SLC 60

SEMINAIRE LOTHARINGIEN DE COMBINATOIRE

DESCENTS AND DECREASES

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Strobl, Wolfgangsee

THE SOURCE

Mostly from the paper

Decreases and descents

Sem. Lothar. Combin., 58, 2007, Art. B58a, 17 pages

jointly written with Guo-Niu Han.

Please, download...

FIRST PART: THE SOURCE

Also from the 194-page volume

Statistical Distributions on words and q-calculus on permutations

that can be downloaded from

http://www-irma.u-strasbg.fr/∼foata/paper/pub109.html

NEED FOR MORE INVERSION FORMULAS

For words by various order statistics.

Transfer methods to go over to permutation groups.

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Moebius inversion formula: **déjà vu**

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Search for q-analog of MacMahon's Theorem (Zeilberger...)

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Search for q-analog of MacMahon's Theorem (Zeilberger...)

It provides no handy tool, and even no q-extension!

NEED FOR MORE INVERSION FORMULAS

For words by various order statistics.

Transfer methods to go over to permutation groups.

The idea is then to enrich MacMahon's content

with appropriate q-homomorphisms

But how?

BACK TO MACMAHON

Consider the matrix

$$C' = \begin{pmatrix} \binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \cdots & \binom{0}{r-1} & \binom{0}{r} \\ \binom{1}{0} & \binom{1}{1} & \binom{1}{2} & \cdots & \binom{1}{r-1} & \binom{1}{r} \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \cdots & \binom{2}{r-1} & \binom{2}{r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{r-1}{0} & \binom{r-1}{1} & \binom{r-1}{2} & \cdots & \binom{r-1}{r-1} & \binom{r-1}{r} \\ \binom{r}{0} & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & \binom{r}{r} \end{pmatrix}$$

and expand the determinant $\det(I - C')$, the multiplication of the entries being the juxtaposition product of the biletters.

BACK TO MACMAHON

For instance,

$$\binom{1}{3}\binom{2}{4}\binom{1}{2} = \binom{121}{342}.$$

BACK TO MACMAHON

We obtain (alternate sum of the principal minors):

$$\det(I - C') = \sum_{J \subset [0,r]} (-1)^{|J|} \sum_{\sigma \in \mathfrak{S}_J} (-1)^{\operatorname{inv} \sigma} \begin{pmatrix} \sigma_{i_1} & \sigma_{i_2} & \cdots & \sigma_{i_l} \\ i_1 & i_2 & \cdots & i_l \end{pmatrix},$$

where $J = \{i_1 < i_2 < \cdots < i_l\}$ and $\mathfrak{S}_J$ is the

permutation group acting on the set J.

BACK TO MACMAHON

Next, expand the inverse $1/\det(I-C')$ assuming that two biletters $\binom{i}{j}$, $\binom{i'}{j'}$ commute whenever $i \neq i'$.

Theorem. *We have:*

$$\frac{1}{\det(I-C')} = \sum_{w \in [0,r]^*} \binom{\overline{w}}{w}$$

$$= 1 + \binom{0}{0} + \cdots + \binom{r}{r} + \cdots + \binom{1\,1\,2\,3\,3}{2\,3\,1\,1\,3} + \cdots$$

$\overline{w} :=$ nondecreasing rearrangement of w:

if $w = x_1 \cdots x_n$, then $\overline{w} := \overline{x}_1 \cdots \overline{x}_n \quad (\overline{x}_1 \leq \cdots \leq \overline{x}_n)$.

BACK TO MACMAHON

Let $A = (a_{i,j})$ $(0 \leq i, j \leq r)$ be a matrix with ring entries and consider the homomorphism ϕ generated by $\phi\binom{i}{j} = X_i a(i,j)$. Then

$$\phi \det(I - C') = \begin{vmatrix} 1 - a_{0,0}X_0 & -a_{0,1}X_0 & \cdots & -a_{0,r}X_0 \\ -a_{1,0}X_1 & 1 - a_{1,1}X_1 & \cdots & -a_{1,r}X_1 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{r,0}X_r & -a_{r,1}X_r & \cdots & 1 - a_{r,r}X_r \end{vmatrix}$$

For $w = x_1 \cdots x_n$ let $X(w) := X_{x_1} \cdots X_{x_n}$. Then

$$\phi\begin{pmatrix} \overline{w} \\ w \end{pmatrix} = a(\overline{x}_1, x_1) \cdots a(\overline{x}_n, x_n) X(w).$$

BACK TO MACMAHON

The image under ϕ:

$$\frac{1}{\begin{vmatrix} 1-a_{0,0}X_0 & -a_{0,1}X_0 & \cdots & -a_{0,r}X_0 \\ -a_{1,0}X_1 & 1-a_{1,1}X_1 & \cdots & -a_{1,r}X_1 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{r,0}X_r & -a_{r,1}X_r & \cdots & 1-a_{r,r}X_r \end{vmatrix}} = \sum_{w \in [0,r]^*} a(\overline{x}_1, x_1) \cdots a(\overline{x}_n, x_n) X(w).$$

(MacMahon Master Theorem, 1913).

BACK TO MACMAHON

In fact, MacMahon essentially needed the result for

$$A = \begin{pmatrix} 1 & t & \cdots & t \\ 1 & 1 & \cdots & t \\ \vdots & \vdots & \ddots & t \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

and then

$$\frac{1}{\begin{vmatrix} 1 - X_0 & -tX_0 & \cdots & -tX_0 \\ -X_1 & 1 - X_1 & \cdots & -tX_1 \\ \vdots & \vdots & \ddots & \vdots \\ -X_r & -X_r & \cdots & 1 - X_r \end{vmatrix}} = \sum_{w \in [0,r]^*} t^{\operatorname{exc} w} X(w).$$

(``exc" defined shortly.)

THE ENRICHMENT

The MacMahon identity involves a single statistic ``exc".

Is there an inversion formula that involves several statistics?

STATISTICAL DISTRIBUTIONS ON WORDS

Let $[0, r]^*$ denote the set of all words $w = x_1 x_2 \cdots x_n$, whose letters x_i belong to $[0, r] = \{0, 1, \ldots, r\}$.

Work out with the word:

$$w = 3\ 2\ 5\ 5\ 8\ 6\ 6\ 3\ 0\ 0\ 1\ 4\ 0\ 3\ 8$$

SEVERAL WORD STATISTICS

The descents (indicated by bullets):

$$\begin{array}{cccccccccccccccc} w = & 3 & 2 & 5 & 5 & 8 & 6 & 6 & 3 & 0 & 0 & 1 & 4 & 0 & 3 & 8 \\ & \bullet & & & & \bullet & & \bullet & \bullet & & & & \bullet & & & \end{array}$$

The set of descents of w, $\mathrm{DES}\, w$, defined as

$$\mathrm{DES}\, w := \{ i : 1 \leq i \leq n-1,\ x_i > x_{i+1} \}.$$

SEVERAL WORD STATISTICS

The decreases:

$$w = 3\ 2\ 5\ 5\ 8\ 6\ 6\ 3\ 0\ 0\ 1\ 4\ 0\ 3\ 8$$
$$\bullet \qquad\qquad \bullet\ \circ\ \bullet\ \bullet \qquad\qquad \bullet$$

The set of decreases of w, $\mathrm{DEC}\, w$, defined as

$$\mathrm{DEC}\, w := \{i : 1 \le i \le n-1 \text{ and } x_i = x_{i+1} = \cdots = x_j > x_{j+1}\}$$

for some j such that $i \le j \le n-1$.

($\circ$ decrease which is not a descent.)

SEVERAL WORD STATISTICS

The rises:

$$\begin{array}{ccccccccccccccccc} w & = & 3 & 2 & 5 & 5 & 8 & 6 & 6 & 3 & 0 & 0 & 1 & 4 & 0 & 3 & 8 \\ & & & \bullet & & \bullet & & & & & & \bullet & \bullet & & \bullet & \bullet & \end{array}$$

The set of rises of w, $\mathrm{RISE}\, w$, defined as

$$\mathrm{RISE}\, w := \{i : 1 \le i \le n-1,\ x_i < x_{i+1}\}.$$

SEVERAL WORD STATISTICS

The increases:

$$\begin{array}{cccccccccccccccc} w = & 3 & 2 & 5 & 5 & 8 & 6 & 6 & 3 & 0 & 0 & 1 & 4 & 0 & 3 & 8 \\ & & \bullet & \circ & \bullet & & & & & \circ & \bullet & \bullet & & \bullet & \bullet & \end{array}$$

The set of increases of w, $\mathrm{INC}\, w$, defined as

$$\mathrm{INC}\, w := \{i : 1 \leq i \leq n-1 \text{ and } x_i = x_{i+1} = \cdots = x_j < x_{j+1}\}$$

for some j such that $i \leq j \leq n-1$.

($\circ$ increase which is not a rise.)

SEVERAL WORD STATISTICS

The records:

$$w = 3\,2\,5\,5\,8\,6\,6\,3\,0\,0\,1\,4\,0\,3\,8$$
$$\bullet \quad\; \bullet\;\bullet\;\bullet \qquad\qquad\qquad\qquad\qquad\quad \bullet$$

The set of records of w, $\mathrm{REC}\,w$, defined as

$$\mathrm{REC}\,w := \{i : 1 \le i \le n,\ x_j \le x_i \text{ for all } j \text{ such that } 1 \le j \le i-1\}$$

STATISTICAL DISTRIBUTIONS ON WORDS

Example. For the word $w = 3\,2\,5\,5\,8\,6\,6\,3\,0\,0\,1\,4\,0\,3\,8$ the sets DES, DEC, INC, RISE, REC of w are indicated by bullets.

w	=	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES	=	●				●		●	●				●			
DEC	=	●				●	○	●	●				●			
RISE	=		●		●						●	●		●	●	●
INC	=		●	○	●					○	●	●		●	●	●
REC	=	●		●	●	●										●

STATISTICAL DISTRIBUTIONS ON WORDS

DESCENTS:

$w =$	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES $=$	•				•		•	•				•			
DEC $=$	•				•	•	•	•				•			
RISE $=$		•		•						•	•		•	•	•
INC $=$		•	•	•					•	•	•		•	•	•
REC $=$	•		•	•	•										•
	X_3				X_8		X_6	X_3				X_4			

If i is a DESCENT, map x_i onto X_{x_i}.

RISES not RECORDS (RISE \ REC):

$w =$	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES $=$	•				•		•	•				•			
DEC $=$	•				•	•	•	•				•			
RISE $=$		•		•						•	•		•	•	•
INC $=$		•	•	•					•	•	•		•	•	•
REC $=$	•		•	•	•										•
		Y_2								Y_0	Y_1		Y_0	Y_3	

If i is a RISE, but not a RECORD, map x_i onto Y_{x_i}.

DECREASES not DESCENTS (DEC $\setminus$ DES):

w	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES	•				•		•	•				•			
DEC	•				•	•	•	•				•			
RISE		•		•						•	•		•	•	•
INC		•	•	•					•	•	•		•	•	•
REC	•		•	•	•										•

$$Z_6$$

If i is a DECREASE, but not a DESCENT, map x_i onto Z_{x_i}.

INCREASES not RISES, not RECORDS ($(\mathrm{INC} \setminus \mathrm{RISE}) \setminus \mathrm{REC}$):

w =	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES =	•				•		•	•				•			
DEC =	•				•	•	•	•				•			
RISE =		•		•						•	•		•	•	•
INC =		•	•	•					•	•	•		•	•	•
REC =	•		•	•	•										•
									T_0						

If i is an INCREASE, but not a RISE and not a RECORD, map x_i onto T_{x_i}.

STATISTICAL DISTRIBUTIONS ON WORDS

RISES and RECORDS ($\mathrm{RISE} \cap \mathrm{REC}$):

w	$=$	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES	$=$	•				•		•	•				•			
DEC	$=$	•				•	•	•	•				•			
RISE	$=$		•		•						•	•		•	•	•
INC	$=$		•	•	•					•	•	•		•	•	•
REC	$=$	•		•	•	•										•
					Y'_5											Y'_8

If i is a RISE and a RECORD, map x_i onto Y'_{x_i}.

INCREASES and RECORDS but not RISES

$((\mathrm{INC} \setminus \mathrm{RISE}) \cap \mathrm{REC})$:

w	$=$	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES	$=$	•				•		•	•				•			
DEC	$=$	•				•	•	•	•				•			
RISE	$=$		•		•						•	•		•	•	•
INC	$=$		•	•	•					•	•	•		•	•	•
REC	$=$	•		•	•	•										•
				T'_5												

If i is an INCREASE and a RECORD, but not a RISE, map x_i onto T'_{x_i}.

Altogether,

w	$=$	3	2	5	5	8	6	6	3	0	0	1	4	0	3	8
DES	$=$	•				•		•	•				•			
DEC	$=$	•				•	•	•	•				•			
RISE	$=$		•		•						•	•		•	•	•
INC	$=$		•	•	•					•	•	•		•	•	•
REC	$=$	•		•	•	•										•
$\psi(w)$	$=$	X_3	Y_2	T'_5	Y'_5	X_8	Z_6	X_6	X_3	T_0	Y_0	Y_1	X_4	Y_0	Y_3	Y'_8

X for ``des",
Z for ``dec" not ``des",
Y' for ``rise" and ``rec",

Y for ``rise" not ``rec",
T for ``inc" not ``rise" not ``rec",
T' for ``inc" and ``rec" not ``rise".

STATISTICAL DISTRIBUTIONS ON WORDS

Each place i belongs to one and only one of the sets

$$\mathrm{DES}, \quad \mathrm{RISE} \setminus \mathrm{REC}, \quad \mathrm{DEC} \setminus \mathrm{DES},$$
$$(\mathrm{INC} \setminus \mathrm{RISE}) \setminus \mathrm{REC}, \quad \mathrm{RISE} \cap \mathrm{REC}, \quad (\mathrm{INC} \setminus \mathrm{RISE}) \cap \mathrm{REC}\,.$$

STATISTICAL DISTRIBUTIONS ON WORDS

With six sequences of commuting variables (X_i), (Y_i), (Z_i), (T_i), (Y'_i), (T'_i) $(i = 0, 1, 2, \ldots)$ and for each word $w = x_1x_2\ldots x_n$ define the *weight* $\psi(w)$ of $w = x_1x_2\ldots x_n$ to be

$$\psi(w) := \prod_{i\in\mathrm{DES}} X_{x_i} \prod_{i\in\mathrm{RISE}\setminus\mathrm{REC}} Y_{x_i} \prod_{i\in\mathrm{DEC}\setminus\mathrm{DES}} Z_{x_i}$$
$$\times \prod_{i\in(\mathrm{INC}\setminus\mathrm{RISE})\setminus\mathrm{REC}} T_{x_i} \prod_{i\in\mathrm{RISE}\cap\mathrm{REC}} Y'_{x_i} \prod_{i\in(\mathrm{INC}\setminus\mathrm{RISE})\cap\mathrm{REC}} T'_{x_i},$$

($i \in \mathrm{RISE} \setminus \mathrm{REC}$ stands for $i \in \mathrm{RISE}(w) \setminus \mathrm{REC}(w)$, etc.)

Now let C be the $(r+1) \times (r+1)$ matrix

$$C = \begin{pmatrix} 0 & \dfrac{X_1}{1-Z_1} & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & 0 & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & 0 & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & 0 & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & \dfrac{Y_{r-1}}{1-T_{r-1}} & 0 \end{pmatrix}.$$

Theorem. *The generating function for the set* $[0, r]^*$ *by the weight* ψ *is given by*

$$\sum_{w \in [0,r]^*} \psi(w) = \frac{\displaystyle\prod_{0 \le j \le r} \left(1 + \frac{Y'_j}{1 - T'_j}\right)}{\det(I - C)},$$

where I *is the identity matrix of order* $(r+1)$.

Remember $\quad C = C(X_j, Y_j, Z_j, T_j) = \begin{pmatrix} 0 & \frac{X_j}{1-Z_j} \\ \frac{Y_j}{1-T_j} & 0 \end{pmatrix}.$

Theorem. *The generating function for the set* $[0,r]^*$ *by the weight* ψ *is given by*

$$\sum_{w\in[0,r]^*} \psi(w) = \frac{\prod_{0\le j\le r}\Bigl(1+\dfrac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)},$$

where I *is the identity matrix of order* $(r+1)$.

Proof. Use a non-commutative version of the MacMahon Master Theorem and the first fundamental transformation on words. ∎

FIRST HOMOMORPHISM

Let α be generated by the substitutions

$$X_j \leftarrow usq^j, \qquad Y_j \leftarrow uq^j, \qquad Z_j \leftarrow usq^j,$$

$$T_j \leftarrow uq^j, \qquad Y'_j \leftarrow uRq^j, \qquad T'_j \leftarrow uRq^j.$$

Then

$$\alpha\,\psi(w) = u^{\lambda w} q^{\operatorname{tot} w} s^{\operatorname{dec} w} R^{\operatorname{inrec} w},$$

where $\operatorname{inrec} w$ is the number of records, which are also increases.

FIRST HOMOMORPHISM

The image under α yields:

$$\sum_{w\in[0,r]^*} s^{\operatorname{dec} w} R^{\operatorname{inrec} w} u^{\lambda w} q^{\operatorname{tot} w} = \frac{\dfrac{(us;q)_{r+1}}{(uR;q)_{r+1}}}{1 - \displaystyle\sum_{0\le l\le r} \frac{(us;q)_l}{(u;q)_l} usq^l},$$

the right-hand side being also equal to (by telescoping)

$$\frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

FIRST HOMOMORPHISM

The identity

$$\sum_{w\in[0,r]^*} s^{\mathrm{dec}\,w} R^{\mathrm{inrec}\,w} u^{\lambda w} q^{\mathrm{tot}\,w}$$

$$= \frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

can also be derived by means of

the Kim-Zeng transformation.

SECOND HOMOMORPHISM

For each word $w = x_1 x_2 \cdots x_n \in [0, r]^*$ let

$\operatorname{evdec} w := \#\,\text{even decreases of } w$;

$\operatorname{odd} w := \#\,\text{odd letters of } w$.

Let β be generated by the substitutions

$$X_j, Z_j \leftarrow u s^{2\chi(j \text{ even})} (sZ)^{\chi(\operatorname{odd} w)} q^j;$$
$$Y_j, T_j, Y'_j, T'_j \leftarrow u q^j.$$

Then

$$\beta\,\psi(w) = u^{\lambda w} q^{\operatorname{tot} w} s^{2 \operatorname{evdec} w + \operatorname{odd} w} Z^{\operatorname{odd} w}.$$

SECOND HOMOMORPHISM

Then

$$\sum_{w\in[0,r]^*} \alpha\psi(w) = F_r(u; s, q, Z),$$

where

$$\begin{aligned}
&F_r(u; s, q, Z) \\
&= \frac{(us^2q^2; q^2)_{\lfloor r/2\rfloor}\,(1-s^2q^2)\,(u; q^2)_{\lfloor (r+1)/2\rfloor}\,(u; q^2)_{\lfloor r/2\rfloor}}{(u; q^2)_{\lfloor r/2\rfloor+1}\Big((u; q^2)_{\lfloor (r+1)/2\rfloor}\big((u; q^2)_{\lfloor r/2\rfloor} - s^2q^2\,(us^2q^2; q^2)_{\lfloor r/2\rfloor}\big) \\ \quad + sqZ\,(u; q^2)_{\lfloor r/2\rfloor}\big((u; q^2)_{\lfloor (r+1)/2\rfloor} - (us^2q^2; q^2)_{\lfloor (r+1)/2\rfloor}\big)\Big)},
\end{aligned}$$

a necessary step in the derivation of a multivariable distribution for B_n.

TWO EASY DERIVATIONS

By simple specialization:

$$\sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w} = \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}.$$

Also (but goes back to MacMahon)

$$\sum_{w\in[0,r]^*} s^{\operatorname{des} w} u^{\lambda w} = \frac{1-s}{(1-u+us)^{r+1} - s}.$$

TWO EASY DERIVATIONS

Interesting to note:

$$\sum_{n\geq 0} \frac{u^n}{(1-t)^{n+1}} \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma}$$
$$= \sum_{r\geq 0} t^r \sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w}$$
$$= \sum_{r\geq 0} t^r \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}.$$