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ISOMETRY CLASSES OF GENERALIZED ASSOCIAHEDRA

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Abstract. Let W be a finite Coxeter group. Generalized associahedra are convex polytopes constructed from a permutahedron of W and an orientation of the Coxeter graph of W. They play a fundamental role in the theory of finite type cluster algebras initiated by Fomin and Zelevinsky, and also appear in algebraic topology. In this article, we show that the isometries of these polytopes are given by certain automorphisms of oriented Coxeter graphs.

To the memory Pierre Leroux

1. Introduction

Studying homotopy theory of loop spaces, Jim Stasheff [10, 11] constructed a cell complex whose vertices correspond to the possible compositions of n binary operations. Furthermore this cell complex can be realized as a simple polytope, the associahedron (also called Stasheff polytope). There is a natural relation between the permutahedron (weak lattice on permutations) and the associahedron: the permutahedron can naturally be written as an intersection of halfspaces indexed by weights for the A_n root system. If one intersects a certain, carefully chosen subset of these halfspaces, one can obtain the associahedron (see Example 2.2 below). Shnider-Sternberg [9] and Loday [5] give us a beautiful explicit construction of the associahedron from the permutahedron along these lines.

Generalized associahedra were introduced by S. Fomin and A. Zelevinsky in their work on cluster algebras [2]. The geometry of these objects encodes nice algebraic structures. Therefore one important question is to find good polytopal realizations of the generalized associahedra. This was first answered in [1] by Chapoton, Fomin and Zelevinsky. Then, N. Reading [8] constructed a family of fans, the Cambrian fans $\{\mathcal{F}_c\}$ indexed by Coxeter elements c of a given finite Coxeter group W. More recently, we have constructed [3] a family of generalized associahedra $\mathsf{Asso}_c(W)$, one for each Cambrian fan $\mathcal{F}_c$. These generalized associahedra are realized from the corresponding permutahedron by removing some halfspaces according to a rule

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specified by c, linking questions about generalized associahedra to questions about the better known permutahedron.

It is now natural to ask how many distinct (up to isometry) generalized associahedra we get. This is what we answer here. Our main theorem (Theorem 2.3) describes completely the isometry classes of generalized associahedra as realized in [3]. The isometry classes depend of the choice of the starting permutahedron. As a byproduct we obtain a classification of the isometry classes of Cambrian fans (Corollary 2.6): the Cambrian fans indexed by Coxeter elements c and c' are isometric if and only if $\mu(c') = c$ or $\mu(c') = c^{-1}$ for some μ an automorphism of the Coxeter graph of W. In Section 2 we introduce the necessary definitions and state our main theorem (Theorem 2.3). The proof is found in Section 4. Section 3 is dedicated to some auxiliary results needed for this proof. For most of the paper, we make the simplifying assumption that the Coxeter system in question is irreducible; in Section 5, we explain how to deal with the reducible case.

2. Background and main theorem

We assume some basic familiarity with Coxeter groups and root systems and follow the notation of [4]. Let (W, S) be a finite Coxeter system acting by reflections on an $\mathbb{R}$-Euclidean space $(V, \langle\cdot,\cdot\rangle)$ with length function $\ell : W \to \mathbb{N}$. Without loss of generality, we assume that the action of W is essential relative to V, that is, has no nontrivial space fixed pointwise.

Let Φ be a root system corresponding to (W, S), with all roots having equal length. (In particular, we do not assume Φ is crystallographic.) The simple roots Δ form a basis of V, and the reflection s maps α_s to $-\alpha_s$ and fixes the hyperplane $H_s = \{v \in V \mid \langle v, \alpha_s\rangle = 0\}$. Let $\Delta^* = \{v_s \mid s \in S\}$ be the set of fundamental weights of Δ, that is, $\langle v_t, \alpha_s\rangle = 1$ if $s = t$ and $\langle v_t, \alpha_s\rangle = 0$ otherwise. As V is finite dimensional we identify V and V^*.

2.1. **The permutahedron.**

We now aim for a definition of the W-permutahedron and pick a point $u \in V$ contained in the complement of the reflection hyperplanes of W. Without loss of generality, we choose

$$u := \sum_{s\in S} \kappa_s v_s, \qquad \kappa_s > 0.$$

For $w \in W$ we write

$$M(e) := u \qquad \text{and} \qquad M(w) := w\,(M(e))$$

and obtain the permutahedron $\mathsf{Perm}_u(W)$ as convex hull of $\{M(w) \mid w \in W\}$. The index u will often be omitted for brevity. Equivalently, we have

$$\mathsf{Perm}(W) = \bigcap_{s\in S} \bigcap_{x\in W} \mathscr{H}_{(x,s)}$$

where

$$\mathscr{H}_{(x,s)} := \{v \in V \mid \langle v, x(v_s)\rangle \le \langle M(e), v_s\rangle\}.$$

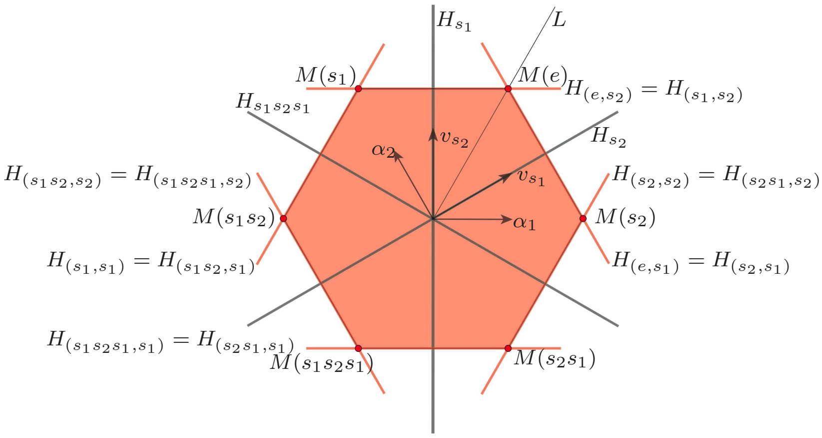

FIGURE 1. The permutahedron $\mathsf{Perm}(S_3)$ obtained as convex hull of the S_3-orbit of $M(e) \in L$ or as intersection of the half spaces $\mathscr{H}_{(x,s)}$.

We also make use of the hyperplane $H_{(x,s)} = \{v \in V \mid \langle v, x(v_s)\rangle = \langle M(e), v_s\rangle\}$.

Denote by W_I the *standard parabolic subgroup of* W generated by $I \subseteq S$. Note that $H_{(w,s)} = H_{(x,s)}$ if and only if $w \in xW_{S\setminus\{s\}}$. Also, $M(w) \in H_{(x,s)}$ if and only if $H_{(x,s)} = H_{(w,s)}$. Hence we have a simple way to describe the vertices:

$$\{M(w)\} = \bigcap_{s\in S} H_{(w,s)}.$$

Example 2.1 (Realization of $\mathsf{Perm}(A_2)$)**.** We consider the Coxeter group $W = S_3$ of type A_2 acting on $\mathbb{R}^2$. The reflections s_1 and s_2 generate W. The simple roots that correspond to s_1 and s_2 are α_1 and α_2. They are normal to the reflection hyperplanes H_{s_1} and H_{s_2}. The dual vectors to the simple roots correspond to the vectors v_1 and v_2. Fix a ray $L = \{\mu(\kappa_1 v_1 + \kappa_2 v_2) \mid \mu > 0\}$ where $\kappa_1, \kappa_2 > 0$. We choose $M(e) \in L$ and obtain the permutahedron as convex hull of the W-orbit of $M(e)$. Alternatively, the permutahedron can be described as intersection of the half spaces $\mathscr{H}_{(x,s)}$ with bounding hyperplanes $H_{(x,s)}$ for $x \in W$ and $s \in S$. All the objects are indicated in Figure 1.

2.2. **Generalized associahedra.** For c a Coxeter element in W, that is to say, the product of the simple reflections of W taken in some order, and $I \subseteq S$, we denote by $c_{(I)}$ the subword of c obtained by taking only the simple reflections in I. So $c_{(I)}$ is a Coxeter element of W_I. Reading defined the *c-sorting word* of $w \in W$ in [6, Section 2] as the unique subword of the infinite word $c^\infty = cccccc\ldots$ that is a reduced expression for w and is the lexicographically smallest sequence of positions occupied by this subword. In particular, the c-sorting word of w is such that $w = c_{(K_1)}c_{(K_2)}\ldots c_{(K_p)}$ with non-empty $K_i \subseteq S$ and $\ell(w) = \sum_{i=1}^{p} |K_i|$. As example we consider the Coxeter group $W = S_4$ of type A_3 generated by the simple reflections $S = \{s_1, s_2, s_3\}$, where s_1, s_3 commute, and the Coxeter element $c = s_2s_1s_3$. The c-sorting word of the longest element $w_0 \in W$ is $s_2s_1s_3s_2s_1s_3 = c_{(S)}c_{(S)}$. If we choose the Coxeter element $c = s_1s_2s_3$ instead of $s_2s_1s_3$, then the c-sorting word of w_0 is $s_1s_2s_3s_1s_2s_1 = c_{(S)}c_{(\{s_1,s_2\})}c_{(\{s_1\})}$.

The sequence $c_{(K_1)}, \ldots, c_{(K_p)}$ associated to the c-sorting word for w is called *the c-factorization* of w. The c-factorization of w is independent of the chosen reduced word for c but depends on the Coxeter element c. In general the c-factorization does not yield a nested sequence $K_1, \ldots, K_p$ of subsets of S. An element $w \in W$ is called *c-sortable* if $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_p$. Reading proves in [6] that the longest element $w_0 \in W$ is c-sortable for any chosen Coxeter element c.

Given a specific reduced word $\mathbf{v}$, we say that u is a prefix up to commutation of $\mathbf{v}$ if some reduced word for u appears as a prefix of a word which can be obtained from $\mathbf{v}$ by the commutation of commuting reflections. In [3] we define an element $w \in W$ to be a *c-singleton* if it is a prefix up to commutation of the c-factorization of w_0. We illustrate this notion by considering again the Coxeter group $W = S_4$ and the Coxeter element $c = s_2s_3s_1$. The c-singletons are

$$\begin{array}{lll} e, & s_2s_3, & s_2s_1s_3s_2s_1, \\ s_2, & s_2s_1s_3, & s_2s_1s_3s_2s_3, \text{ and} \\ s_2s_1, & s_2s_1s_3s_2, & w_0 = s_2s_3s_1s_2s_3s_1. \end{array}$$

For example s_2s_1 is a not a prefix of the c-factorization $w_0 = s_2s_3s_1s_2s_3s_1$, but it is a prefix up to commutation because it appears as a prefix after commuting the simple reflections s_1, s_3.

The halfspace $\mathscr{H}_{(x,s)}$ is said to be *c-admissible* if the hyperplane $H_{(x,s)}$ contains $M(w)$ for some c-singleton w. We have shown in [3] that the intersection of all c-admissible halfspaces $\mathscr{H}_{(x,s)}$ is a *generalized associahedron* $\mathsf{Asso}_c(W)$ whose normal fan is the c-Cambrian fan $\mathcal{F}_c$ (see [8] for a definition of $\mathcal{F}_c$).

Example 2.2. The Coxeter group $W = S_3$ generated by the reflections s_1 and s_2 has two Coxeter elements: $c_1 = s_1s_2$ and $c_2 = s_2s_1$. The c_1-singletons are e, s_1, s_1s_2, and $s_1s_2s_1$ while the c_2-singletons are e, s_2, s_2s_1,and $s_2s_1s_2$. Starting with the permutahedron $\mathsf{Perm}(S_3)$, we obtain the two associahedra $\mathsf{Asso}_{c_1}(S_3)$ and $\mathsf{Asso}_{c_2}(S_3)$ shown in Figure 2 as intersection of the c_1- and c_2-admissible halfspaces.

2.3. **Main result.**

For most of the paper, we will assume that (W, S) is irreducible. The case where (W, S) is reducible requires a straightforward (but not immediate) extension of the results in the irreducible case; we describe this in the final section.

An automorphism of the Coxeter graph associated to (W, S) is a bijection μ on S such that the order of $\mu(s)\mu(t)$ equals the order of st for all $s, t \in S$. In particular, μ induces an automorphism on W.

Let $u = \sum_{s\in S} \kappa_s v_s$ be a point in V. We will say that u *is balanced* if $\kappa_s = \kappa_t$ for all $s, t \in S$. An automorphism μ of the Coxeter graph is a *u-automorphism* if $\kappa_s = \kappa_{\mu(s)}$ for all $s \in S$. In particular, if u is balanced, then any automorphism of a Coxeter graph is a u-automorphism.

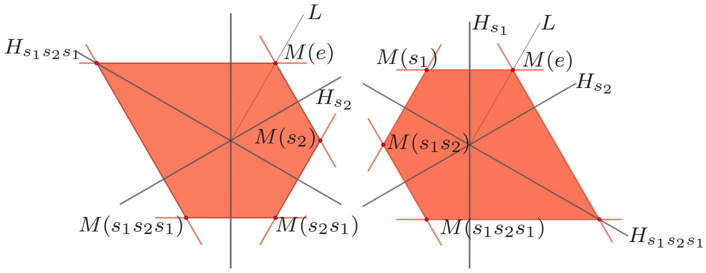

FIGURE 2. The two associahedra $\mathsf{Asso}_{c_1}(S_3)$ (left) and $\mathsf{Asso}_{c_2}(S_3)$ (right) obtained from the permutahedron $\mathsf{Perm}(S_3)$ by keeping the c-admissible halfspaces $\mathscr{H}_{(x,s)}$.

Theorem 2.3. *Let (W, S) be an irreducible finite Coxeter system and c_1, c_2 be two Coxeter elements in W. Suppose that $u = \sum_{s\in S} \kappa_s v_s$ for some $\kappa_s > 0$. The following statements are equivalent.*

(1) $\mathsf{Asso}_{c_1}(W) = \varphi\left(\mathsf{Asso}_{c_2}(W)\right)$ *for some linear isometry φ on V.*
(2) *There is a u-automorphism μ of the Coxeter graph of (W, S) such that $\mu(c_2) = c_1$ or $\mu(c_2) = w_0c_1^{-1}w_0$.*

Observe that $w_0c^{-1}w_0$ may or may not equal c (for instance in A_3 take $c = s_1s_3s_2$). So the second condition in Theorem 2.3 may be redundant and the associahedra may actually be identical (not just isometric). Moreover, if the coefficients κ_s are chosen generically, that is distinct, then the isometry classes are of cardinality 1 or 2. As stated in the next corollary, the isometry classes reach their maximal cardinality if u is balanced.

Corollary 2.4. *Let (W, S) be an irreducible finite Coxeter system and c_1, c_2 be two Coxeter elements in W. If u is balanced, then the following statements are equivalent.*

(1) $\mathsf{Asso}_{c_1}(W) = \varphi\left(\mathsf{Asso}_{c_2}(W)\right)$ *for some linear isometry φ on V.*
(2) *There is an automorphism μ of the Coxeter graph of (W, S) such that $\mu(c_2) = c_1$ or $\mu(c_2) = c_1^{-1}$.*

Proof. It follows from Theorem 2.3 and a rewriting of the second assumption in accordance with the fact that the map $s \mapsto w_0sw_0$ is an automorphism of the Coxeter graph. □

If u is balanced, then $\mathsf{Asso}_{c_1}(W) = \varphi\left(\mathsf{Asso}_{c_2}(W)\right)$ for some linear isometry φ on V if and only if there is a u-automorphism θ of the Bruhat ordering of (W, S) such that $\theta(c_2) = c_1$. This follows by inspection for $|S| = 1, 2$ and, for $|S| \geq 3$, from a characterization of automorphisms of Bruhat orderings due to van den Hombergh, see Section 8.8 of [4] and the fact that a Coxeter element c defines an orientation of the Coxeter graph Γ: orient the edge $\{s_i, s_j\}$ from s_i to s_j if and only if s_i is to the left of s_j for any reduced word for c.

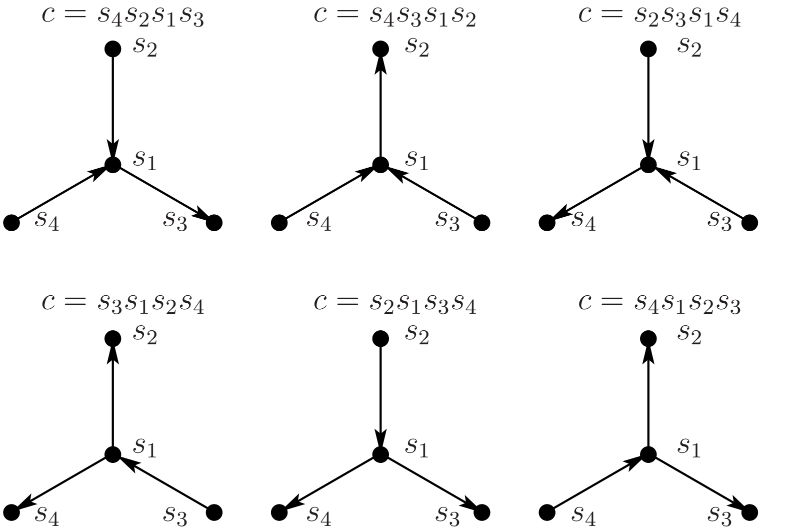

FIGURE 3. Six Coxeter elements and their associated oriented Coxeter graphs of the Coxeter group of type D_4 that yield isometric associahedra.

Theorem 2.3 combined with the classification of irreducible finite Coxeter groups yields that the cardinality of an isometry class in the case where u is balanced is either two, four, or six. We briefly discuss the situation.

Example 2.5. We use the notation and hypothesis of Corollary 2.4.

(1) Let (W, S) be a Coxeter system of type A_n $(n \geq 2)$, E_6, F_4, or $I_2(m)$. Then there is precisely one non-trivial automorphism μ of the Coxeter graph. Hence there are either two or four elements in the isometry class of $\mathsf{Asso}_c(W)$. The cardinality equals two if $\mu(c) \in \{c, c^{-1}\}$ and equals four if $\mu(c) \notin \{c, c^{-1}\}$.

(2) Let (W, S) be a Coxeter system of type B_n $(n \geq 2)$, E_7, E_8, H_3, or H_4. Then the conjugation by w_0 is the identity, and Id is the only automorphism of the associated Coxeter graph. So each isometry class has cardinality two, only the Coxeter elements c and c^{-1} yield isometric associahedra.

(3) Let (W, S) be a Coxeter system of type D. If $|S| > 4$ then there is only one non-trivial automorphism μ of the Coxeter graph and the isometry class of $\mathsf{Asso}_c(W)$ has cardinality two if $\mu(c) \in \{c, c^{-1}\}$ and four otherwise. If $|S| = 4$, the group of automorphisms of the Coxeter graph is generated by the non-trivial automorphisms μ and ν with $\mu^2 = \mathrm{Id}$ and $\nu^3 = \mathrm{Id}$. The isometry class of $\mathsf{Asso}_c(W)$ consists either of two or six elements, see Figure 3 for six Coxeter elements that yield isometric associahedra.

Theorem 2.3 allows a classification of the isometric Cambrian fans as well. The proof will be given at the end of Section 4.

Corollary 2.6. *The following propositions are equivalent:*

(1) *The Cambrian fans* $\mathcal{F}_c$ *and* $\mathcal{F}_{c'}$ *are isometric;*
(2) $\mathsf{Asso}_c(W)$ *and* $\mathsf{Asso}_{c'}(W)$ *are isometric if* u *is balanced;*
(3) *there is an automorphism* μ *of the Coxeter graph of* (W, S) *such that* $\mu(c') = c$ *or* $\mu(c') = c^{-1}$.

Remark 2.7. In fact, the condition that u be balanced in (2) above can be weakened to require only that u satisfies $\kappa_s = \kappa_{w_0 s w_0}$, for all $s \in S$ (see Eq. (1) in the proof).

3. Preliminary results

From now on, we fix $u = \sum_{s\in S} \kappa_s v_s$ for some constants $\kappa_s > 0$. Remember that $M(e) = u$.

Let (W, S) be a Coxeter system and consider the polyhedron P defined by:

$$P := \bigcap_{s\in S} \mathscr{H}_{(e,s)} \cap \bigcap_{s\in S} \mathscr{H}_{(w_0,s)}.$$

Remark 3.1. In fact, P is a full-dimensional convex polytope because W acts essentially on V and the cones $\bigcap_{s\in S} \mathscr{H}_{(e,s)}$ and $\bigcap_{s\in S} \mathscr{H}_{(w_0,s)}$ are strictly convex, pointed with apex $M(e)$ and $M(w_0)$, and both contain $\mathsf{Perm}(W)$. In other words, P is obtained from $\mathsf{Perm}(W)$ by removing, from the definition of $\mathsf{Perm}(W)$ as an intersection of halfspaces, all halfspaces $\mathscr{H}_{(x,s)}$ that satisfy $M(e) \notin H_{(x,s)}$ and $M(w_0) \notin H_{(x,s)}$.

Proposition 3.2. *Let* $\varphi : V \to V$ *be a linear isometry that maps* P *to itself and has the fixed point* $M(e)$. *Then* φ *induces a* u*-automorphism* μ *of the Coxeter graph of* (W, S) *such that* $\varphi(v_s) := v_{\mu(s)}$ *for every* $s \in S$.

Proof. For φ satisfying our hypothesis, we have $\varphi(M(e)) = M(e) \in H_{(e,s)}$. Hence φ induces a bijection on the set $\{\mathscr{H}_{(e,s)} \,|\, s \in S\}$. Since v_s is a normal vector to $\mathscr{H}_{(e,s)}$ for any $s \in S$, we have $\varphi(v_s) = k_s v_{t_s}$ for some $k_s > 0$ and $t_s \in S$. Hence $\sum_{s\in S} k_s \kappa_s v_{t_s} = \varphi(M(e)) = M(e) = \sum_{s\in S} \kappa_s v_s$ and then $\kappa_{t_s} = k_s \kappa_s$.

On the other hand, since φ is an isometry fixing P and $M(e)$, it induces a bijection on the set of edges of P which have $M(e)$ as one of their vertices. Each of these edges is contained in a line $l_s := \bigcap_{r\in S\setminus\{s\}} H_{(e,r)}$, for $s \in S$. So φ induces a bijection on the set $\{l_s \,|\, s \in S\}$. Since v_r is a fundamental weight of Δ, and is a normal vector of $H_{(e,r)}$, the hyperplane $H_{(e,r)}$ is spanned by the simple roots $\{\alpha_u \,|\, u \in S \setminus \{r\}\}$, and thus l_s is spanned by the simple root α_s. As the simple roots are all of the same length and φ preserves the norm of vectors, $\varphi(\alpha_s) = \pm\alpha_{r_s}$ for some $r_s \in S$. Now,

$$1 = \langle v_s, \alpha_s \rangle = \langle \varphi(v_s), \varphi(\alpha_s) \rangle = \pm k_s \langle v_{t_s}, \alpha_{r_s} \rangle.$$

We conclude that $r_s = t_s$, $k_s = 1$ and $\kappa_s = \kappa_{t_s}$ for all $s \in S$. Therefore φ induces a bijection on the set $\{v_s \,|\, s \in S\}$. In other words $\varphi(\Delta^*) = \Delta^*$, and, since ϕ is an isometry, $\varphi(\Delta) = \Delta$ and the angle between α_s, α_r is preserved. That is, φ induces a u-automorphism of the Coxeter graph of W, since the order of st in W is entirely determined by the angle between α_s and α_t, and since $\kappa_s = \kappa_{t_s}$ for all $s \in S$. $\square$

Remark 3.3. In the proof of the previous proposition, we have made use of the assumption (stated when we introduced the root system Φ) that all roots of Φ are of equal length. Note that this is not an important restriction, since any root system can be rescaled to have all its roots of equal length. For example, the vectors $(1, 0)$ and $(-1, 1)$ are simple roots for the crystallographic root system B_2. Instead of these, we would take $(1, 0)$ and $(-1/\sqrt{2}, 1/\sqrt{2})$ as the simple roots. The reader should be well aware that the assumption that the simple roots are of the same length does not imply that the fundamental weights are of the same length, see for instance in A_3.

Proposition 3.4. *For every u-automorphism μ of the Coxeter graph, there is a unique linear isometry φ_μ that fixes P and $M(e)$ defined by $\varphi_\mu(\alpha_s) := \alpha_{\mu(s)}$ for every $s \in S$.*

Proof. The map φ_μ is well-defined since Δ is a basis of V. As μ is an automorphism of the Coxeter graph and $\langle \alpha_{\mu(s)}, \alpha_{\mu(t)} \rangle$ depends only on the order of st, we have $\langle \alpha_{\mu(s)}, \alpha_{\mu(t)} \rangle = \langle \alpha_s, \alpha_t \rangle$ for $s, t \in S$. In other words φ_μ is an isometry since Δ is a basis of V.

From duality it is clear that $v_s = \sum_{r \in S} \langle v_r, v_s \rangle \alpha_r$, for all $s \in S$. Moreover, the matrices $[\langle v_r, v_s \rangle]_{s,t}$ and $[\langle \alpha_r, \alpha_s \rangle]_{s,t}$ are inverse to each other and the permutation $\mu \colon S \to S$ is such that $[\langle \alpha_{\mu(r)}, \alpha_{\mu(s)} \rangle]_{s,t} = [\langle \alpha_r, \alpha_s \rangle]_{s,t}$. Hence $[\langle v_{\mu(r)}, v_{\mu(s)} \rangle]_{s,t} = [\langle v_r, v_s \rangle]_{s,t}$. Thus, for $s \in S$ we have

$$\varphi_\mu(v_s) = \sum_{r \in S} \langle v_s, v_r \rangle \alpha_{\mu(r)} = \sum_{r \in S} \langle v_{\mu(s)}, v_{\mu(r)} \rangle \alpha_{\mu(r)} = \sum_{r' \in S} \langle v_{\mu(s)}, v_{r'} \rangle \alpha_{r'} = v_{\mu(s)}.$$

Now as $\kappa_s = \kappa_{\mu(s)}$ for all $s \in S$, φ_μ fixes $M(e)$, and therefore P. $\square$

Similarly, for every isometry φ that fixes P and $M(e)$ there is a u-automorphism μ such that $\varphi(v_s) := v_{\mu(s)}$ for every $s \in S$, by Proposition 3.2.

Corollary 3.5. *Let μ be an automorphism of the Coxeter graph of (W, S) and φ be a linear isometry that maps P to itself and has $M(e)$ as fixed point. Suppose that μ and φ are related via $\varphi(v_s) = v_{\mu(s)}$ for all $s \in S$. Then $\varphi = \varphi_\mu$ and μ is a u-automorphism. Moreover,*

$$\varphi\left(w(v_s)\right) = \left(\mu(w)\right)\left(v_{\mu(s)}\right) \qquad \textit{and} \qquad \varphi(\mathscr{H}_{(w,s)}) = \mathscr{H}_{(\mu(w),\mu(s))}$$

for $w \in W$, $s \in S$. In particular, $\varphi(\mathsf{Perm}(W)) = \mathsf{Perm}(W)$.

Proof. As Δ^* is a basis of V, $\varphi = \varphi_\mu$. Moreover, since φ fixes $M(e)$ we have

$$\sum_{\mu(s) \in S} \kappa_{\mu(s)} v_{\mu(s)} = \sum_{s \in S} \kappa_s v_s = M(e) = \varphi_\mu(M(e)) = \sum_{s \in S} \kappa_s v_{\mu(s)}.$$

By identification, we have $\kappa_s = \kappa_{\mu(s)}$ for all $s \in S$, which proves that μ is a u-automorphism.

We prove the first remaining claim by induction on the length of w. If $w = e$ the claim is $\varphi(v_s) = v_{\mu(s)}$ and was shown in the proof of Proposition 3.2. Now assume $\ell(w) > 0$. There is $t \in S$ such that $w = w't$ with $\ell(w') < \ell(w)$. The action of t on V is a reflection in reflection hyperplane H_t. Hence we have

$$t(v_s) = v_s - \frac{2\langle v_s, \alpha_t\rangle}{\langle \alpha_t, \alpha_t\rangle}\alpha_t.$$

Also, since φ is an isometry that maps α_r to $\alpha_{\mu(r)}$ and v_r to $v_{\mu(r)}$, we have

$$\begin{aligned}\varphi_\mu(w(v_s)) &= \varphi_\mu\left(w'\left(t(v_s)\right)\right)\\ &= \mu(w')\left(v_{\mu(s)} - \frac{2\langle v_{\mu(s)}, \alpha_{\mu(t)}\rangle}{\langle \alpha_{\mu(t)}, \alpha_{\mu(t)}\rangle}\alpha_{\mu(t)}\right)\\ &= \mu(w')\mu(t)\left(v_{\mu(s)}\right) = \left(\mu(w't)\right)\left(v_{\mu(s)}\right)\\ &= \mu(w)\left(v_{\mu(s)}\right).\end{aligned}$$

We now prove the second claim.

$$\begin{aligned}\varphi_\mu(\mathscr{H}_{(w,s)}) &= \{\varphi_\mu(v) \in V \mid \langle v, w(v_s)\rangle \le \langle M(e), v_s\rangle\}\\ &= \{v \in V \mid \langle v, \varphi_\mu(w(v_s))\rangle \le \langle \varphi_\mu(M(e)), \varphi_\mu(v_s)\rangle\}\\ &= \{v \in V \mid \langle v, \mu(w)(v_{\mu(s)}))\rangle \le \langle M(e), v_{\mu(s)}\rangle\}\\ &= \mathscr{H}_{(\mu(w),\mu(s))}.\end{aligned}$$

□

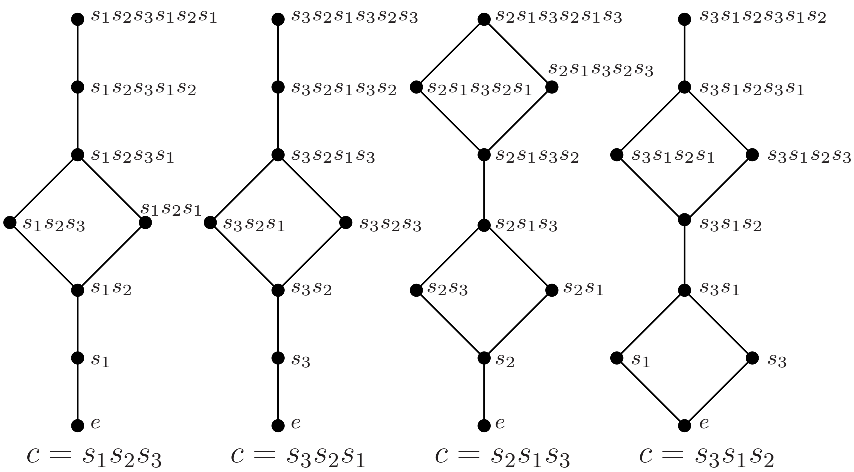

FIGURE 4. There are four Coxeter elements in S_4, each yields a distributive lattice $\mathcal{G}_c$ of c-singletons.

The c-singletons form a distributive sublattice of the right weak order, see [3]. We denote the Hasse diagram of this poset by $\mathcal{G}_c$, see Figure 4 for examples. They also form a sublattice of the c-Cambrian lattice.

There is an important linear isometry on V that fixes P and interchanges $M(e)$ and $M(w_0)$: the map g defined by $v \longmapsto w_0(v)$.

Reading proves in [7, Proposition 1.3] that the map $w \longmapsto ww_0$ is an anti-isomorphism from the c-Cambrian lattice to the c^{-1}-Cambrian lattice. In particular, the map $w \longmapsto ww_0$ is an anti-isomorphism between the lattices of c-singletons and c^{-1}-singletons by restriction, that is, w is a c-singleton if and only if ww_0 is a c^{-1}-singleton. Since the map $w \mapsto w_0ww_0$ is an isomorphism from the c-Cambrian lattice to the w_0cw_0-Cambrian lattice, the map $w \mapsto w_0w = (w_0ww_0)w_0$ is an anti-isomorphism from the c-Cambrian lattice to the $w_0c^{-1}w_0$-Cambrian lattice. In other words, $\mathscr{H}_{(x,s)}$ is c-admissible for all $s \in S$ if and only if $\mathscr{H}_{(w_0x,s)}$ is $w_0c^{-1}w_0$-admissible for all $s \in S$. Therefore we obtain the following proposition.

Proposition 3.6. *Let c be a Coxeter element of the finite Coxeter system (W,S). Then*
$$\mathsf{Asso}_c(W) = g\left(\mathsf{Asso}_{w_0c^{-1}w_0}(W)\right),$$
that is, the generalized associahedra $\mathsf{Asso}_c(W)$ and $\mathsf{Asso}_{w_0c^{-1}w_0}(W)$ are isometric.

Let T be the reflections of W and $I(w)$ be the inversions of $w \in W$ defined as
$$T := \bigcup_{w\in W} wSw^{-1} \qquad \text{and} \qquad I(w) := \{t \in T \mid \ell(tw) < \ell(w)\}.$$
A *parabolic subgroup* is a subgroup that is the conjugate of a standard parabolic subgroup of W. Given the Coxeter system (W,S) and a parabolic subgroup W', there is a natural way to distinguish a set of *simple generators* of W', see [6]. We shall only make use of the case that W' is standard parabolic, in which case, the simple generators of W' are simply $W' \cap S$.

Theorem 3.7. *For $w \in W$ and any Coxeter element c of W, the following statements are equivalent:*
(i) *w and ww_0 are both c-singletons;*
(ii) *$w \in \{e, w_0\}$;*
(iii) *wcw^{-1} is a Coxeter element of W and $w\mathcal{G}_c = \mathcal{G}_{wcw^{-1}}$;*
(iv) *$w\mathcal{G}_c = \mathcal{G}_{c'}$ for some Coxeter element c'.*

Proof. (i) $\Rightarrow$ (ii): Suppose w and ww_0 are c-singletons. A c-singleton u is c-sortable and c-antisortable by Proposition 2.2 of [3], that is, the element u is c-sortable and uw_0 is c^{-1}-sortable. Hence w is c-sortable and c^{-1}-sortable. From [6, Theorem 4.1] we know that $g \in W$ is c-sortable and c^{-1}-sortable if and only if $I(g) \cap (W' \setminus \{t_1\}) \neq \emptyset$ implies $t_2 \in I(g)$ for any irreducible dihedral parabolic subgroup W' of W (that is $|W'| > 4$) with simple generators $t_1, t_2 \in T$.

Assume that $w \neq e$. There exists $s \in I(w) \cap S$. Pick $t \in S$ such that the standard parabolic subgroup W' generated by $\{s,t\}$ is dihedral and of cardinality > 4. We first show that $s,t \in I(w)$. We have to distinguish two cases:

(1) If $I(w) \cap (W' \setminus \{s\}) \neq \emptyset$ then $t \in I(w)$ because w is c-sortable and c^{-1}-sortable. Hence $s,t \in I(w)$.

(2) Assume $I(w) \cap (W' \setminus \{s\}) = \emptyset$. We first observe that $I(ww_0) = I(w_0) \setminus I(w)$. Hence $I(ww_0) \cap (W' \setminus \{t\}) \neq \emptyset$ which implies $s \in I(ww_0)$ since ww_0 is also c-sortable and c^{-1}-sortable. In particular, $s \in I(w) \cap I(ww_0)$ which is impossible.

Since (W, S) is irreducible, the Coxeter graph associated to (W, S) is connected. Now repeat this process along paths starting at s to conclude that $S \subseteq I(w)$. Hence $w = w_0$.

(ii) $\Rightarrow$ (iii): For $w = e$ the result is clear. Recall that the conjugation by w_0 is an automorphism φ of the Coxeter system (W, S). So the w_0cw_0-factorization of w_0 is induced by φ from the c-factorization of w_0. The claim for $w = w_0$ follows.

(iii) $\Rightarrow$ (iv): Set $c' := wcw^{-1}$.

(iv) $\Rightarrow$ (i): Since e and w_0 are c'-singletons, we conclude that w^{-1} and $w^{-1}w_0$ are both c-singletons. Apply $(i) \Rightarrow (ii)$ to deduce that $w^{-1} = e$ or $w^{-1} = w_0$. In particular, w and ww_0 are both c-singletons. $\square$

Remark 3.8. Two maximal cones C and C' in the c-Cambrian fan $\mathcal{F}_c$ are antipodal if $C = -C'$. Theorem 3.7 implies that a pair of antipodal maximal cones that correspond to c-singletons is unique and the corresponding elements are e and w_0.

4. Proof of Theorem 2.3

Assume there is an isometry φ on V such that $\mathsf{Asso}_{c_1}(W) = \varphi(\mathsf{Asso}_{c_2}(W))$. Let w be a c_2-singleton. Then

(1) $M(w) = \bigcap_{s \in S} H_{(w,s)}$ is a vertex of $\mathsf{Asso}_{c_2}(W)$,

(2) $\varphi(M(w)) = \bigcap_{s \in S} \varphi(H_{(w,s)})$ is a vertex of $\mathsf{Asso}_{c_1}(W)$,

(3) $\varphi(M(w)) = M(w')$ for some c_1-singleton w' since φ is an isometry.

(For (3), note that c-singleton cones are the only cones in the Cambrian fan which consist of a single chamber from the Coxeter fan, and thus an isometry must take singleton cones to singleton cones.)

Apply these results to $w = e$ to obtain a c_1-singleton w'_e with $M(w'_e) = \varphi(M(e))$. Moreover, $w'_e w_0$ is also a c_1-singleton with $M(w'_e w_0) = \varphi(M(w_0))$. Hence $w'_e \in \{e, w_0\}$ by Theorem 3.7 and φ is a linear isometry of V that fixes P and either fixes $M(e)$ and $M(w_0)$ or interchanges $M(e)$ and $M(w_0)$. If φ fixes $M(e)$ and P then there is an induced u-automorphism μ of the Coxeter graph of (W, S) by Proposition 3.2 and $\mu(c_2) = c_1$. If φ interchanges $M(e)$ and $M(w_0)$, then we consider $\widetilde{\varphi} := g \circ \varphi$. We have $\widetilde{\varphi}(\mathsf{Asso}_{c_2}(W)) = \mathsf{Asso}_{w_0 c_1^{-1} w_0}(W)$ by Proposition 3.6 and $\widetilde{\varphi}$ is an isometry that fixes P, $M(e)$, and $M(w_0)$. Hence $\widetilde{\varphi}$ induces a u-automorphism μ of the Coxeter system (W, S) by Proposition 3.2 and we get $\mu(c_2) = w_0 c_1^{-1} w_0$.

Assume there is a u-automorphism μ of the Coxeter graph (W, S). Without loss of generality, we may assume that $\mu(c_2) = c_1$ because $\mathsf{Asso}_c(W)$ and $\mathsf{Asso}_{w_0 c^{-1} w_0}(W)$ are isometric via g by Proposition 3.6.

We have to specify an isometry φ_μ on V such that $\varphi_\mu(\mathsf{Asso}_{c_2}(W)) = \mathsf{Asso}_{c_1}(W)$. This is done according to Proposition 3.4: Define $\varphi_\mu : V \to V$ by $\varphi_\mu(\alpha_s) := \alpha_{\mu(s)}$ for all $s \in S$ or equivalently by $\varphi_\mu(v_s) := v_{\mu(s)}$.

It remains to show that φ_μ maps c_2-admissible halfspaces to c_1-admissible halfspaces. From Corollary 3.5 we know how the facet defining halfspaces $\mathscr{H}_{(w,s)}$ are permuted by the isometry φ_μ, that is, $\varphi_\mu(\mathscr{H}_{(w,s)}) = \mathscr{H}_{(\mu(w),\mu(s))}$ for $w \in W$ and $s \in S$. The automorphism μ on W preserves the length function ℓ, so we have $\mu(w_0) = w_0$ and any prefix of the c-factorization of w_0 up to commutation is a prefix of the $\mu(c)$-factorization of w_0 up to commutation. In other words, μ induces a lattice isomorphism between the c_2-singletons and the $\mu(c_2)$-singletons. Hence $\mathscr{H}_{(x,s)}$ is c_2-admissible if and only if $\mathscr{H}_{(\mu(x),\mu(s))}$ is $\mu(c_2)$-admissible. This shows that $\varphi_\mu(\mathsf{Asso}_c(W)) = \mathsf{Asso}_{\mu(c)}(W)$ and ends the proof of Theorem 2.3. $\square$

Proof of Corollary 2.6. The assertion that (2) is equivalent to (3) is Theorem 2.3. That (2) implies (1) follows from the definition of normal fans.

Now we show that (1) implies (3). As the c-Cambrian fan $\mathcal{F}_c$ and the c'-Cambrian fan $\mathcal{F}_{c'}$ are isometric, there is an isometry φ such that the image under φ of each cone in $\mathcal{F}_c$ is a cone of $\mathcal{F}_{c'}$. By Remark 3.8, the pair of antipodal singleton cones $C(e), C(w_0)$ corresponding to e, w_0 are the unique singleton antipodal cones in both Cambrian fans, and either φ fixes them or exchanges them. If u is balanced, then apply Corollary 3.5 with μ to be the conjugation by w_0 to obtain

$$M(w_0) = \sum_{s\in S} \kappa w_0(v_s) = \sum_{s\in S} \kappa(-v_{w_0 s w_0}) = -M(e). \tag{1}$$

As $C(e)$ is $\mathbb{R}_{>0}$-spanned by Δ^* and $C(w_0)$ is $\mathbb{R}_{>0}$-spanned by $-\Delta^*$, either φ fixes $M(e)$ and $M(w_0)$ or interchanges them. In both cases, $\varphi(P) = P$. So either φ or $g \circ \varphi$ fixes $M(e)$ and P. Conclude by Proposition 3.2 as in the first part of the proof of Theorem 2.3. $\square$

5. The reducible case

The reducible case does not follow immediately from an application of the irreducible case. Rather, one goes through the same steps as in the proof of the irreducible case, but with some slight added complication. We sketch the process below.

Let $\mathcal{D}$ denote the set of irreducible components of the Coxeter graph of (W, S). For any $\mathcal{A} \subset \mathcal{D}$, let $w_{\mathcal{A}}$ be the longest word for the subgroup generated by the components in $\mathcal{A}$. Let $L = \{w_{\mathcal{A}} \mid \mathcal{A} \subset \mathcal{D}\}$. All the elements of L are c-singletons (for any c).

Up to just before Proposition 3.6, the argument goes through in exactly the same way. Then, instead of constructing a single isometry g, we construct an isometry $g_{\mathcal{A}}$ for each $\mathcal{A} \subset \mathcal{D}$, defining $g_{\mathcal{A}}(v) = w_{\mathcal{A}}(v)$. Write $c^{\mathcal{A}}$ for the Coxeter element obtained from c by reversing the order of the reflections in c coming from components in $\mathcal{A}$. The generalization of Proposition 3.6 then asserts that

$\mathsf{Asso}_c(W) = g_{\mathcal{A}}(\mathsf{Asso}_{w_{\mathcal{A}}c^{\mathcal{A}}w_{\mathcal{A}}}(W))$ and, in particular, these associahedra are isometric. Theorem 3.7 goes through with condition (ii) replaced by the condition that $w \in L$.

References

[1] F. Chapoton, S. Fomin and A. Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2003), 537–566.

[2] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Annals of Mathematics **158** (2003), 977–1018.

[3] C. Hohlweg, C. Lange, H. Thomas, *Permutahedra and generalized associahedra*, 31 pages, 2007; `arχiv:math.CO/0709.4241`.

[4] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[5] J.-L. Loday, *Realization of the Stasheff polytope*, Arch. Math. **83** (2004), 267–278.

[6] N. Reading, *Clusters, Coxeter-sortable elements and noncrossing partitions*, 28 pages, 2005, to appear in Trans. Amer. Math. Soc.; `arχiv:math.CO/0507186`.

[7] N. Reading, *Sortable elements and Cambrian lattices,* Algebra Universalis **56**(3-4) (2007), 411–437.

[8] N. Reading and D. Speyer, *Cambrian fans*, 40 pages, 2006; `arχiv:math.CO/0606201`.

[9] S. Shnider and S. Sternberg, *Quantum groups: From coalgebas to Drinfeld algebras*, Graduate texts in mathematical physics, International Press, 1994.

[10] J. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.

[11] J. Stasheff, *From operads to "physically" inspired theories*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math., 202, Amer. Math. Soc., 1997, pp. 53–81.

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