(w) - i.$$

Equivalently,

- $f_i \leq f_{\ell(w)-i}$, for all $i < \ell(w)/2$
- $f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil}$

Bruhat interval f-vectors

Gives new inequalities already for the special case of Young's lattice:



Lower intervals $[\emptyset, \lambda]$: Ferrers' diagrams contained in shape λ, and ordered by containment

Recall definition: (W, S) is *crystallographic* if $m(s, s') \in \{2, 3, 4, 6, \infty\}$ for all distinct generators s and s'.

Fact: Crystallographic $\Leftrightarrow$ appears as Weyl group of a Kac-Moody algebra

Fact: Crystallographic $\Rightarrow$ $\exists$ Schubert varieties

Let (W,S) be crystallographic, $J \subseteq S$.

For each $w \in W^J$ there exists a complex projective variety (called *Schubert variety*) $\overline{X}_w$ containing closed subvarieties $\overline{X}_u$ for all $u \in [e,w]^J$, which are disjoint unions

$$\overline{X}_u = \biguplus_z X_z,$$

where $z \in [e,u]^J$.

Furthermore, X_u is a subvariety of $\overline{X}_w$ isomorphic to affine space $\mathbf{A}^{\ell(u)}$.

Idea of proof of Thm:

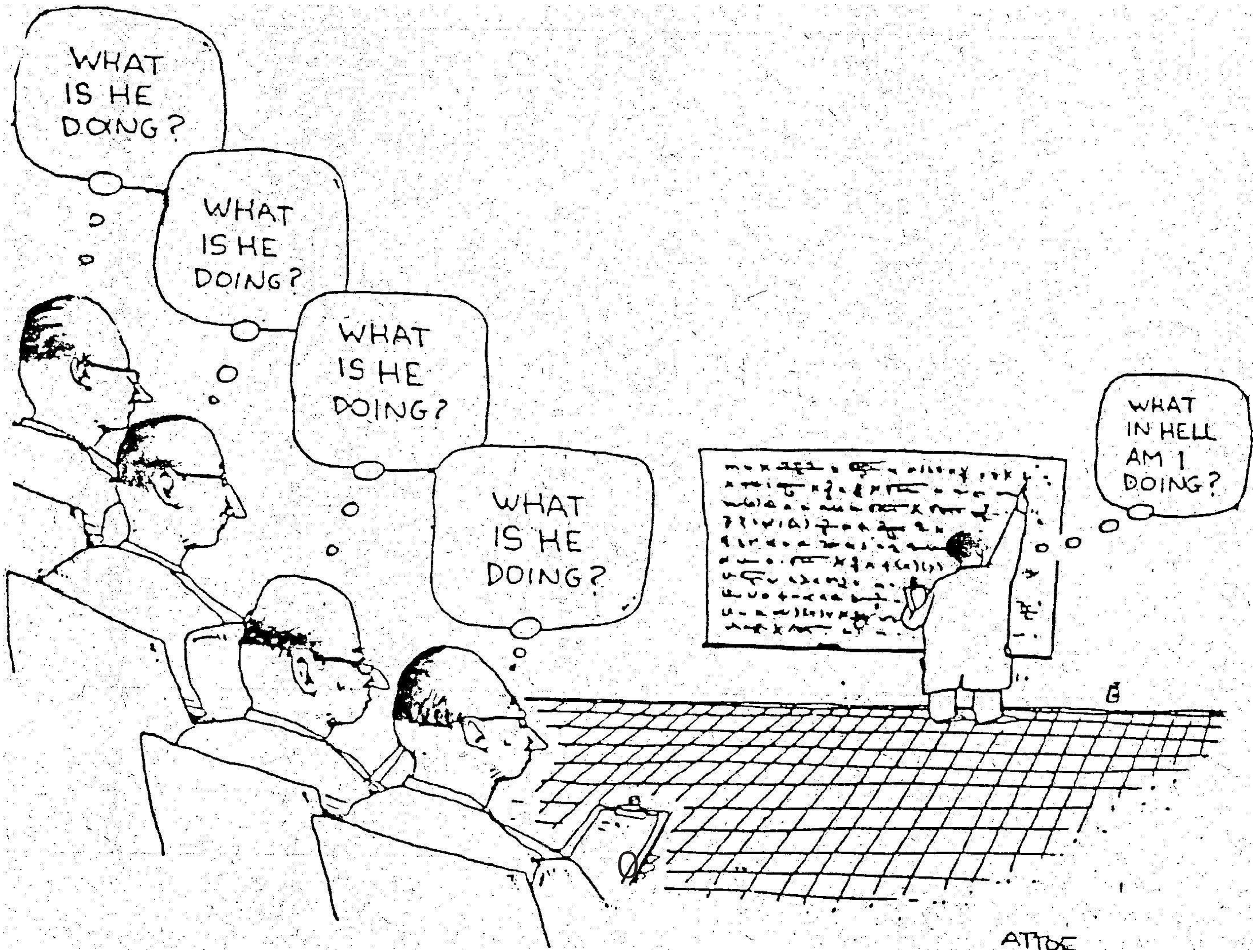
* Use ℓ-adic étale cohomology $H^*(X, \mathbb{Q}_\ell)$ and intersection cohomology $IH^*(X, \mathbb{Q}_\ell)$.

* There is a $H^*(X, \mathbb{Q}_\ell)$-module map $\varphi : H^*(X, \mathbb{Q}_\ell) \to IH^*(X, \mathbb{Q}_\ell)$

* For Schubert varieties $X = X_w$ this map φ is injective.

* $f_i^w = \dim_{\mathbb{Q}_\ell} H^{2i}(X_w, \mathbb{Q}_\ell)$

Bruhat interval f-vectors



Idea of proof of Thm (cont'd)

Let $X = X_w$. The map φ is an $H^*(X, \mathbb{Q}_\ell)$-module map
$\Rightarrow$ for $i \leq j \leq m - i$ it commutes with multiplication by $c_1(\mathcal{L})^{j-i}$
$\Rightarrow$ commutative diagram

$$\begin{array}{ccc} H^{2i}(X,\mathbb{Q}_\ell) & \longrightarrow & IH^{2i}(X,\mathbb{Q}_\ell) \\ \Big\downarrow{\scriptstyle \cap c_1(\mathcal{L})^{j-i}} & & \Big\downarrow{\scriptstyle \cap c_1(\mathcal{L})^{j-i}} \\ H^{2j}(X,\mathbb{Q}_\ell) & \longrightarrow & IH^{2j}(X,\mathbb{Q}_\ell). \end{array}$$

The horisontal maps φ are injective, and the right vertical map is an injection by hard Lefschetz.

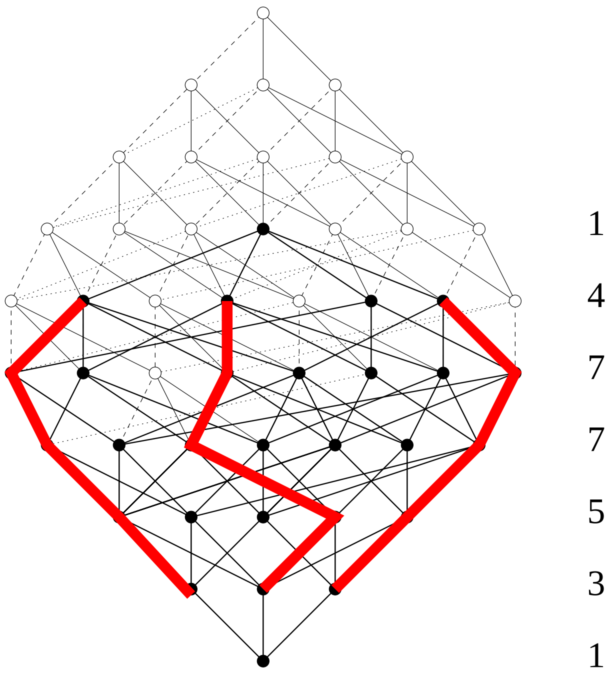
Idea of proof of Thm (cont'd)

For $i \leq j \leq m - i$, we have a commutative diagram

$$\begin{array}{ccc} H^{2i}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2i}(X, \mathbb{Q}_\ell) \\ \downarrow{\scriptstyle \cap c_1(\mathcal{L})^{j-i}} & & \downarrow{\scriptstyle \cap c_1(\mathcal{L})^{j-i}} \\ H^{2j}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2j}(X, \mathbb{Q}_\ell). \end{array}$$

The horisontal maps φ are injective, and the right vertical map is an injection by hard Lefschetz. Hence the left vertical map is injective, giving

$$f_i^w = \dim_{\mathbb{Q}_\ell} H^{2i}(X, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell} H^{2j}(X, \mathbb{Q}_\ell) = f_j^w.$$



Theorem 2. *Let (W,S) be crystallographic, $J \subseteq S$. Fix $w \in W^J$ and i such that $0 \leq i < \ell(w)/2$. Then, in $[e,w]^J$ there exist $f_i^{w,J}$ pairwise disjoint chains*

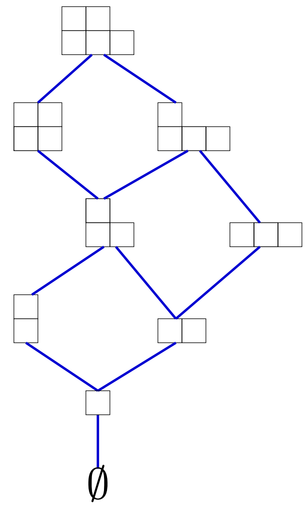
$$u_i < u_{i+1} < \cdots < u_{\ell(w)-i}$$

such that $\ell(u_j) = j$.

References:

Stanley (1980) did the $J = S$, $w = w_0$ case for finite groups

Interpretation of Theorem 2 for Young's lattice:



Given a partition λ of n and $k \le n/2$. Suppose that there are $b = b(\lambda, k)$ partitions of k below λ. Then there exist b standard Young tableaux of shape λ, $T^1, \ldots, T^b$, such that

$$\text{shape}(T_p^i) \ne \text{shape}(T_p^j)$$

for all $i \ne j$ and all $p = k, k+1, \ldots, n-k$.

Here T_p is the subtableau gotten by erasing the boxes with numbers $> p$.

Question: what about the case of equality in some of the relations $f_i \leq f_{\ell(w)-i}$?

From now on: Only the $J = \emptyset$ case.
Then $W^J = W$, so we drop "J" from the notation.

Bruhat interval f-vectors

Fix $w \in W$, and let $m := \lfloor (\ell(w) - 1)/2 \rfloor$. Let

$$P_{e,w}(q) = 1 + \beta_1 q + \cdots + \beta_m q^m$$

be the Kazhdan-Lusztig polynomial of the interval $[e, w]$.

Known:

* $\beta_i \geq 0$ if W is crystallographic,

* $P_{e,w}(q) = 1 \iff X_w$ is rationally smooth

* X_w is rationally smooth $\iff f_i^w = f_{\ell(w)-i}^w, \ \forall i$
(Carrell-Peterson '94)

* For simply-laced W: smooth $\Leftrightarrow$ rationally smooth

Theorem 3. *Suppose that* W *is crystallographic,* $w \in W$ *and* $1 \leq k \leq m$*. Then the following conditions are equivalent:*

(a) $f_i^w = f_{\ell(w)-i}^w$, *for* $i = 1, \ldots, k$,

(b) $\beta_i = 0$, *for* $i = 1, \ldots, k$.

Remark: The equivalence of (a) and (b) in the case $k = m$ gives the Carrell-Peterson criterion for rational smoothness of the Schubert variety X_w.

Theorem 4. *Suppose that W is crystallographic, $w \in W$ and $1 \leq k \leq m$. Then the following conditions are equivalent:*

(a) $f_i^w = f_{\ell(w)-i}^w$, *for* $i = 1, \ldots, k$,

(b) $\beta_i = 0$, *for* $i = 1, \ldots, k$.

Furthermore, if $k < m$ then (a) and (b) imply

(c) $\beta_{k+1} = f_{\ell(w)-k-1}^w - f_{k+1}^w$.

Idea of proof: Based on

* Monotonicity theorem for K-L polynomials (extending Braden-MacPherson '01).

* Polynomial $F_w(q) = \sum_{x \leq w} q^{\ell(x)} P_{x,w}(q)$ is palindromic (Kazhdan-Lusztig '79).

More can be said about the increasing inequalities

$$f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil},$$

namely, the sequence cannot grow too fast.

Recall:

For $n, k \geq 1$ there is a unique expansion

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

with $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$. Let

$$\begin{aligned} \partial^k(n) &= \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}, \\ \partial^k(0) &= 0. \end{aligned}$$

Theorem (Macaulay-Stanley)
For an integer sequence $(1, m_1, m_2, \ldots)$ the following are equivalent (and this defines an M-sequence):

(1) $\partial^k(m_k) \leq m_{k-1}$, for all $k \geq 1$,

(2) some order ideal of monomials contains exactly m_k monomials of degree k,

(3) $dim(A_k) = m_k$ for some graded commutative algebra $A = \oplus_{k\geq 0} A_k$ which is generated by A_1.

More can be said about the increasing inequalities

$$f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil},$$

namely, the sequence cannot grow too fast.

Theorem 5. *In the case of finite Weyl groups every f-vector $f^w = \{f_0, f_1, \ldots, f_{\ell(w)}\}$ is an M-sequence.*

Idea of proof of Thm: Based on

* The f-vector $f^w = \{f_0, f_1, \dots, f_{\ell(w)}\}$ is coeff-sequence of Poincaré polynomial of $H^*(X_w)$, the cohomology algebra of the Schubert variety X_w (over $\mathbb{C}$).

* So, we only need that $H^*(X_w)$ is generated in degree one (dim $= 2$).

* For $w = w_0$ this is classical: $H^*(X_{w_0}) \cong$ coinvariant algebra of W.

* For $w \neq w_0$ there is algebra surjection $H^*(X_{w_0}) \to H^*(X_w)$

Remarks:

** M-sequence property fails for the affine group $\widetilde{C}_2$:

$$\sum q^{\ell(w)} = 1 + 3q + 5q^2 + 8q^3 + \cdots$$

Consequence: $H^*(X_w)$ not necessarily generated in degree one for affine Schubert varieties X_w.

** M-sequence property fails for general intervals in finite B_4:

$$\sum_{w \leq x \leq w_0} q^{\ell(x)-\ell(w)} = 1 + 4q + 11q^2 + \cdots$$

for certain $w \in B_4$.

The increasing inequalities $f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil}$ have decreasing counterparts at the upper end of the Bruhat interval
— but the information we are able to give about this is much weaker.

Theorem 6. *For all $k \geq 1$ there exists a number N_k, such that for every finite Coxeter group (W, S) and every $w \in W$ such that $\ell(w) \geq N_k$ we have that*

$$f^w_{\ell(w)-k} \geq f^w_{\ell(w)-k+1} \geq \cdots \geq f^w_{\ell(w)}.$$

Questions

1. Do f^w-vectors satisfy more inequalities?
(Noticed by D. Stanton: unimodality fails on some intervals in Young's lattice. Unimodality might be true for full intervals, i.e. $J = \emptyset$.)

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2. Are the theorems true for general (non-crystallographic) Coxeter groups?

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3. Does there exist some $\alpha < 1$ such that

$$f^w_{\lfloor \alpha \cdot \ell(w) \rfloor} \geq f^w_{\lfloor \alpha \cdot \ell(w) \rfloor + 1} \geq \cdots \geq f^w_{\ell(w)}.$$

Will $\alpha = \frac{3}{4}$ do?

Questions

1. Does f^w-vector satisfy more inequalities?

2. Are Theorems 1–2 true for general (non-crystallographic) Coxeter groups?

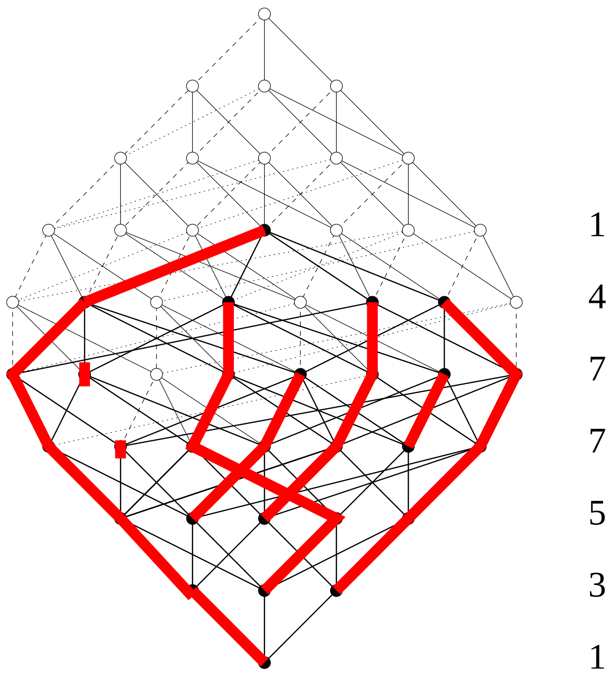
3. Does there exist some $\alpha < 1$ such that

$$f^w_{\lfloor \alpha \cdot \ell(w) \rfloor} \geq f^w_{\lfloor \alpha \cdot \ell(w) \rfloor + 1} \geq \cdots \geq f^w_{\ell(w)}.$$

Will $\alpha = \frac{3}{4}$ do?

4. What can be said about the shape of general intervals $[u, w]^J$? (I.e., for $u \neq e$)

Bruhat interval f-vectors



Def: An upper chain decomposition is a partition of $[e, w]^J$ into pairwise disjoint saturated chains

$$u_i < u_{i+1} < \cdots < u_k$$

such that $\ell(j) = j$ for all $j = i, \ldots, k$, and $k \geq \ell(w) - i$.

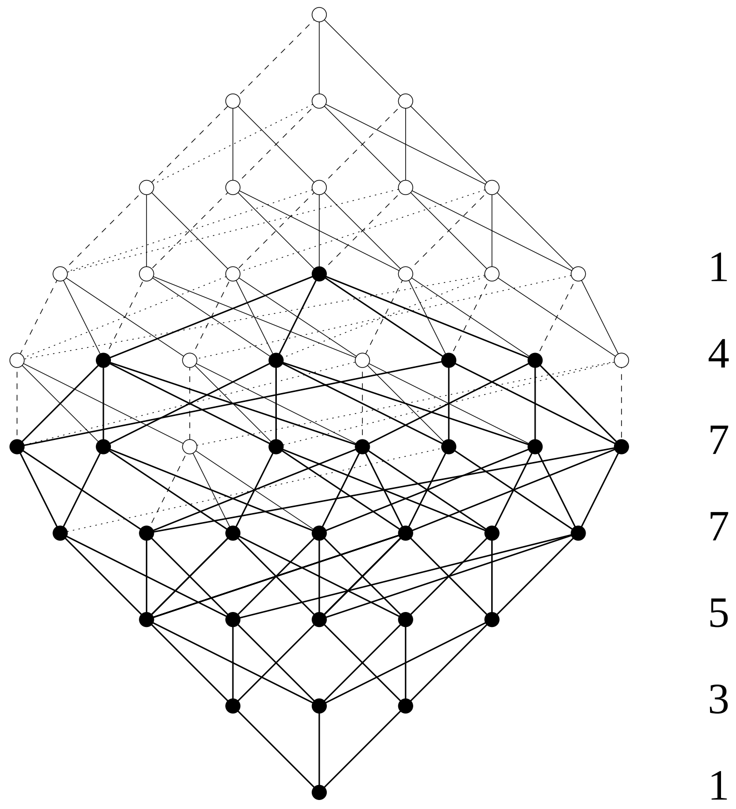
Questions

5. Do the intervals $[e, w]^J$ admit upper chain decompositions?

Note:

1. This would imply Thms 6 and 7.
2. Specializes to symmetric chain decomposition, if f^w-vector is symmetric.
3. Symmetric chain decomposition question still open for intervals $[\emptyset, \lambda]$ in Young's lattice, λ of rectangular shape.

Shape of Bruhat intervals



THE END

TOPIC 2: Global interval structure

Theorem 7. *(Verma 68)*

$$\sum_{u \leq x \leq w} (-1)^{\ell(x)} = 0$$

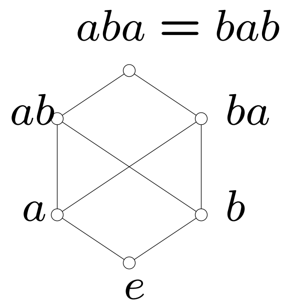
Equivalently, intervals $[u, w]$ "have the Euler characteristic of a sphere": # odd card chains − # even card chains = $(-1)^{\text{dim}}$

Theorem 8. *(Bj-Wachs 82)*

Intervals $[u, w]^J$ are "lexicographically shellable".

Interval structure

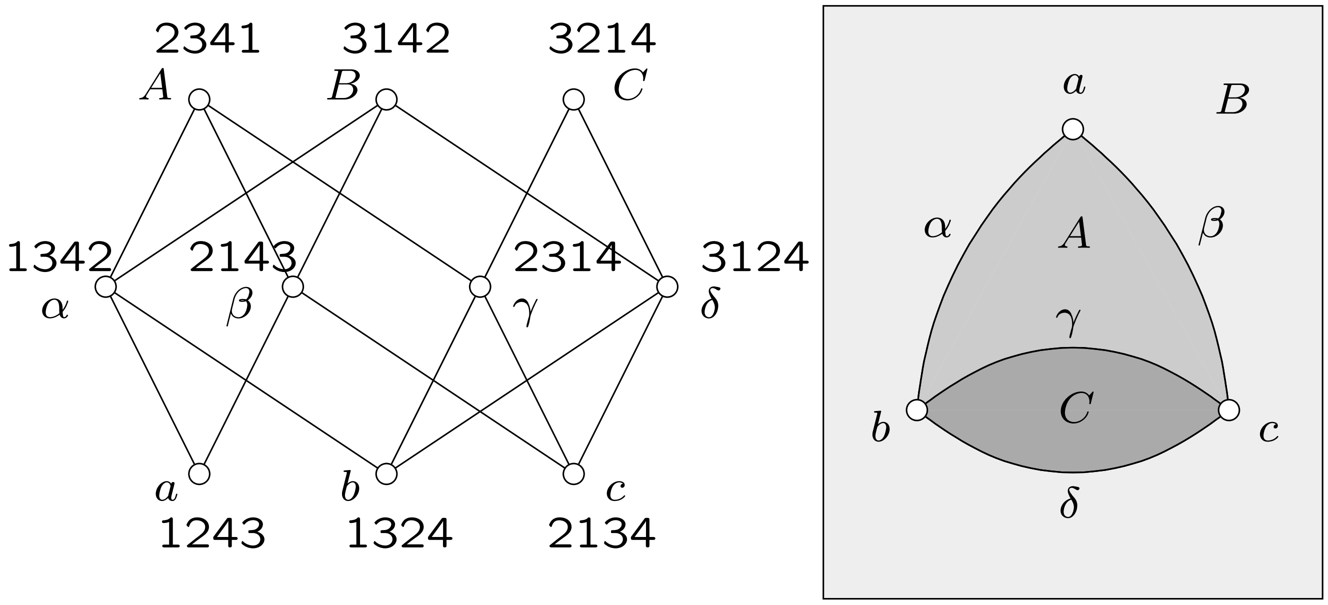
Example: Lex. shelling of $W = S_3$, generators $S = \{a, b\}$



Choosing "aba" as reduced expression for the top element the induced labels of the four maximal chains are

$$\lambda(aba \;\triangleright\; *ba \;\triangleright\; **a \;\triangleright\; ***) = (1,2,3),$$
$$\lambda(aba \;\triangleright\; *ba \;\triangleright\; *b* \;\triangleright\; ***) = (1,3,2),$$
$$\lambda(aba \;\triangleright\; ab* \;\triangleright\; *b* \;\triangleright\; ***) = (3,1,2),$$
$$\lambda(aba \;\triangleright\; ab* \;\triangleright\; a** \;\triangleright\; ***) = (3,2,1).$$

Interval structure



Regular CW interpretation of the Bruhat interval [1234, 3241] in S_4.

Theorem 9. *(Bj. 84) Let* $[u,w]$ *be a Bruhat interval. Then* $\exists$ *regular CW decomposition* $\Gamma_{u,w}$ *of the* $(\ell(w)-\ell(u)-2)$*-dimensional sphere with cells* σ_x*,* $u<x<w$*, such that*

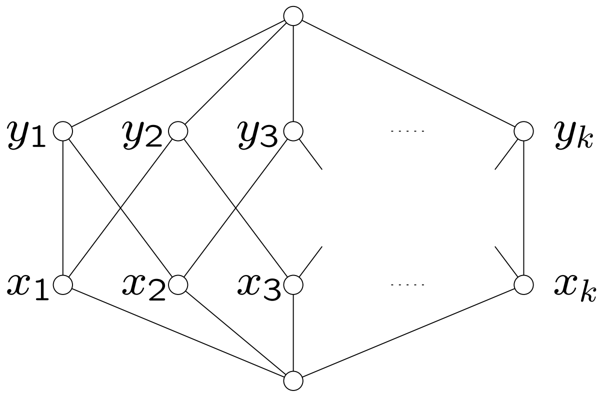
$$\dim(\sigma_x) = \ell(x) - \ell(u) - 1$$

and

$$\sigma_x \subseteq \overline{\sigma_z} \quad \Leftrightarrow \quad x \leq z.$$

Proof idea: via lexicographic shellability of Bruhat order

Interval structure



A k-crown.

Special case: All Bruhat intervals of length 3 are k-crowns, $k \geq 2$.

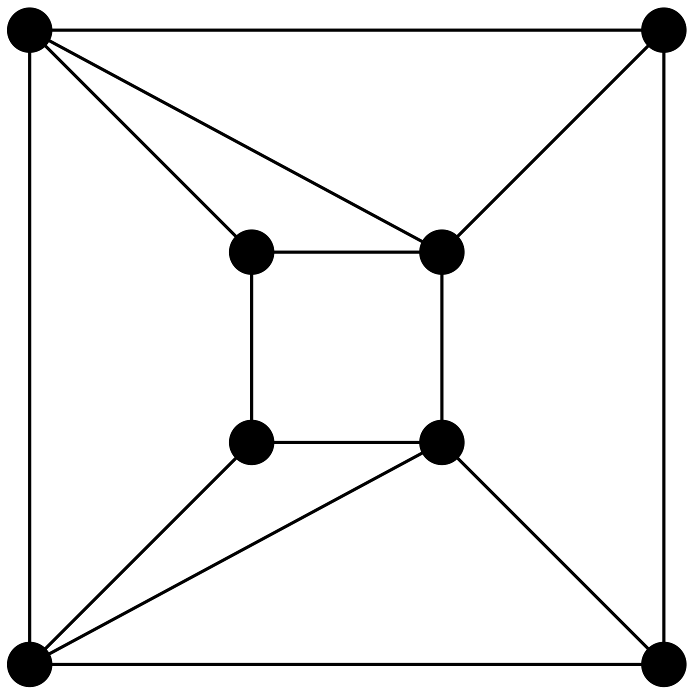
Finite case $\Rightarrow$ only $k = 2, 3, 4$ possible, for type H also $k = 5$.

Theorem 10. *(Dyer 91). For each m, there exist only finitely many isomorphism classes of length m intervals in finite Coxeter groups.*

Theorem 11. *(Hultman 03). There are 24 types of length 4 intervals in finite Weyl groups.*

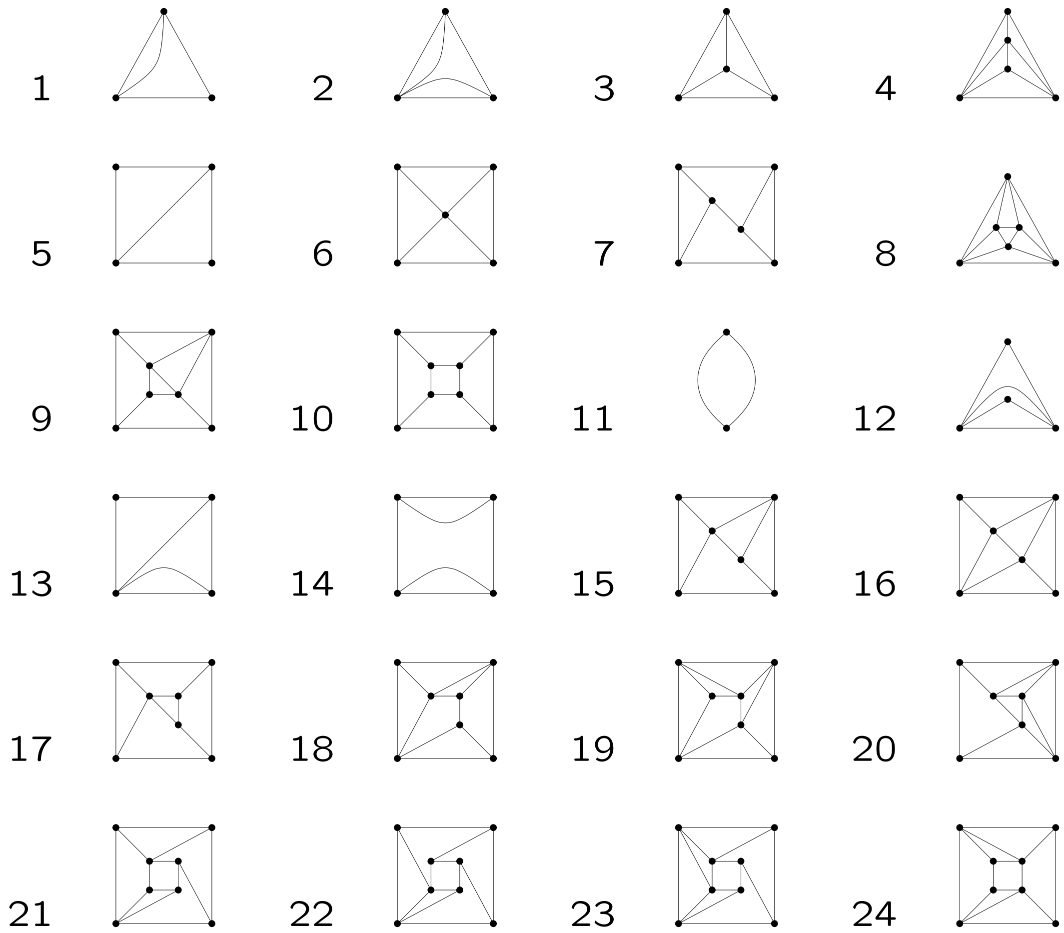
Only 7 of them occur in the symmetric groups.
All 24 show up in F_4.

Interval structure



A Bruhat interval of length 4
(rendered as a CW complex)

Interval structure



All length 4 intervals that appear in finite Weyl groups.

Interval structure

What about intervals of length > 4?

$\exists$ classification (up to (anti)isomorphism), due to F. Incitti:

* Length $5, 6, 7$ non-lattice intervals in type A:
there are 217 such of length 7

* Length 5 non-lattice intervals in type B: there are 46 such

* Length 5 non-lattice intervals in type D: there are 12 such

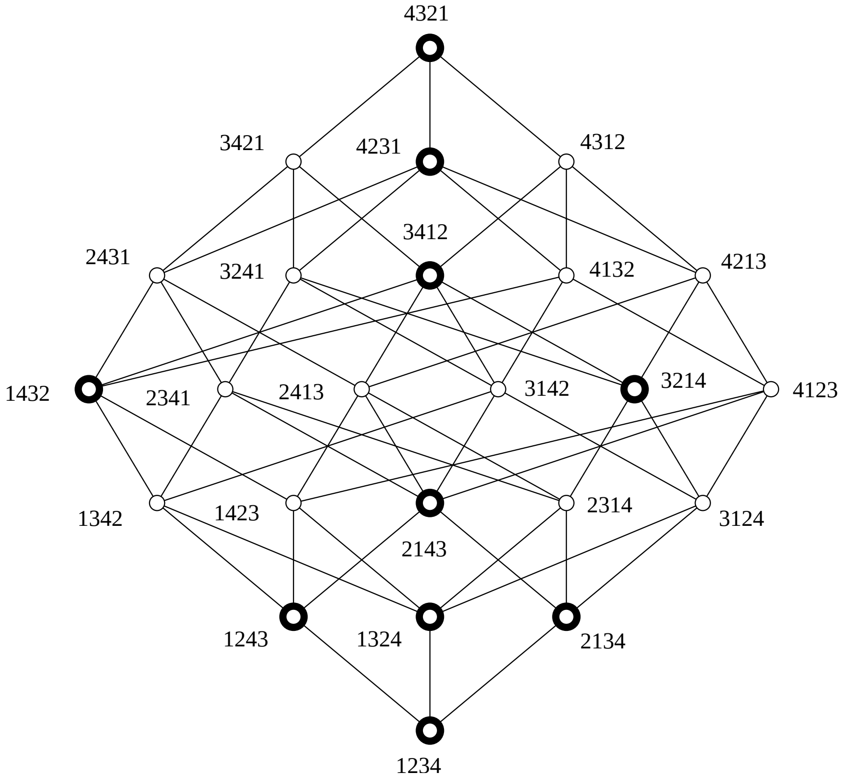
TOPIC 3: Some other Bruhat-related posets coming from algebraic geometry

Let Invol(W) $\stackrel{\text{def}}{=}$ involutions of W with induced Bruhat order.

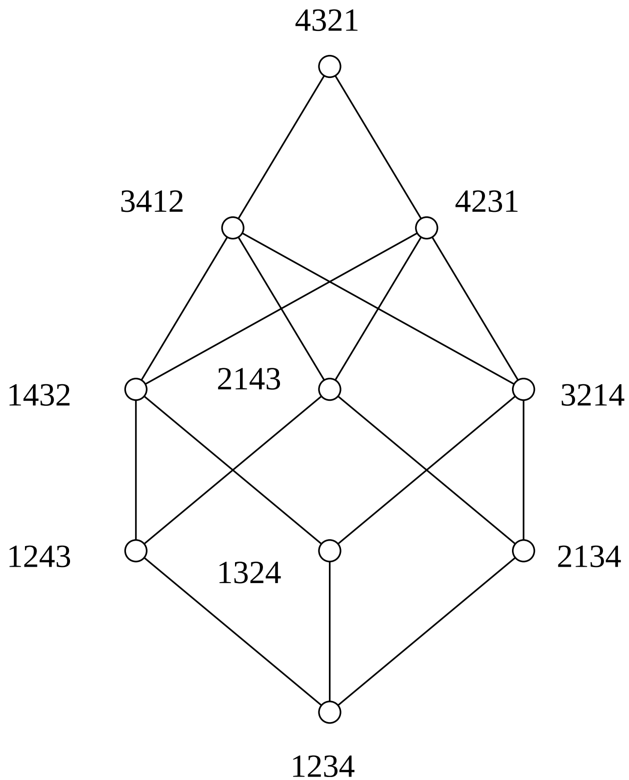
Studied by Richardson-Springer '94 in connection with certain orbit spaces arising from real reductive Lie geoups, then combinatorially by Incitti '03 and Hultman '04.

Has wonderful properties as poset, much as W itself: pure, intervals = regular CW spheres, ...

Poset rank function: $\text{rk}(w) = \frac{\ell(w)+a\ell(w)}{2}$, where $a\ell(w)$ is absolute length.



Involutions in S_4



Invol(S_4)

Open problems:

1. Are there only finitely many isomorphism types of intervals of length k in Invol(W), W finite ?

2. Are intervals in Invol(W) shellable?
(Proved for finite groups of classical type by Incitti)

Greater generality:

Let $\theta : W \to W$ be induced by an involutive automorphism of the Coxeter diagram. The set of *twisted involutions* (with respect to θ) is $\{w \in W \mid \theta(w) = w^{-1}\}$.

Invol(W) poset is part of the more general construction of *poset of twisted involutions*, containing both Invol(W) and Bruhat order on W itself as special cases.

Let $\mathrm{Int}(W) \stackrel{\text{def}}{=}$ Bruhat intervals of W ordered by containment.

Inherits good properties as poset from W:
pure, shellable, intervals = regular CW spheres, ...

Studied by Lusztig and Rietsch '98 as poset of cells of a decomposition of the totally nonnegative part of a flag variety over $\mathbb{R}$ (for a reductive algebraic group).

Further poset-related work on totally nonnegative Grassmannians and partial flag varieties by Postnikov, Rietsch and Williams. Related work by Fomin and Shapiro.

"Theorem-Definition" (R-polynomials):
There is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *satisfying the following conditions:*

(i) $R_{u,v}(q) = 0$ *if* $u \not\leq v$*;*

(ii) $R_{u,v}(q) = 1$ *if* $u = v$*;*

(iii) if $vs < v$ *then*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } us < u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } us > u. \end{cases}$$

"Theorem-Definition" (Kazhdan-Lusztig polynomials):
There exists a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *satisfying the following conditions:*

(i) $P_{u,v}(q) = 0$ *if* $u \not\leq v$*;*

(ii) $P_{u,v}(q) = 1$ *if* $u = v$*;*

(iii) $\deg(P_{u,v}(q)) \leq \frac{1}{2}(\ell(u,v) - 1)$, *if* $u < v$*;*

(iv)

$$q^{\ell(u,v)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{a \in [u,v]} R_{u,a}(q)\, P_{a,v}(q), \qquad \text{if } u \leq v.$$

Two famous conjectures for K-L polynomials:

1. Nonnegativity: $P_{u,v}(q) \in \mathbb{N}[q]$

2. Combinatorial invariance:

$$[u, v] \cong [x, y] \implies P_{u,v}(q) = P_{x,y}(q)$$

Two famous conjectures for K-L polynomials:

1. Nonnegativity: $P_{u,v}(q) \in \mathbb{N}[q]$

KNOWN: True for all crystallographic groups
$(m(s, s') \in \{2, 3, 4, 6, \infty\})$

2. Combinatorial invariance:

$$[u, v] \cong [x, y] \implies P_{u,v}(q) = P_{x,y}(q)$$

KNOWN: Partial results by Brenti-Caselli-Marietti and du Cloux