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TRIANGLE-FREE TRIANGULATIONS

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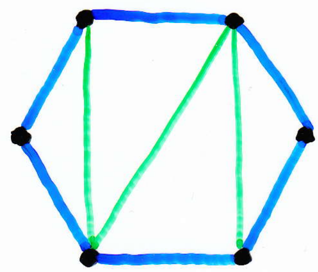
TFT

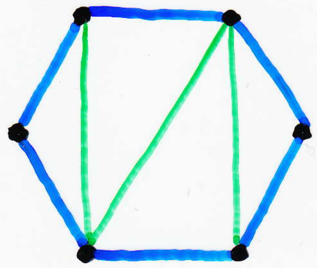
TRIANGULATION OF CONVEX n-GON



DEF: A TRIANGULATION IS TRIANGLE-FREE (TFT) IF IT HAS NO TRIANGLES

TFT

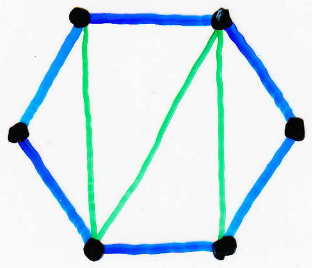
TRIANGULATION OF CONVEX n-GON



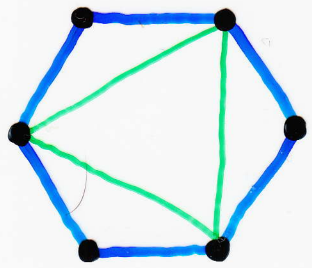
DEF: A TRIANGULATION IS TRIANGLE-FREE (TFT) IF IT HAS NO TRIANGLES WITH INTERNAL EDGES ONLY.

INTERNAL EDGE = DIAGONAL OF n-GON

EXTERNAL EDGE = EDGE OF n-GON

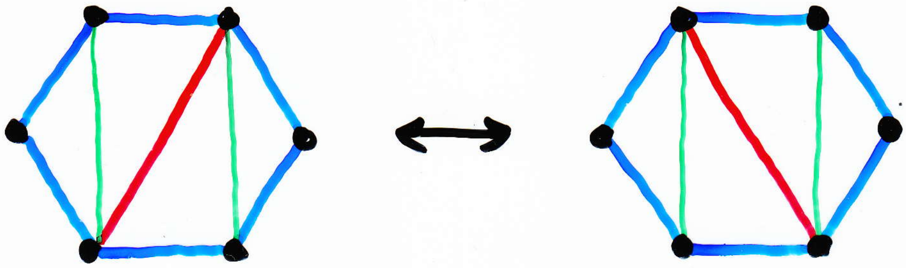


TFT



NON-TFT

FLIP :



FLIP GRAPH $G(n)$:

NODES = TFT(n)

(SET OF ALL TFT'S OF n-GON)

ARCS = FLIPS

Q : WHAT IS THE DIAMETER OF THE GRAPH $G(n)$?

MOTIVATION

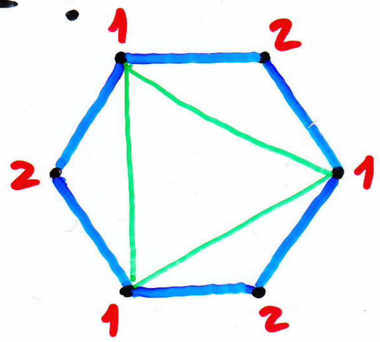
1. DIAMETER PROBLEM FOR FLIP GRAPH OF ALL TRIANGULATIONS OF n-GON:

$$\leq 2n-10 \quad (n \geq 13)$$
$$= 2n-10 \quad (n \gg 1)$$

(CONJECTURED BY KUPITZ;
PROVED BY SLEATOR-TARJAN-THURSTON [JAMS '88])

2. FLIP GRAPH OF MONOCHROMATIC-TRIANGLE-FREE TRIANGULATIONS [SAGAN '08] $\#\text{VERTICES} = 2^t C_{t,3}$ $(n = 2t)$

DIAMETER = ?



NO

3. ALGEBRAIC INTERPRETATION OF FLIPS (AS A GROUP ACTION).

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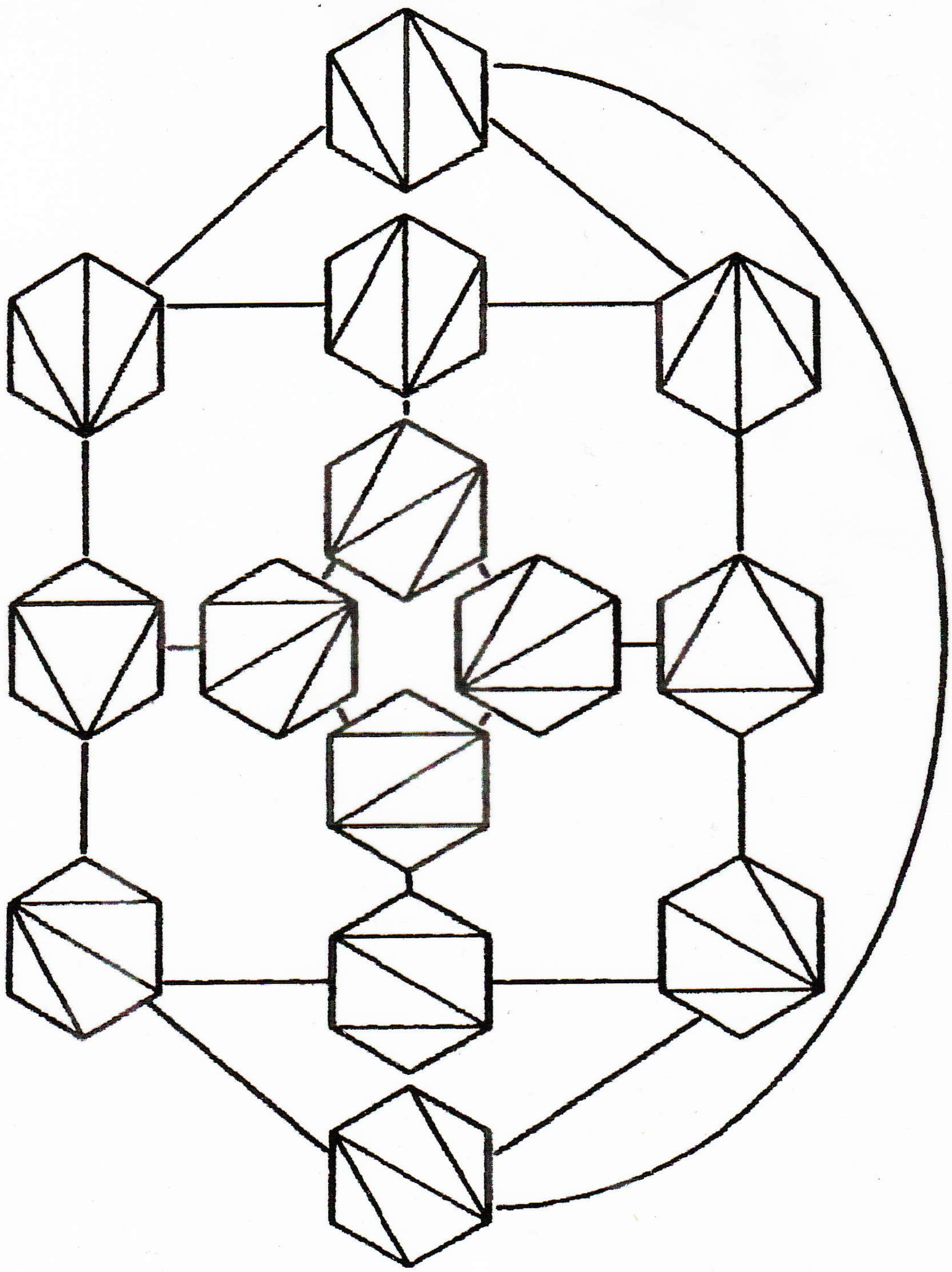
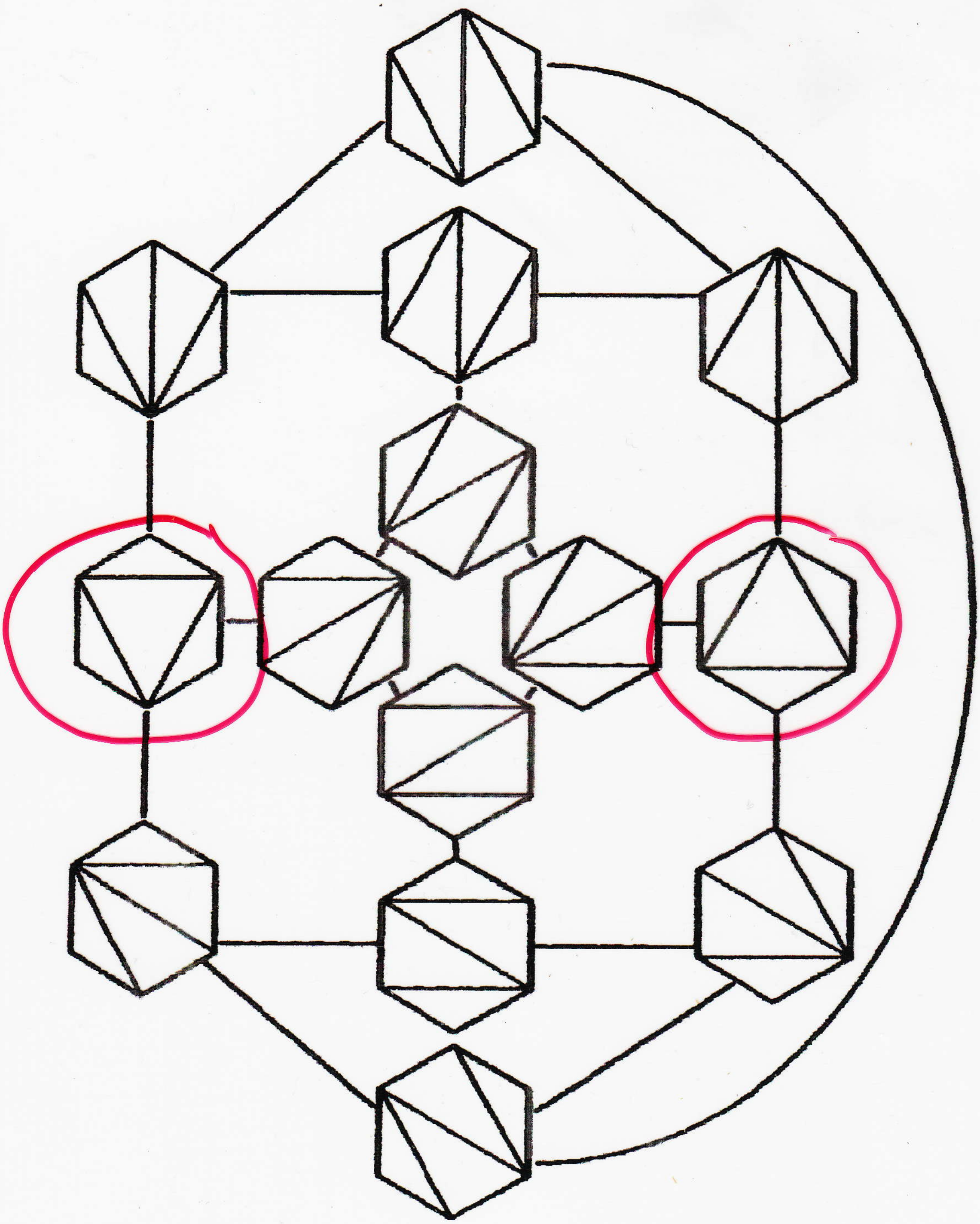
MAIN RESULT

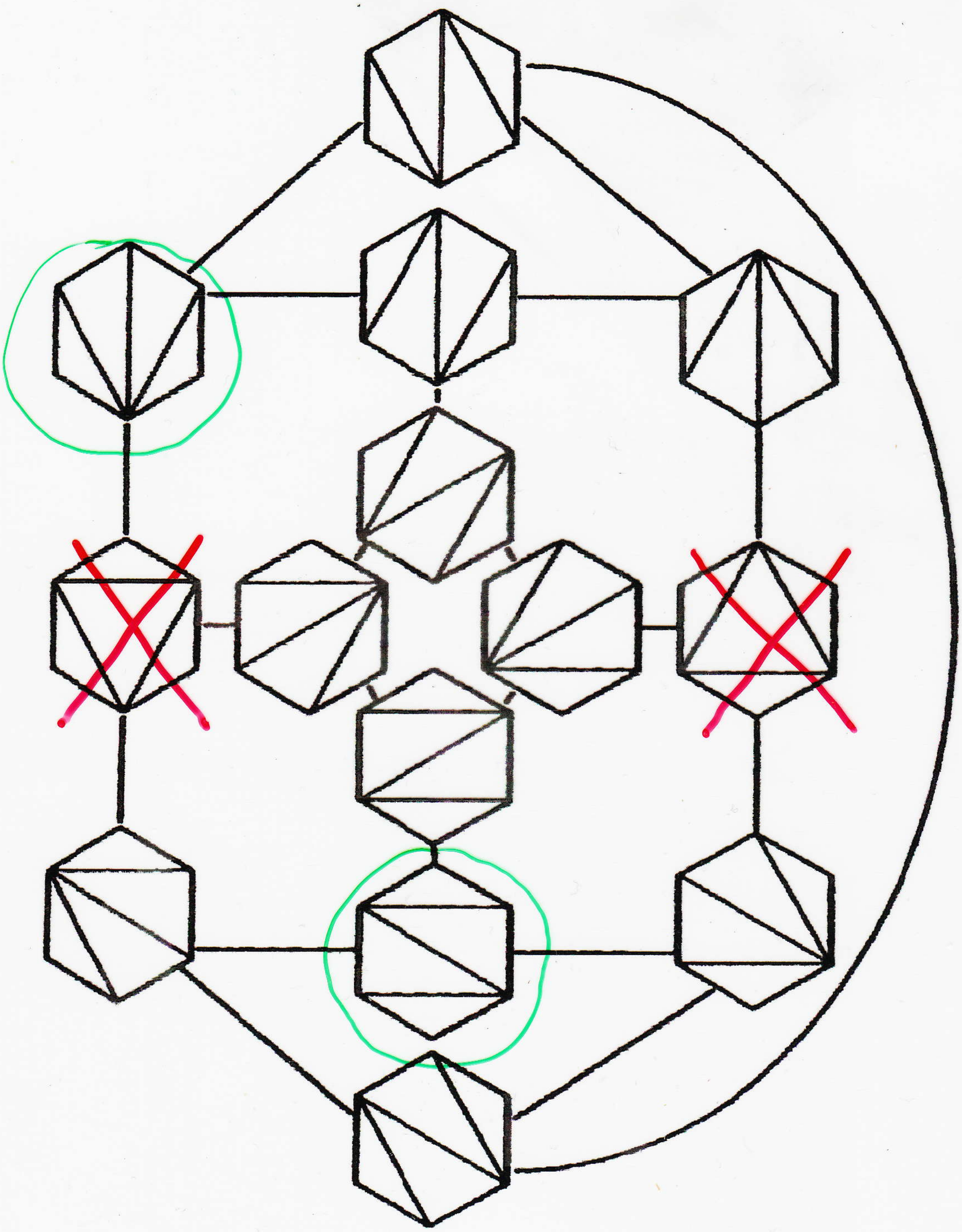
THM: [AFR] FOR $n \geq 3$,

$$\operatorname{diam} G(n) = \left\lfloor \frac{n(n-3)}{4} \right\rfloor$$





d = 4



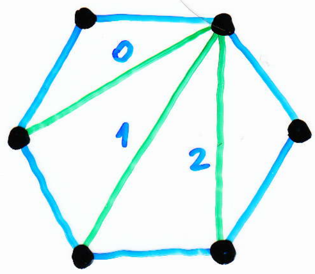
TFT

d = 4

LABEL THE INTERNAL EDGES $0, \dots, n-4$

(TWO POSSIBILITIES PER TFT).

LABEL FLIPS $s_0, \dots, s_{n-4}$.



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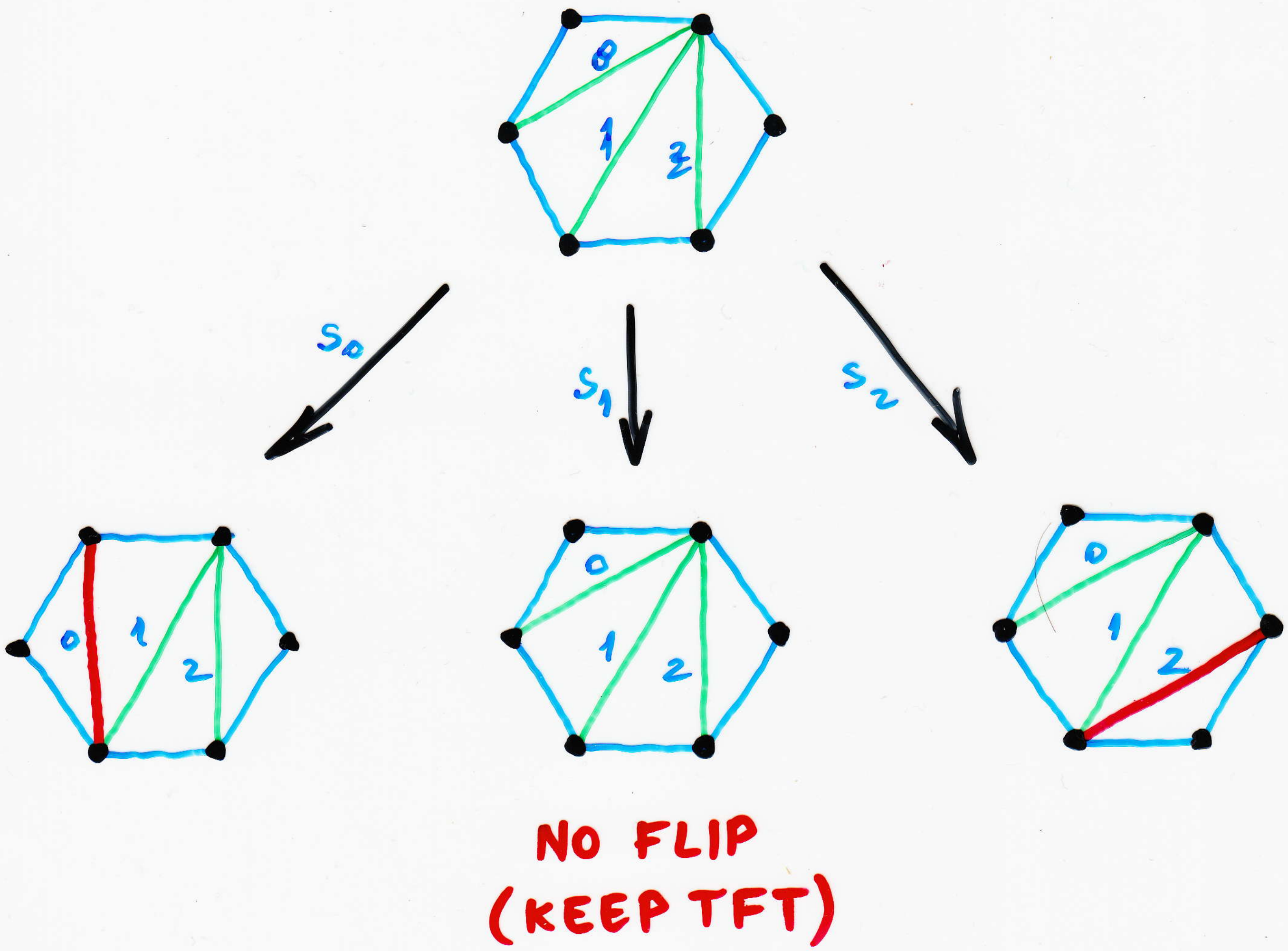

$\underline{T(n)}$:= 2-COVER OF $G(n)$

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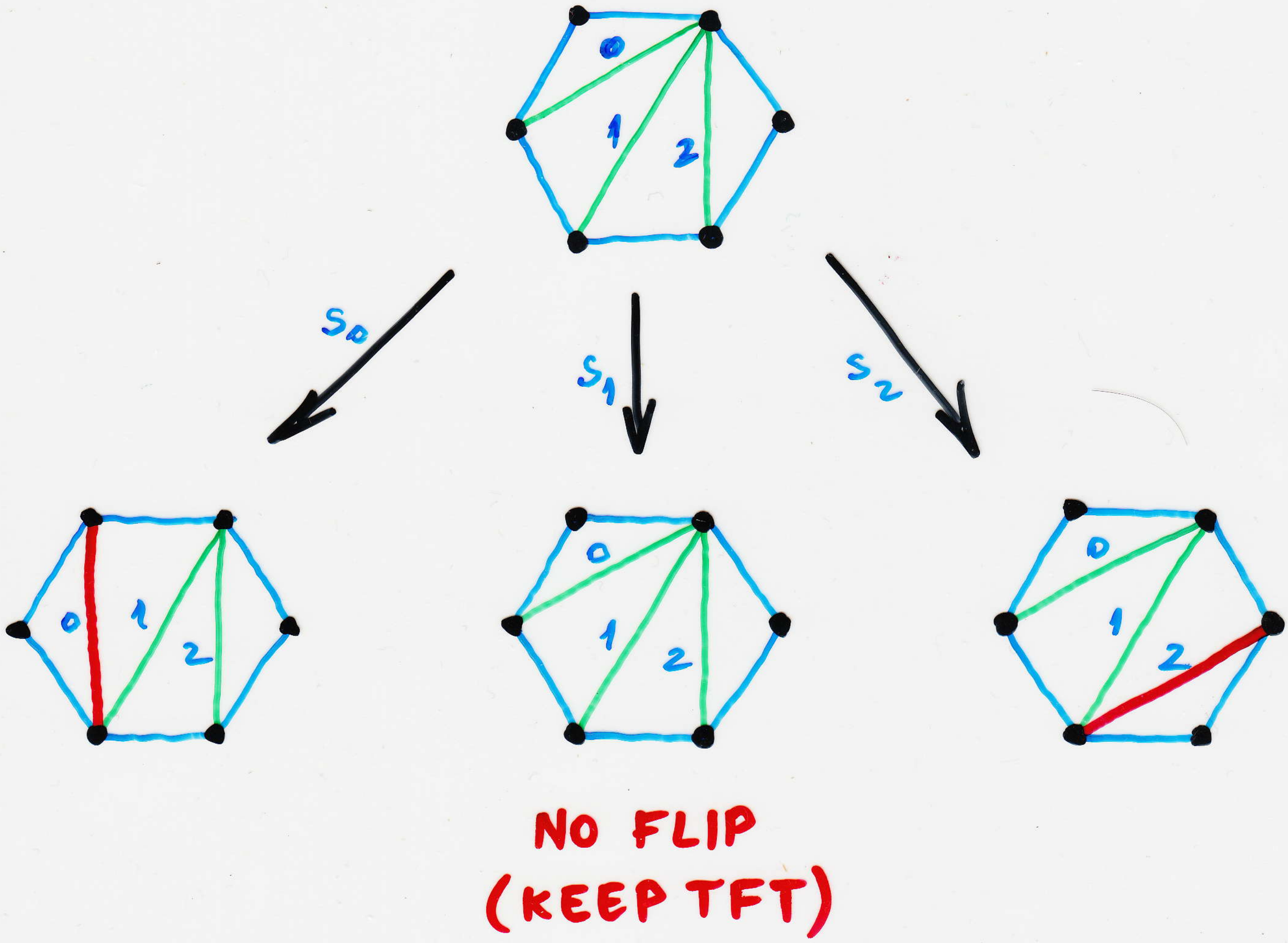
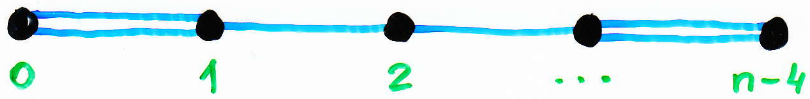


$\underline{T(n)}$:= 2-COVER OF $G(n)$

→ GROUP ACTION OF ...



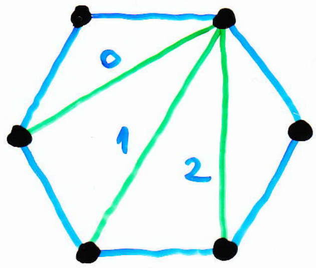
$\tilde{C}_{n-4}$

AFFINE WEYL GROUP

$$\tilde{C}_{n-4} = \langle s_0, \dots, s_{n-4} \mid s_i^2 = 1 \ (\forall i), \ (s_i s_j)^2 = 1 \ (|i-j|>1), \ (s_i s_{i+1})^3 = 1 \ (0 < i < n-5), \ (s_i s_{i+1})^4 = 1 \ (i = 0, n-5) \rangle$$

CLAIM: $\tilde{C}_{n-4}$ ACTS TRANSITIVELY ON $T'(n)$ BY LABELLED FLIPS.

St_{n-4} := STABILIZER OF STAR TFT

T_0 = 

LEMMA: $St_{n-4} \leq \tilde{C}_{n-4}$ IS GENERATED BY $g_0, s_1, \ldots, s_{n-5}, g_{n-4}$,

WHERE

$$g_0 := s_0 s_1 \cdots s_{n-6} s_{n-4} s_{n-5} s_{n-6} \cdots s_1 s_0$$

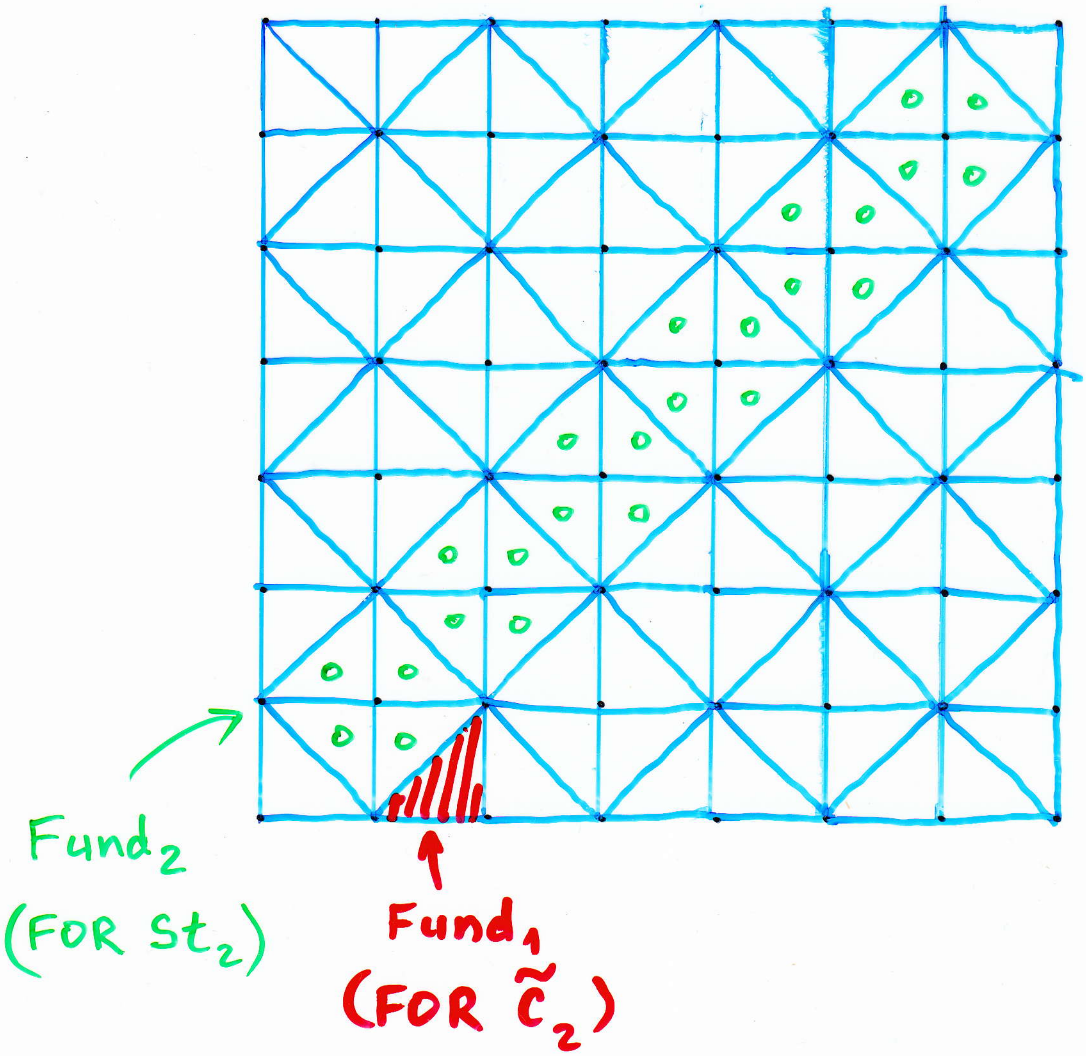
$$g_{n-4} := (s_{n-4} \cdots s_0)^n$$

FUNDAMENTAL REGIONS:

Fund_1 (FOR $\tilde{C}_{n-4}$) = SIMPLEX Δ_{n-4}

Fund_2 (FOR St_{n-4}) = PRISM $\Delta_{n-5} \times I$

$n = 6$:



LEMMA:

$$\frac{\mathrm{vol}(\mathrm{Fund}_2)}{\mathrm{vol}(\mathrm{Fund}_1)} = 2^{n-4} \cdot n$$

PROOF: DETERMINANTS.

EX: $n = 6$, $2^{n-4} \cdot n = 24$.

COROLLARY:

$$2 \cdot \#\mathrm{TFT}(n) = \#\mathrm{VERT}\,\Gamma(n) = 2^{n-4} \cdot n$$

$\Gamma(n)$ IS A SCHREIER GRAPH (COSETS OF St_{n-4} IN $\tilde{C}_{n-4}$).

COSET REPRESENTATIVES:

$a_i := s_i \cdots s_0 \quad (0 \leq i \leq n-4)$

$R_{n-4} := \{ a_0^{\epsilon_0} \cdots a_{n-4}^{\epsilon_{n-4}} \mid \epsilon_i \in \{0,1\} \ (i \neq n-4),$
$0 \leq \epsilon_{n-4} < n-4 \}$

LEMMA: R_{n-4} IS A COMPLETE SET OF REPRESENTATIVES FOR THE COSETS OF St_{n-4} IN $\tilde{C}_{n-4}$.

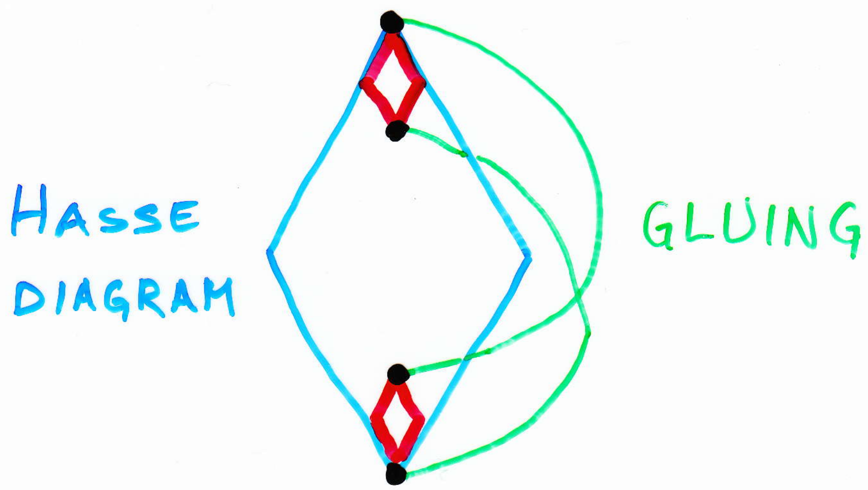
LEMMA: R_{n-4} IS A SELF-DUAL LOWER INTERVAL $[id, w]$ IN LEFT WEAK ORDER ON $\tilde{C}_{n-4}$.

COROLLARY: R_{n-4} IS A GRADED LATTICE.

LEMMA: EDGES OF $T(n)$ ARE

(1) EITHER EDGES OF THE HASSE DIAGRAM OF LEFT WEAK ORDER,

(2) OR IN A SMALL SET OF "VERTEX GLUINGS".



LEMMA:

$$\operatorname{diam}(\Gamma(n)) = \frac{n(n-3)}{2}$$

COROLLARY:

$$\operatorname{diam}(G(n)) = \left\lfloor \frac{n(n-3)}{4} \right\rfloor$$

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THANK YOU

MERCI

OBRIGADO

ありがとうございました。

GRACIAS