Linear equivalence and identical Young tableau bijections for the conjugation symmetry of Littlewood-Richardson coefficients

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March 31, 2008

Overview

1. The conjugation symmetry map
2. Calculus on words and tableaux
3. Hanlon-Sundaram bijection ξ^{HS}, Benkart-Sottile-Stroomer bijection ξ^{BSS}, and Azenhas-Zaballa bijection ξ^{AZ}
4. Linear equivalence

[HS] Philip Hanlon, Sheila Sundaram, On a bijection between Littlewood-Richardson fillings of conjugate shape J. Combin. Theory Ser. A 60 (1992), no. 1, 1–18.

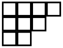
[BSS] Georgia Benkart, Frank Sottile, Jeffrey Stroomer, Tableau switching: algorithms and applications, J. of Combin. Theory Ser. A 76 (1996), no.1, 11–34.

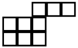
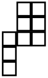
[Z] Ion Zaballa, Increasing and decreasing Littlewood-Richardson sequences and duality, preprint, University of Basque Country, 1996.

[A] Olga Azenhas, The admissible interval for the invariant factors of a product of matrices, Linear and Multilinear Algebra 46 (1999), no. 1-2, 51–99.

[PV] Igor Pak, Ernesto Vallejo, Reductions of Young tableau bijections, available at arXiv:math/0408171

1. Conjugation symmetry map

$\nu = (4,3,2) =$  partition, $Y(\nu) = \begin{array}{l}1111\\222\\33\end{array}$ Yamanouchi tableau

$\mu = (2) \subseteq \lambda = (5,3,3), \quad \lambda/\mu =$ , $\quad (\lambda/\mu)^t := \lambda^t/\mu^t =$ 

$$T = \begin{array}{lll} & 111 \\ 122 & \\ 233 & \end{array}, \qquad w(T) = 111221332 \text{ lattice permutation}$$

$T \overset{k}{\equiv} Y(\nu)$ if and only if $w(T)$ is a lattice permutation

$$\begin{array}{rcl} \xi : LR(\lambda/\mu,\nu) & \longrightarrow & LR(\lambda^t/\mu^t,\nu^t) \\ T & \mapsto & \xi(T) = [Y(\nu^t)]_k \cap [\widehat{T}^t]_d \end{array}$$

2. Calculus on words and tableaux

$\lambda/\mu = \ydiagram{2+3,3,3}$

LR-reading λ/μ : $\begin{array}{ccccc} & & 3 & 2 & 1 \\ 6 & 5 & 4 & & \\ 9 & 8 & 7 & & \end{array}$, column LR-reading λ/μ : $\begin{array}{ccccc} & & 3 & 2 & 1 \\ 8 & 6 & 4 & & \\ 9 & 7 & 5 & & \end{array}$

$$\alpha_{\lambda/\mu} = (5\,6\,8\,7) \in \mathcal{S}_8$$

$T = \begin{array}{l} 111 \\ 122 \\ 233 \end{array}$, $\quad w(T) = 111221332, \quad w_{col}(T) = 111232312$

$$P(w(T)) = P(w_{col}(T)) = Y(\nu) = \begin{array}{l} 1111 \\ 222 \\ 33 \end{array}$$

$$Q(w(T)) = \begin{array}{l} 1236 \\ 459 \\ 78 \end{array}, \qquad Q(w_{col}(T)) = \begin{array}{l} 1238 \\ 469 \\ 57 \end{array}$$

T tableau,

(i) $w(T) \overset{k}{\equiv} w_{col}(T)$.

(ii) $\alpha_{\lambda/\mu} Q(w(T)) = Q(w_{col}(T))$ [Fulton, Young Tableaux, Appendix A.3].

Tableau switching

Let $T = \begin{array}{ccc} & & 1 \\ & 2 & 2 \\ 3 & 4 & \end{array}$, with shape λ/μ and let $S = \begin{array}{cc} \color{red}{1} & \color{red}{1} \\ \color{red}{2} & \end{array}$ with shape μ over the alphabet $\{\color{red}{1}, \color{red}{2}\}$.

$$S \cup T = \begin{array}{ccc} \color{red}{1} & \color{red}{1} & 1 \\ \color{red}{2} & 2 & 2 \\ 3 & 4 & \end{array} \xrightarrow{s} A \cup B = \begin{array}{ccc} 1 & 2 & \color{red}{1} \\ 2 & 4 & \color{red}{2} \\ 3 & \color{red}{1} & \end{array}.$$

Note: $S = B^n$ and $A = T^n$.

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Note: $S = B^n$ and $A = T^n$.

Theorem (BSS)

$S \cup T \xrightarrow{s} A \cup B$. Then:

(i) A and B are tableaux, the shape of A is the inner shape of B.

(ii) $A \cup B$ has the same shape as $S \cup T$.

(iii) $B \overset{k}{\equiv} S$ and $A \overset{k}{\equiv} T$.

(v) $A \cup B \xrightarrow{s} S \cup T$.

Dual Knuth equivalence

Two tableaux T and U of the same shape are dual equivalent, $T \overset{d}{\equiv} U$, if any sequence of jeu de taquin slides that can be applied to one of them can also be applied to the other, and the sequence of shape changes is the same for both.

$$T = \begin{array}{cccc} & & \blacksquare & 2 \\ & 1 & 3 & 6 \\ 4 & 5 & 7 & \end{array} \longrightarrow \begin{array}{cccc} & & 2 & 6 \\ & 1 & 3 & \blacksquare \\ 4 & 5 & 7 & \end{array}$$

$$U = \begin{array}{cccc} & & \blacksquare & 1 \\ & 2 & 2 & 2 \\ 3 & 3 & 4 & \end{array} \longrightarrow \begin{array}{cccc} & & 1 & 2 \\ & 2 & 2 & \blacksquare \\ 3 & 3 & 4 & \end{array}$$

Theorem

Let T and U be tableaux of the same shape. Then, $T \overset{d}{\equiv} U$ if and only if $Q(w(\mathrm{T})) = Q(w(\mathrm{U}))$.

Corollary

Two tableaux of the same normal shape are dual equivalent.

Theorem (Haiman 1992)
Let T and Q be tableaux with the same normal shape and let W be a tableau whose inner shape is the shape of T. Then,

$$\left.\begin{array}{r} T \cup W \stackrel{s}{\rightarrow} Z \cup X \\ Q \cup W \stackrel{s}{\rightarrow} Z \cup Y \end{array}\right\} \Rightarrow X \stackrel{d}{\equiv} Y$$

Knuth and dual Knuth equivalence

Theorem (Haiman 1992**)**

Let $\mathcal{D}$ be a dual Knuth equivalence class and $\mathcal{K}$ be a Knuth equivalence class, both corresponding to the same normal shape. Then, there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$.

Algorithm to construct $\mathcal{D} \cap \mathcal{K}$:

Let $U \in \mathcal{D}$ and let $V \in \mathcal{K}$ be the only tableau with normal shape in this class.

$$\begin{array}{ccc} W \cup U & & W \cup X \\ s\downarrow & & \uparrow s \\ U^n \cup Z & \rightarrow & V \cup Z \end{array}$$

$X \overset{d}{\equiv} U$ and $X \overset{k}{\equiv} V$.
$\mathcal{D} \cap \mathcal{K} = \{X\}$.

BSS bijection

$$
\begin{array}{rccc}
\xi^{BSS}: & LR(\lambda/\mu,\nu) & \to & LR(\lambda^t/\mu^t,\nu^t) \\
& T & \mapsto & \xi^{BSS}(T) \in [Y(\nu^t)]_K \cap [\widehat{T}^t]_d
\end{array}
$$

(1) $$T = \begin{array}{ccccc} & & 1 & 1 & 1 \\ 1 & 2 & 2 & & \\ 2 & 3 & 3 & & \end{array} \to \widehat{T} = \begin{array}{ccccc} & & 2 & 3 & 4 \\ 1 & 6 & 7 & & \\ 5 & 8 & 9 & & \end{array} \to \widehat{T}^t = \begin{array}{ccc} & 1 & 5 \\ & 6 & 8 \\ 2 & 7 & 9 \\ 3 & & \\ 4 & & \end{array}, \quad \nu = (4,3,2)$$

(2)

$$
\begin{array}{ccc}
W \cup \widehat{T}^t = \begin{array}{ccc} {\color{red}1} & 1 & 5 \\ {\color{red}2} & 6 & 8 \\ 2 & 7 & 9 \\ 3 & & \\ 4 & & \end{array} & & \begin{array}{ccc} {\color{red}1} & 1 & 1 \\ {\color{red}2} & 2 & 2 \\ 1 & 3 & 3 \\ 2 & & \\ 4 & & \end{array} = W \cup \xi^{BSS}(T) \\
s \downarrow & & \uparrow s \\
(\widehat{T}^t)^n \cup Z = \begin{array}{ccc} 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & {\color{red}1} \\ 4 & & \\ {\color{red}2} & & \end{array} & \longrightarrow & \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & {\color{red}1} \\ 4 & & \\ {\color{red}2} & & \end{array} = Y(\nu^t) \cup Z
\end{array}
$$

Evacuation and reversal

$$\lambda/\mu = \ydiagram{2+3,3,3} \quad \overset{\textit{rotate}}{\longleftrightarrow} \quad (\lambda/\mu)^* = \ydiagram{2+3,2+3,3}$$

Dual word: given $w = w_1 w_2 \cdots w_\ell \in \{1, \ldots, t\}^*$ let $w^* := w_\ell^* \cdots w_2^* w_1^*$, where $i^* = t - i + 1$.

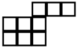
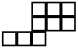
$$w = 1113213 \overset{*}{\leftrightarrow} w^* = 1321333$$

$$S = \begin{array}{ccccc} & & 1 & 1 & 1 \\ & 1 & 2 & 3 & \\ 3 & & & & \end{array} \overset{\textit{rotate}}{\longleftrightarrow} \begin{array}{ccccc} & & & & 3 \\ & 3 & 2 & 1 & \\ 1 & 1 & 1 & & \end{array} \overset{*}{\leftrightarrow} S^* = \begin{array}{ccccc} & & & & 1 \\ & 1 & 2 & 3 & \\ 3 & 3 & 3 & & \end{array}$$

$$T = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \overset{\textit{rotate}}{\longleftrightarrow} \begin{array}{ccc} & & 3 \\ & 3 & 2 \\ 2 & 1 & 1 \end{array} \overset{*}{\leftrightarrow} T^* = \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 2 & 3 & 3 \end{array}$$

Evacuation: T tableau of <u>normal shape</u>, $T^E := T^{*n}$

Evacuation and reversal

$\lambda/\mu =$  $\overset{\textit{rotate}}{\longleftrightarrow}$ $(\lambda/\mu)^* =$ 

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$$T = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \overset{\textit{rotate}}{\longleftrightarrow} \begin{array}{ccc} & & 3 \\ & 3 & 2 \\ 2 & 1 & 1 \end{array} \overset{*}{\leftrightarrow} T^* = \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 2 & 3 & 3 \end{array} \to T^{*n} = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & \\ 3 & & \end{array}$$

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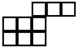
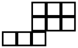
Evacuation and reversal

$\lambda/\mu =$  $\overset{\text{rotate}}{\longleftrightarrow}$ $(\lambda/\mu)^* =$ 

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Evacuation: T tableau of <u>normal shape</u>, $T^E := T^{*n}$

- $T^E = T^{a*}$
- Schützenberger involution

Reversal: T tableau of any shape, T^e is the only tableau in $[T^*]_k \cap [T]_d$

Note: $[T^*]_k \cap [T]_d = [T^{nE}]_k \cap [T]_d$. If T has normal shape then $T^E = T^e$.

Reversal in the case of LR tableaux:

• Crystal reflection operators on lattice permutations

$$w = 1(1(12)2)(1332) \xrightarrow{\sigma_1} \sigma_1(w) = 211221332$$

$\mathcal{Q}(w) = \mathcal{Q}(\sigma_1(w))$

$\sigma_0(w) = \sigma_1\sigma_2\sigma_1(w) = 311222333$

$\sigma_0(Y) = Y^E$

$$T = \begin{array}{ccccc} & & 1 & 1 & 1 \\ 1 & 2 & 2 & & \\ 2 & 3 & 3 & & \end{array} \xrightarrow{e} \begin{array}{ccccc} & & 1 & 1 & 3 \\ 2 & 2 & 2 & & \\ 3 & 3 & 3 & & \end{array} = T^e$$

Note: In general, $\sigma_0(w) \overset{k}{\not\equiv} w^*$

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$\sigma_0(Y) = Y^E$

$$T = \begin{array}{ccccc} & & 1 & 1 & 1 \\ 1 & 2 & 2 & & \\ 2 & 3 & 3 & & \end{array} \xrightarrow{e} \begin{array}{ccccc} & & 1 & 1 & 3 \\ 2 & 2 & 2 & & \\ 3 & 3 & 3 & & \end{array} = T^e$$

Note: In general, $\sigma_0(w) \stackrel{k}{\not\equiv} w^*$

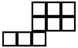
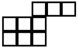
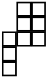
Jeu de taquin on two row/column tableaux

$$\begin{array}{ccccc} \blacksquare & \blacksquare & \blacksquare & 1 & 3 \\ 1 & 2 & 3 & 4 & \end{array} \rightsquigarrow \begin{array}{ccccc} \blacksquare & 1 & \blacksquare & 3 & \blacksquare \\ 1 & 2 & 3 & 4 & \end{array} \overset{\theta}{\longleftrightarrow} \begin{array}{ccccc} 1 & 1 & 3 & 3 & \blacksquare \\ \blacksquare & 2 & \blacksquare & 4 & \end{array} \rightsquigarrow \begin{array}{ccccc} 1 & 1 & 3 & 3 & \blacksquare \\ 2 & 4 & \blacksquare & \blacksquare & \end{array}$$

$$T = \begin{array}{ccccc} & & 1 & 1 & 1 \\ 1 & 2 & 2 & & \\ 2 & 3 & 3 & & \end{array}, \qquad \text{LR-reading: } \begin{array}{ccccc} & & 3 & 2 & 1 \\ 6 & 5 & 4 & & \\ 9 & 8 & 7 & & \end{array}, \qquad Q(w(T)) = L = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 9 & \\ 7 & 8 & & \end{array}$$

$$\theta_0(L) = \theta_1\theta_2\theta_1(L) = L^a = \begin{array}{cccc} & & 2 & 3 \\ & 4 & 5 & 6 \\ 1 & 7 & 8 & 9 \end{array} \leftrightarrow T^e = \begin{array}{ccccc} & & 1 & 1 & 3 \\ 2 & 2 & 2 & & \\ 3 & 3 & 3 & & \end{array}$$

$$\begin{array}{ccccc} T & \leftrightarrow & \sigma_0(T) & \leftrightarrow & T^e \\ \updownarrow & & \updownarrow & & \updownarrow \\ L & \leftrightarrow & \theta_0(L) & \leftrightarrow & L^a \end{array}$$

$\lambda/\mu =$  $\xrightarrow{\textit{rotate}} (\lambda/\mu)^* =$  $\xrightarrow{t} (\lambda/\mu)^\diamond := (\lambda/\mu)^{*t} =$ 

Zaballa map: $\diamond$ operation on lattice permutation words:

$$w = 111221332 \xrightarrow{\diamond} w^\diamond = 123124123$$

$$U = \begin{array}{ccccc} & & 1 & 1 & 1 \\ & & 2 & 2 & 2 \\ 1 & 3 & 3 & & \end{array} \xrightarrow{\diamond} \begin{array}{ccccc} & & 3 & 2 & 1 \\ & & 3 & 2 & 1 \\ 4 & 2 & 1 & & \end{array} \xrightarrow[\textit{rotate}]{t} \begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & 3 \\ 2 & & \\ 4 & & \end{array} = U^\diamond$$

$w_{col}(U^\diamond) = w(U)^\diamond$

Lemma

$$\begin{array}{ccc} [Y(\nu)]_k & \xrightarrow{\diamond} & [Y(\nu^t)]_k \\ w & \mapsto & w^\diamond \end{array}$$ such that $w^\diamond \overset{k}{\equiv} Y(\nu^t)$ and $Q(w^\diamond) = Q(w)^t$.

ξ^{HS} and ξ^{AZ}

$$\begin{array}{ccccccc} T & \longrightarrow & T^e & \longrightarrow & T^{e*} = U & \longrightarrow & U^{\diamond} \\ Q(T) = L & \longrightarrow & L^a & \longrightarrow & L^E & \longrightarrow & (\alpha_{(\lambda/\mu)^*} L^E)^t \end{array}$$

where $(L^a)^* = L^E = Q(U)$ and
$Q(U^{\diamond}) = Q(w_{col}(U))^t = (\alpha_{(\lambda/\mu)^*} Q(U))^t = (\alpha_{(\lambda/\mu)^*} Q(T)^E)^t$

$(\alpha_{(\lambda/\mu)^*} Q(T)^E)^t = ((\alpha_{\lambda/\mu} Q(T))^E)^t = (Q(w_{col}(T))^E)^t = (Q(w_{col}(\widehat{T}))^E)^t =$
$Q(w_{col}(\widehat{T})^{rev}) = Q(\widehat{T}^t)$

Theorem

Given the LR tableau T of shape λ/μ and weight ν, $\xi^{SH}(T) = \xi^{AZ}(T)$ is the unique tableau Knuth equivalent to $Y(\nu^t)$ and dual equivalent to $\widehat{T}^t$,

$$\begin{array}{rcl} \xi^X : LR(\lambda/\mu, \nu) & \longrightarrow & LR(\lambda^t/\mu^t, \nu^t) \\ T & \mapsto & \xi^X(T) \in [Y(\nu^t)]_k \cap [\widehat{T}^t]_d \end{array}$$

$X = BSS, HS, AZ.$

Linear equivalence

I. Pak, E. Vallejo, Reductions of Young tableau bijections, available at arXiv:math/0408171

We view a bijection $\tau : \mathcal{A} \longrightarrow \mathcal{B}$ as an algorithm which inputs $A \in \mathcal{A}$ and outputs $B = \tau(A) \in \mathcal{B}$. The computational complexity is, roughly, the number of steps in the bijection.

$$\alpha \text{ reduces linearly to } \beta : \ \alpha \hookrightarrow \beta \Leftrightarrow C(\alpha) \leq aC(\beta) + b$$

$$\alpha \text{ linear equivalent to } \beta : \ \alpha \sim \beta \Leftrightarrow \alpha \hookrightarrow \beta \hookrightarrow \alpha$$

$D = (d_1, \ldots, d_n)$ array of integers, $m = m(D) := \max_i d_i$. The *bit–size* of D, denoted by $\langle D \rangle$, is the amount of space required to store D; for simplicity from now on we assume that $\langle D \rangle = n\lceil log_2 m + 1\rceil$.

Elementary circuits

- Suppose $\delta_1 : \mathcal{A}_1 \longrightarrow \mathcal{X}_1$, $\gamma : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $\delta_2 : \mathcal{X}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_2 have linear cost, and consider $\chi = \delta_2 \circ \gamma \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *trivial* and denote it by $I(\delta_1, \gamma, \delta_2)$.
- Suppose $\gamma_1 : \mathcal{A} \longrightarrow \mathcal{X}$ and $\gamma_2 : \mathcal{X} \longrightarrow \mathcal{B}$, and let $\chi = \gamma_2 \circ \gamma_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *sequential* and denote it by $S(\gamma_1, \gamma_2)$.
- Suppose $\delta_1 : \mathcal{A} \longrightarrow \mathcal{X}_1 \times \mathcal{X}_2$, $\gamma_1 : \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$, $\gamma_2 : \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$, and $\delta_1 : \mathcal{Y}_1 \times \mathcal{Y}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_1 have linear cost. Consider $\chi = \delta_2 \circ (\gamma_1 \times \gamma_2) \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$: we call this circuit *parallel* and denote it by $P(\delta_1, \gamma_1, \gamma_2, \delta_2)$.

Parallel-sequencial circuits

For a fixed bijection α, we say that $\sqsupset$ is an α*–based ps–circuit* if one of the following holds:

- $\sqsupset = \delta$, where δ is a bijection having linear cost.
- $\sqsupset = I\left(\delta_1, \alpha, \delta_2\right)$, where δ_1, δ_2 are bijections having linear cost.
- $\sqsupset = P\left(\delta_1, \gamma_1, \gamma_2, \delta_2\right)$, where γ_1, γ_2 are α–based ps–circuits and δ_1, δ_2 are bijections having linear cost.
- $\sqsupset = S\left(\gamma_1, \gamma_2\right)$, where γ_1, γ_2 are α–based ps–circuits.

A map β is *linearly reducible* to α, write $\beta \hookrightarrow \alpha$, if there exist a finite α–based ps-circuit $\sqsupset$ which computes β. We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β, and β is linearly reducible to α.

Theorem (PV)

The following maps are linearly equivalent:

(1) RSK correspondence.

(2) Jeu de Taquin map.

(3) Littlewood-Robinson map.

(4) Tableau switching map s.

(5) Evacuation for normal shapes (Schützenberger involution) E.

(6) Reversal e.

(7) First and Second Fundamental Symmetry maps ρ_1 and ρ_2.

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Theorem

(8) First and Second Fundamental Symmetry maps ρ_1 and ρ_2 are identical [Danilov-Koshevoy].
(9) First and Third Fundamental Symmetry maps ρ_1 and ρ_3 are identical [A.].
(10) ξ^{AZ}, ξ^{BSS}, ξ^{HS} are identical and

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(5) Evacuation for normal shapes (Schützenberger involution) E.

(6) Reversal e.

(7) First and Second Fundamental Symmetry maps ρ_1 and ρ_2.

Theorem

(8) First and Second Fundamental Symmetry maps ρ_1 and ρ_2 are identical [Danilov-Koshevoy].

(9) First and Third Fundamental Symmetry maps ρ_1 and ρ_3 are identical [A.].

(10) ξ^{AZ}, ξ^{BSS}, ξ^{HS} are identical and

$$\xi^{AZ} \hookrightarrow E : T \overset{RSK}{\rightarrow} L \overset{E}{\rightarrow} L^E \overset{RSK^{-1}}{\rightarrow} U \rightarrow U_k \circ \cdots \circ U_1 \overset{RSK}{\rightarrow} \mathcal{Q}(w_{col}(U)) \rightarrow \mathcal{Q}(w_{col}(U))^t \overset{RSK^{-1}}{\rightarrow} U^{\diamond}.$$