Euler's q-difference table for $C_\ell \wr S_n$

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Euler

$$g_n^n = n!, \qquad g_n^m = g_n^{m+1} - g_{n-1}^m \quad (0 \leq m \leq n-1).$$

$n \backslash m$	0	1	2	3	4	5
0	0!					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

(g_n^m)

Combinatorial interpretation. Let $[n] := \{1, \ldots, n\}$.

J. Riordan, G. Kreweras, D. Dumont and A. Randrianarivony,

$$g_n^m = \#\{\sigma \in S_n | \,\mathrm{FIX}\, \sigma \subset \{n-m+1, n-m, \ldots, n-1, n\}\}.$$

In particular,

$$g_n^0 = \#\mathcal{D}_n \quad \text{and} \quad g_n^n = \#\mathcal{S}_n.$$

The first column gives the derangement numbers:

$$g_n^0 = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

A derangement is a fixed-point free permutation.

Let $\sigma = x_1 x_2 \ldots x_n$ be a permutation of $[n]$.

$$\operatorname{maj} \sigma = \sum_{x_i > x_{i+1}} i.$$

MacMahon

$$\sum_{\sigma \in \mathcal{S}_n} q^{\operatorname{maj} \sigma} = n!_q,$$

where $n!_q := 1 \cdot (1+q) \cdot (1+q+\cdots+q^{n-1})$.

In 1989 Gessel, Wachs, ...

$$\sum_{\sigma \in \mathcal{D}_n} q^{\operatorname{maj} \sigma} = n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{k!_q}.$$

In 1997 Clarke-Han-Z: For $\sigma \in S_n$ let $\tilde{\sigma}$ be the restriction of σ to $[n] \setminus \mathrm{FIX}(\sigma)$. If $\sigma \in S_n$ with $\mathrm{FIX}(\sigma) = \{i_1, i_2, \ldots, i_k\}$, then the statistic maf is defined by

$$\mathrm{maf}\,\sigma = \sum_{j=1}^{k} (i_j - j) + \mathrm{maj}\,\tilde{\sigma}.$$

For example, let $\sigma = 3\,2\,1\,6\,5\,9\,4\,8\,7$. Then $\mathrm{FIX}(\sigma) = \{2, 5, 8\}$ and $\tilde{\sigma} = 3\,1\,6\,9\,4\,7$. Hence

$$\mathrm{maj}\,\tilde{\sigma} = 1 + 4 = 5,$$
$$\mathrm{maf}\,\sigma = (2-1) + (5-2) + (8-3) + 5 = 14.$$

Note that $\mathrm{maf}\,\sigma = \mathrm{maj}\,\sigma$ if σ is a derangement.

A key lemma: There is a bijection $\Psi : S_n \to S_n$ such that $\mathrm{maf}\,\sigma = \mathrm{maj}\,\varphi(\sigma)$.

Theorem (Clarke-Han-Z). Define the q-Euler table

$$\begin{cases} g_{\ell,n}^{n}(q) = n!_q & (m=n); \\ g_{\ell,n}^{m} = g_{\ell,n}^{m+1} - q^{n-m-1} g_{\ell,n-1}^{m} & (0 \le m \le n-1). \end{cases} \tag{1}$$

Then

$$g_{\ell,n}^{m}(q) = \sum_{\sigma \in S_n^m} q^{\mathrm{maf}\,\sigma},$$

where S_n^m is the set of permutations in S_n such that the fixed points are greater than $n-m$. In particular,

$$g_{\ell,n}^{n}(q) = \sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{maf}\,\sigma} = n!_q,$$

$$g_{\ell,n}^{0}(q) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}\,\sigma} = n!_q \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{k!_q}.$$

R. M. Adin and Y. Roichman, The flag major index and group actions on polynomial rings, *European J. Combin.* **22** (2001).

J. Haglund, N. Loehr and J. B. Remmel, Statistics on wreath products, perfect matchings and signed words, *European J. Combin.* **26** (2005).

Chak-On Chow: On derangemen polynomials of type B, SLC, 55 (2006).

D. Foata and G.N. Han, A series of papers, 2006-2008.

Fanja Rakotondrajo, On Euler's difference table, FPSAC'07.

W. Y. C. Chen and J. C. Y. Zhang, The skew and relative derangements of type B, Electronic J. Combinatorics, 14 (2007).

Euler's difference table for $C_\ell \wr S_n$ For a fixed integer $\ell \geq 1$, Euler's difference table associated to the sequence $\{\ell^n n!\}_{n\geq 0}$ is the array $(g_{\ell,n}^m)_{n,\,m\geq 0}$ defined by

$$\begin{cases} g_{\ell,n}^n = \ell^n\, n! & (m=n); \\ g_{\ell,n}^m = g_{\ell,n}^{m+1} - g_{\ell,n-1}^m & (0 \leq m \leq n-1). \end{cases} \tag{2}$$

A q-analogue Consider the array $\{g_{\ell,n}^m(q)\}$ defined by

$$\begin{cases} g_{\ell,n}^n(q) = [\ell]_q [2\ell]_q \cdots [n\ell]_q; \\ g_{\ell,n}^m(q) = g_{\ell,n}^{m+1}(q) - q^{\ell(n-m-1)} g_{\ell,n-1}^m(q) & \qquad (0 \leq m \leq n-1), \end{cases}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$.

Euler's difference table of type B

$n\backslash m$	0	1	2	3	4	5
0	1					
1	1	$2^1\,1!$				
2	5	6	$2^2\,2!$			
3	29	34	40	$2^3\,3!$		
4	233	262	296	336	$2^4\,4!$	
5	2329	2562	2824	3120	3456	$2^5\,5!$

$(g_{2,n}^m)$

The $\ell = 1$ case corresponds to Euler's difference table.

Some explicit formulas

Lemma 1. *Let* $(a_{n,m})_{0\le m\le n}$ *be an array defined by*

$$\begin{cases} a_{0,m} = x_m & (m = n); \\ a_{n,m} = z_m a_{n-1,m+1} + y_n a_{n-1,m} & (0 \le m \le n-1). \end{cases} \tag{3}$$

Then

$$a_{n,m} = \sum_{k=0}^{n} x_{m+k} \left(\prod_{j=0}^{k-1} z_{m+j} \right) e_{n-k}(y_1, y_2, \ldots, y_n), \tag{4}$$

where $e_i(y_1, y_2, \ldots, y_n)$ *is the* i*-th elementary symmetric polynomial of* $y_1, \ldots, y_n$*, i.e.,*

$$(1 + y_1 t)(1 + y_2 t) \cdots (1 + y_n t) = \sum_{i=0}^{n} e_i(y_1, \ldots, y_n) t^i.$$

Theorem 2. *We have*

$$g^{m}_{\ell,n+m}(q) = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^\ell} q^{\ell\binom{n-k}{2}} \prod_{i=1}^{m+k} [i\ell]_q. \tag{5}$$

Proof. Recall the following q-binomial formula:

$$(1+t)(1+qt)\cdots(1+q^{n-1}t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} t^k. \tag{6}$$

Therefore, for $0 \le k \le n$,

$$e_k(1, q, q^2, \ldots, q^{n-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \tag{7}$$

The result follows from (4) and (7) with $y_k = q^{lk}$. ∎

Recall that the two q-exponential functions defined by

$$e(u;q) := \sum_{n\geqslant 0} \frac{u^n}{(q;q)_n}, \qquad E(u;q) := \sum_{n\geqslant 0} \frac{q^{\binom{n}{2}} u^n}{(q;q)_n},$$

satisfy $E(-u;q)e(u;q) = 1$, where

$$(u;q)_n := \begin{cases} 1 & \text{if } n = 0, \\ (1-u)(1-uq)\cdots(1-uq^{n-1}) & \text{if } n \geqslant 1. \end{cases}$$

Corollary 3. *For $n \geqslant 1$, we have*

(i) $$g^{0}_{\ell,n}(q) = [\ell]_q[2\ell]_q\cdots[n\ell]_q \sum_{k=0}^{n} \frac{(-1)^k q^{\ell\binom{k}{2}}}{[\ell]_q[2\ell]_q\cdots[k\ell]_q},$$

(ii) $$\sum_{n\geqslant 0} g^{0}_{\ell,n}(q)\frac{u^n}{[\ell]_q[2\ell]_q\cdots[n\ell]_q} = \frac{E(-u(1-q);q^{\ell})}{1-u},$$

(iii) $$g^{0}_{\ell,n+1}(q) = [\ell n+\ell]_q g^{0}_{\ell,n+1}(q) + (-1)^{n+1} q^{\ell\binom{n+1}{2}}.$$

Combinatorial interpretation.

The wreath product of cyclic group C_ℓ with S_n, $C_\ell \wr S_n$, reduces to the symmetric group S_n when $\ell = 1$ and the hyperoctahedral group B_n when $\ell = 2$.

We can think of the group $C_\ell \wr S_n$ as the group of "colored" permutations where the colors are in the set of ℓ-th roots of unity $\{1, \zeta, \ldots, \zeta^{\ell-1}\}$, where $\zeta = e^{2i\pi/\ell}$.

By definition, the multiplication in $G_{\ell,n} = C_\ell \wr S_n = C_\ell^n \rtimes S_n$, consisting of pairs $(\epsilon, \sigma) \in C_\ell^n \times S_n$, is given by the following rule: for all $\pi = (\epsilon, \sigma)$ and $\pi' = (\epsilon', \sigma')$ in $G_{\ell,n}$,

$$(\epsilon, \sigma) \cdot (\epsilon', \sigma') = ((\epsilon_1 \epsilon'_{\sigma^{-1}(1)}, \epsilon_2 \epsilon'_{\sigma^{-1}(2)}, \ldots, \epsilon_n \epsilon'_{\sigma^{-1}(n)}), \sigma \circ \sigma').$$

One can identify $G_{\ell,n}$ with a permutation group of the colored set:

$$\Sigma_{\ell,n} := \{\zeta^j i \,|\, i \in [n], 0 \le j \le \ell - 1\}$$

via the morphism $(\epsilon, \sigma) \longmapsto \pi$ where

$$\pi(i) = \epsilon_{\sigma(i)}\sigma(i) \qquad \text{and} \qquad \pi(\zeta^j\, i) = \zeta^j \pi(i)$$

for any $i \in [n]$ and $0 \le j \le \ell - 1$.

Clearly the cardinality of $G_{\ell,n}$ equals $\ell^n n!$.

We will write an element $\pi \in G_{\ell,n}$ in two-line notation. For example, if $\pi = (\epsilon, \sigma) \in G_{4,11}$, where $\epsilon = (\zeta^2, 1, 1, \zeta, \zeta^2, \zeta, \zeta, \zeta, 1, \zeta, \zeta^3)$ and $\sigma = 3\,5\,1\,9\,6\,2\,7\,4\,11\,8\,10$, we write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & \zeta^2 5 & \zeta^2 1 & 9 & \zeta 6 & 2 & \zeta 7 & \zeta 4 & \zeta^3 11 & \zeta 8 & \zeta 10 \end{pmatrix}.$$

For small j, we shall write j bars over i instead of $\zeta^j i$. Thus, the above permutation can be written in one-line form as

$$\pi = 3\ \bar{\bar{5}}\ \bar{\bar{1}}\ 9\ \bar{6}\ 2\ \bar{7}\ \bar{4}\ \overline{\overline{\overline{11}}}\ \bar{8}\ \overline{10},$$

or in cyclic notation as

$$\pi = (\bar{\bar{1}},\ 3)\ (2,\ \bar{\bar{5}},\ \bar{6})\ (\bar{4},\ 9,\ \overline{\overline{\overline{11}}},\ \overline{10},\ \bar{8})\ (\bar{7}).$$

Note that when using cyclic notation to determine the image of a number, one ignores the sign on that number and then considers only the sign on the next number in the cycle. Thus, in this example, we ignore the sign ζ^2 on the 5 and note that then 5 maps to $\zeta 6$ since the sign on 6 is ζ. Furthermore, throughout this paper we shall use the following total order on $G_{\ell,n}$: for $i, j \in [\ell]$ and $a, b \in [n]$,

$$\zeta^i a < \zeta^j b \Longleftrightarrow [i > j] \quad \text{or} \quad [i = j \text{ and } a < b].$$

Example For $n = 4$ and $\ell = 3$ we have:

$\bar{\bar{\bar{1}}} < \bar{\bar{\bar{2}}} < \bar{\bar{\bar{3}}} < \bar{\bar{\bar{4}}} < \bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{\bar{4}} < \bar{1} < \bar{2} < \bar{3} < \bar{4} < 0 < 1 < 2 < 3 < 4.$

Defintion 4 (k-circular succession). *Given a permutation $\pi \in G_{\ell,n}$ and a nonnegative integer k, the value $\pi(i)$ is a k-circular succession at position $i \in [n]$ if $\pi(i) \in [n]$ and $\pi(i) = i + k$. In particular a 0-circular succession is also called fixed point.*

Denote by $\mathcal{C}^k(\pi)$ the set of k-circular successions of π and let $c^k(\pi) = \#\ \mathcal{C}^k(\pi)$. In particular $FIX(\pi)$ denotes the set of fixed points of π. For example, for the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{1} & 5 & \bar{\bar{9}} & \bar{6} & 8 & \bar{7} & \bar{\bar{\bar{3}}} & \bar{\bar{4}} & \bar{2} \end{pmatrix} \in G_{4,9}$$

the values 5 and 8 are the two 3-circular successions at positions 2 and 5. Thus $C^3(\pi) = \{5, 8\}$.

Theorem 5. *For any integer* k *such that* $0 \leq k \leq m$, *the entry* $g^m_{\ell,n}$ *equals the number of permutations in* $G_{\ell,n}$ *whose* k*-circular successions are included in* $[m]$. *In particular, by taking* $k = 0$ *and* $k = m$, *respectively, either of the following holds.*

(i) The entry $g^m_{\ell,n}$ *is the number of permutations in* $G_{\ell,n}$ *whose fixed points are included in* $[m]$.

(ii) The entry $g^m_{\ell,n}$ *is the number of permutations in* $G_{\ell,n}$ *without* m*-circular succession.*

For example, the permutations in $G_{2,2}$ whose fixed points are included in [1] are:

$$21, \quad 1\bar{2}, \quad \bar{2}1, \quad 2\bar{1}, \quad \bar{1}\bar{2}, \quad \bar{2}\bar{1};$$

while those without 1-circular succession are:

$$12, \quad \bar{1}2, \quad 1\bar{2}, \quad \bar{1}\bar{2}, \quad \bar{2}1, \quad \bar{2}\bar{1}.$$

Note that Dumont and Randrianarivony proved the $\ell = 1$ case of (i), while Rakotondrajao proved the $\ell = 1$ case of (ii).

For $\sigma \in G_{\ell,n}$, let Der(σ) be the derangement part of σ, i.e., if $y_1 y_2 \cdots y_m$ is obtained by deleting each fixed point of sigma where $y_i = \varepsilon_i |y_i|$ then Der(σ) $= z_1 z_2 \cdots z_m$ where $z_i = \varepsilon_i red(|y_i|)$, red (reduction) is the increasing bijection from $\{|y_1|, |y_2|, \cdots, |y_m|\}$ to $[m]$.

Exemple If $\sigma = 1\,\bar{8}\,3\,4\,6\bar{\bar{\bar{2}}}\,7\,\bar{5}\,9 \in G_{4,9}$ then FIX $\sigma = \{1, 3, 4, 7, 9\}$ and Der(σ) $= \bar{4}\,3\,\bar{\bar{1}}\,\bar{2}$.

Defintion 6. *If $\sigma \in G_{\ell,n}$ then the statistic ω is defined by*

$$\omega(\sigma) = \sum_{j=0}^{k-1} j \cdot \mathit{sgn}_j(\sigma),$$

where $\mathrm{SIGN}_j(\sigma) = \{i \in [n] : \frac{\sigma_i}{|\sigma_i|} = \zeta^j\}$ *and* $\mathit{sgn}_j(\sigma) = |\mathrm{SIGN}_j(\sigma)|$.

Example: $\sigma = \bar{2}\,\bar{\bar{6}}\,1\,\bar{7}\,\bar{\bar{5}}\,\bar{\bar{\bar{4}}}\,3$ we have

$$\mathrm{SIGN}_0(\sigma) = \{3,7\}, \quad \mathrm{SIGN}_1(\sigma) = \{1,4\},$$
$$\mathrm{SIGN}_2(\sigma) = \{2,5\}, \quad \mathrm{SIGN}_3(\sigma) = \{6\};$$

then $\omega(\sigma) = 0 \times 2 + 1 \times 2 + 2 \times 2 + 3 \times 1 = 9$.

Defintion 7. *If* $\sigma \in G_{\ell,n}$ *then*

- *the flag-maj statistic* fmaj *is defined by*

$$\mathrm{fmaj}\,\sigma = \ell \cdot maj\sigma + \omega(\sigma).$$

- *Let* $\mathrm{FIX}(\sigma) = \{i_1, i_2, \ldots, i_k\}$*, the flag-maf statistic* fmaf *is defined by*

$$\mathrm{fmaf}(\sigma) = \ell \cdot \sum_{j=1}^{k} (i_j - j) + \mathrm{fmaj}\,\mathrm{Der}(\sigma).$$

Theorem (Adin-Roichman, Haglund-Loehr-Remmel) The statistic fmaj is mahonian on $G_{l,n}$, i.e.,

$$\sum_{\sigma \in G_{\ell,n}} q^{\mathrm{fmaj}(\sigma)} = \prod_{i=1}^{n} [\ell i]_q. \tag{8}$$

Remark

- *fmaf* equal to *fmaj* in $\mathcal{D}_{\ell,,n}$ the set of derangements of $C_l \wr S_n$.
- For any m such that $0 < m < n$ $fmaf$ and $fmaj$ are not equidistributed on the set $\{\sigma \in G_{l,n} : \mathrm{FIX}(\sigma) \subset \{n-m+1,\ldots,n\}\}$

Theorem 8. *There is a bijection* $\widetilde{\Psi} : G_{\ell,n} \to G_{\ell,n}$ *such that*

$$(\mathsf{fix}, \mathsf{fmaj}, \mathsf{Der})\sigma = (\mathsf{fix}, \mathsf{fmaf}, \mathsf{Der})\widetilde{\Psi}(\sigma) \qquad (9)$$

Our bijection $\widetilde{\Psi}$ is a generalization of the bijection Ψ given by Clarke-Han-Z on the symmetric group.

Therefore, the statistics fmaf and fmaj are equidistributed on $G_{\ell,n}$, i.e.,

$$\sum_{\sigma \in G_{\ell,n}} q^{\mathsf{fmaf}(\sigma)} = \sum_{\sigma \in G_{\ell,n}} q^{\mathsf{fmaj}(\sigma)} = \prod_{i=1}^{n} [\ell i]_q.$$

The statistique fmaf is a new mahonian statistic on $G_{\ell,n}$.

Let $G_{\ell,n}^m$ be the set of permutations σ in $G_{\ell,n}$ such that FIX(σ) $\subset \{n-m+1,\ldots,n-1,n\}$.

Theorem 9. *For $k \geq 0$ we have*

$$g_{\ell,n}^m(q) = \sum_{\sigma \in G_{\ell,n}^m} q^{\mathrm{fmaf}(\sigma)}. \tag{10}$$

For $\ell = 1$ and $m = 0$ we recover the result of Wachs.

For $\ell = 2$ and $m = 0$ we recover the result of Chow.

Foata and Han have recently constructed "another" bijection F which has the same property as that of Ψ on the symetric group.

Theorem 10. *The two bijections Ψ and F are identical.*