Bijections for permutation tableaux

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SLC 60
March 31st, 2008

Permutation Tableaux

Definition

A permutation tableau is a filling of a Ferrers diagram with 0 and 1 such that :

1. *there is at least one 1 in each column ;*
2. *there is no 0 with a 1 above it in the same column and a 1 to its left in the same row.*

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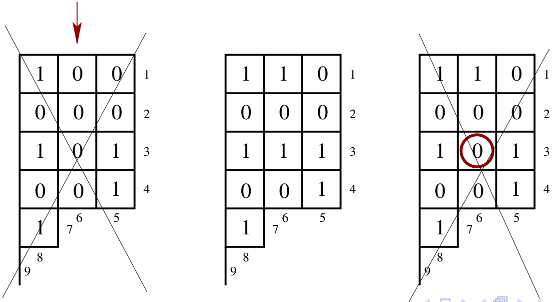

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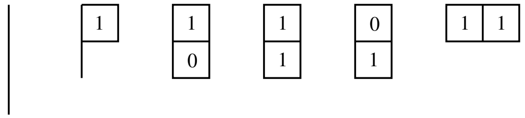
Thus there are 6 tableaux of length 3 :



Theorem (Postnikov)

There are $n!$ tableaux of length n.

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- Origin : algebraic geometry **[Postnikov ~00]**

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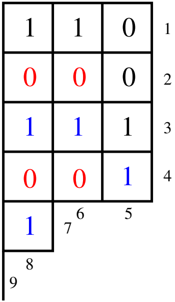
- Origin : algebraic geometry **[Postnikov $\sim$00]**
- Link with a statistical mechanics model, the PASEP
  $\hookrightarrow$ the tableaux explain the underlying combinatorics of the model **[Corteel and Williams 06,07]**
- Close relation with permutations
  $\hookrightarrow$ Several bijections exist already **[Steimgrimsson and Williams 07],[Corteel 06],[Burstein 06]**

Definitions

Let T be a permutation tableau.

- A superfluous 1 is a 1 in T which is not the highest 1 in its column.
- A restricted 0 is a 0 in T such that there is a 1 higher in its column.



Enumeration

Let $t(n,k,\ell)$ be the number of permutation tableaux of length n, with $k+1$ unrestricted rows and ℓ entries 1 in the first row.
Then

$$t(n,k,\ell) = \sum_{j=k}^{n-1} \binom{j}{k} t(n-1,j,\ell-1) + \binom{j}{k-1} t(n-1,j,\ell)$$

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Let $T_n(x,y) = \sum_{k,\ell} t(n,k,\ell) x^k y^\ell$

Theorem

If $n > 1$,

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Enumeration

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Consequences :

- There are $n!$ tableaux of length n
- $t(n, k, \ell) = t(n, \ell, k)$
- The number of tableaux of length n with $\ell + 1$ unrestricted rows is equal to the number of permutations of size n with ℓ cycles.

Permutations

We define some parameters on permutations :

$$\sigma = 28451637$$

- A descent in $\sigma = \sigma_1 \ldots \sigma_n$ is an entry σ_i such that $\sigma_i > \sigma_{i+1}$; the other entries are then called ascents.

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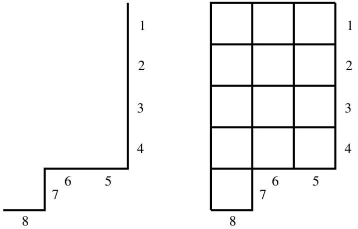
- An occurrence of the pattern $31-2$ in a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the data of indices $i < j$ of σ such that $\sigma_i > \sigma_j > \sigma_{i+1}$.

  28451637, 28451637, 28451637, 28451637

Permutations

We associate to each permutation a Ferrers diagram based on its descents and ascents :

28451637 $\longrightarrow$



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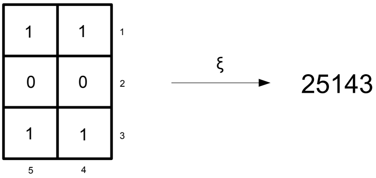
There is a bijection ξ between permutations of length n and permutation tableaux of length n, such that when $T = \xi(\sigma)$:

- *the shapes of σ and T are identical ;*



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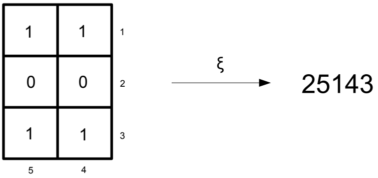

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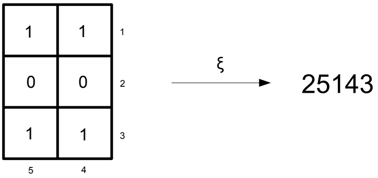
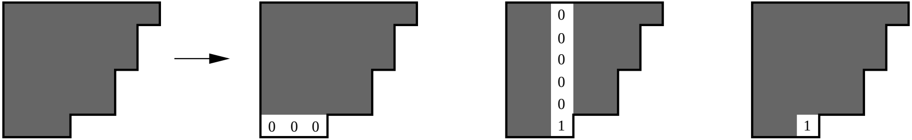
- *the shapes of σ and T are identical ;*
- *the number of superfluous 1s in T is equal to the number of occurrences of $31-2$ in σ.*
- *i is an RL-minimum in σ iff i is the label of an unrestricted row of T.*



Recursive decomposition

Reduction of a tableau according to the content of the bottom right cell :



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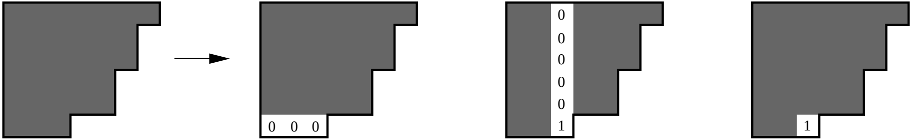


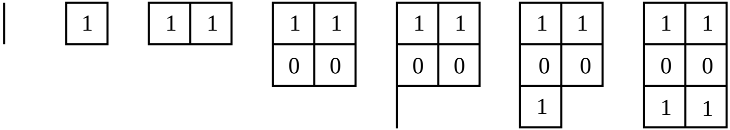
Idea of the bijection : find a recursive decomposition of permutations that mimics this.

$\hookrightarrow$ We will define ξ recursively : supposing we know how to define ξ on a tableau T, how can we define it on the tableau T to which a row, a column or a cell has been added ?

Example

1	1
0	0
1	1

Example



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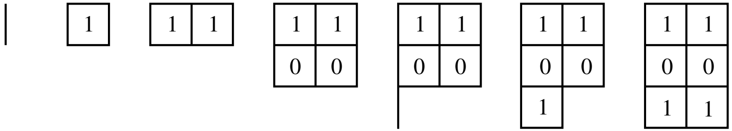

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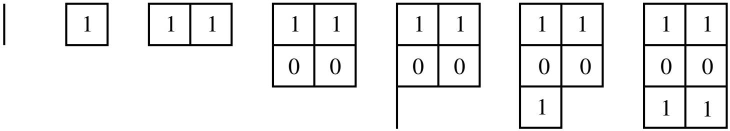

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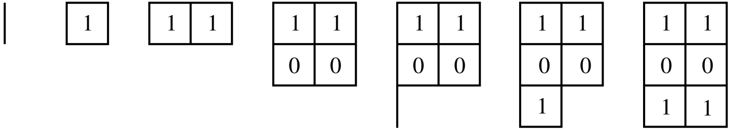

1 21 321

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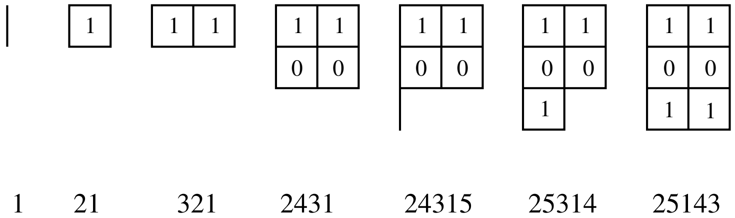
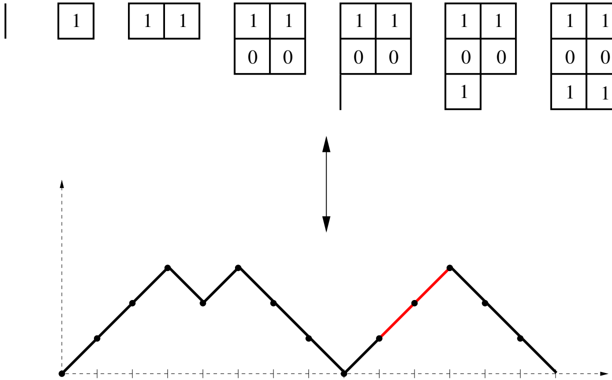


Encoding by a lattice path



Encoding by a lattice path

Bijection ξ + the encoding
$\hookrightarrow$ Allows to enumerate bijectively permutations

- without any occurrence of $31-2$ **[Knuth 75]**

$$\frac{1}{n+1}\binom{2n}{n}$$

- with exactly one occurrence of $31-2$ **[Claesson and Mansour 02]**

$$\binom{2n}{n-3}$$

Permutations with no occurrence of $32-1$

An L-tableau is a tableau such that all its necessary 1 are the leftmost 1 of their rows.

Theorem

L-tableaux of length n are in bijection with permutations of length n avoiding $32-1$*.*
The bijection preserves shapes.

- The same result is true with *R*-tableaux.

Corteel and N., *Bijections for permutation tableaux*, European Journal of Combinatorics, to appear