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EXCEDANCE NUMBERS FOR PERMUTATIONS IN COMPLEX REFLECTION GROUPS

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Abstract. Recently, Bagno, Garber and Mansour [Sém. Lotharingien Combin. **56** (2007), Art. B56d] studied a kind of excedance number on the complex reflection groups and computed its multidistribution with the number of fixed points on the set of involutions in these groups. In this note, we consider the similar problems in more general cases and make a correction of one result obtained by them.

1. Introduction

It is well known that there is a single infinite family of groups $G_{r,s,n}$ and exactly 34 other "exceptional" complex reflection groups [4]. The infinite family $G_{r,s,n}$, where r, s, n are positive integers with $s \mid r$, consists of the groups of $n \times n$ matrices such that

- the entries are either 0 or r^{th} roots of unity;
- there is exactly one nonzero entry in each row and each column;
- the $(r/s)^{\text{th}}$ power of the product of the nonzero entries is 1.

The classical Weyl groups appear as special cases: for $r = s = 1$ we have the symmetric group $G_{1,1,n} = S_n$, for $r = 2, s = 1$ we have the hyperoctahedral group $G_{2,1,n} = B_n$, and for $r = s = 2$ we have the group of even-signed permutations $G_{2,2,n} = D_n$.

We say that a permutation $\pi \in G_{r,s,n}$ is *an involution* if $\pi^2 = 1$. More generally, we define $\mathcal{G}^m_{r,s,n} = \{\sigma \in G_{r,s,n} | \sigma^m = 1\}$. Recently, Bagno, Garber and Mansour [2] studied an excedance number on the complex reflection groups (see [5]) and computed the number of involutions having specific numbers of fixed points and excedances. In this note, we consider the similar problems on the set $\mathcal{G}^m_{r,s,n}$.

This paper is organized as follows. In Section 2, we recall some properties of $G_{r,s,n}$ and define some parameters on $G_{r,n} = G_{r,1,n}$ and hence also on $G_{r,s,n}$. In Section 3,

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we present our main results and compute the corresponding recurrences together with explicit formulas.

2. Preliminaries

Let r and n be any two positive integers. *The group of colored permutations of n digits with r colors* is the wreath product $G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$ consisting of all the pairs (z, τ) where $z \in \mathbb{Z}_r^n$ and $\tau \in S_n$. Let $\tau, \tau' \in S_n$, $z = (z_1, ..., z_n) \in \mathbb{Z}_r^n$ and $z' = (z'_1, ..., z'_n) \in \mathbb{Z}_r^n$, the multiplication in $G_{r,n}$ is defined by $(z, \tau) \cdot (z', \tau') = ((z_1 + z'_{\tau^{-1}(1)}, ..., z_n + z'_{\tau^{-1}(n)}), \tau \circ \tau')$, where $+$ is taken modulo r.

We use some conventions along this paper. For an element $\sigma = (z, \tau) \in G_{r,n}$ with $z = (z_1, ..., z_n)$ we write $z_i(\sigma) = z_i$. For $\sigma = (z, \tau)$, we denote $|\sigma| = (0, \tau), (0 \in \mathbb{Z}_r^n)$.

A much more natural way to present $G_{r,n}$ is the following: consider the alphabet $\Sigma = \{1^{[0]}, \dots, n^{[0]}, 1^{[1]}, \dots, n^{[1]}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$ as the set $[\![n]\!] = \{1, \dots, n\}$ colored by the colors $0, \dots, r-1$. Then, an element of $G_{r,n}$ is called a *colored permutation*, i.e., a bijection $\sigma : \Sigma \to \Sigma$ such that if $\sigma(i) = k^{[t]}$ then $\sigma(i^{[j]}) = k^{[t+j]}$ where $0 \le j \le r-1$ and the addition is taken modulo r. Occasionally, we write j bars over a digit i instead of $i^{[j]}$. For example, an element $(z, \tau) = ((1, 2, 1, 2), (3, 1, 2, 4)) \in G_{3,4}$ will be written as $(\bar{3}\bar{\bar{1}}\bar{2}\bar{\bar{4}})$.

For each $s \mid r$, the *complex reflection group* can also be defined as:

$$G_{r,s,n} := \{\sigma \in G_{r,n} \mid \mathrm{csum}(\sigma) \equiv 0 \bmod s\},$$

where $\mathrm{csum}(\sigma) = \sum\limits_{i=1}^{n} z_i(\sigma)$.

One can define the following well-known statistics on S_n. For any permutation $\sigma \in S_n$, $i \in [\![n]\!]$ is *an excedance of* σ if and only if $\sigma(i) > i$. We denote the number of excedances by $\mathrm{exc}(\sigma)$. Another natural statistic on S_n is the number of fixed points, denoted by $\mathrm{fix}(\sigma)$. We can similarly define some statistics on $G_{r,n}$. The complex reflection group $G_{r,s,n}$ inherits all of them. Given any ordered alphabet Σ', we recall the definition of the *excedance set* of a permutation σ on Σ':

$$\mathrm{Exc}(\sigma) = \{i \in \Sigma' \mid \sigma(i) > i\},$$

and the *excedance number* is defined to be $\mathrm{exc}(\sigma) = |\mathrm{Exc}(\sigma)|$.

We define the color order on the set $\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$ for $0 \le j < i < r$ by $1^{[i]} < 2^{[i]} < \dots < n^{[i]} < 1^{[j]} < 2^{[j]} < \dots < n^{[j]}$. We note that there are some other possible ways of defining orders on Σ, some of them lead to other versions of the excedance number, see for example [1]. For example, given the color order $\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$, we write $\sigma = (2\bar{1}\bar{\bar{3}}) \in G_{3,3}$ in an extended form

$$(\star) \qquad \begin{pmatrix} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 \\ \bar{\bar{2}} & 1 & \bar{3} & \bar{2} & \bar{\bar{1}} & 3 & 2 & \bar{1} & \bar{\bar{3}} \end{pmatrix}$$

which implies that $\mathrm{Exc}(\sigma) = \{\bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}, \bar{1}, \bar{3}, 1\}$ and $\mathrm{exc}(\sigma) = 6$.

Define $\mathrm{Exc}_A(\sigma) = \{i \in [\![n-1]\!] \mid \sigma(i) > i\}$, where the comparison is with respect to the color order, and denote $\mathrm{exc}_A(\sigma) = |\mathrm{Exc}_A(\sigma)|$. For instance, if $\sigma = (\bar{1}\bar{\bar{3}}2\bar{\bar{4}}) \in G_{3,4}$, then $\mathrm{csum}(\sigma) = 5$, $\mathrm{Exc}_{\mathrm{A}}(\sigma) = \{3\}$ and hence $\mathrm{exc}_A(\sigma) = 1$.

Now we can define the colored excedance number for $G_{r,n}$ by

$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) = r \cdot \mathrm{exc}_A(\sigma) + \mathrm{csum}(\sigma).$$

Let Σ ordered by the color order, then we can state that $\mathrm{exc}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma)$ obtained by Bagno and Garber [1] for any $\sigma \in G_{r,n}$.

For $\sigma = (z, \tau) \in G_{r,n}$, $|\sigma|$ is the permutation of $[\![n]\!]$ satisfying $|\sigma|(i) = \tau(i)$. We say that $i \in [\![n]\!]$ is an *absolute fixed point* of $\sigma \in G_{r,n}$ if $|\sigma|(i) = i$. We denote the number of absolute fixed point of $\sigma \in G_{r,n}$ by $\mathrm{fix}(\sigma)$.

3. Main results and proofs

Recall that $\mathcal{G}^m_{r,s,n} = \{\sigma \in G_{r,s,n} | \sigma^m = 1\}$, define

$$\begin{aligned} H^{(m)}_{r,s,n}(u, v, w) &= \sum_{\sigma \in \mathcal{G}^m_{r,s,n}} u^{\mathrm{fix}(\sigma)} v^{\mathrm{exc}_A(\sigma)} w^{\mathrm{csum}(\sigma)}, \\ \mathcal{H}^{(m)}_{r,s}(x; u, v, w) &= \sum_{n \geq 0} H^{(m)}_{r,s,n}(u, v, w) \frac{x^n}{n!}. \end{aligned}$$

It is well known that the Eulerian number, $A_{d-1,k}$, is the number of permutations on $[\![d-1]\!]$ with $k-1$ excedances for $k \in [\![d-1]\!]$, which is also the number of cyclic permutations in S_d with k excedances. A bijective proof of this fact is given in [3, Theorem 1.19].

Our main result can be formulated as follows.

Theorem 3.1. *For any integers $r, m \geq 1$, the generating function $\mathcal{H}^{(m)}_{r,1}(x; u, v, w)$ is*

$$\exp\left\{ \sum_{\{t|0\leq t<r, r|tm\}} xuw^t + \sum_{d|m, d\geq 2} \frac{x^d}{d!} \sum_{k=1}^{d-1} A_{d-1,k} \sum_{i=0}^{k} \binom{k}{i} v^{k-i} \sum_{r|\frac{tm}{d}} U^{(i)}_{d-k,t} w^t \right\},$$

where $U^{(i)}_{d-k,t}$ is the coefficient of x^t in $(x + x^2 + \cdots + x^{r-1})^i (1 + x + \cdots + x^{r-1})^{d-k}$, i.e.,

$$U^{(i)}_{d-k,t} = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \sum_{\ell \geq 0} (-1)^{\ell} \binom{d+j-k}{\ell} \binom{d+j+t-k-\ell r-1}{t-\ell r}.$$

Proof. For any $\pi \in \mathcal{G}^m_{r,1,n}$, the length of each cycle of π is a factor of m, then there exist $k_1, k_2, \ldots, k_{d-1} \in [\![n-1]\!]$ with $d|m$ such that $k_1, k_2, \ldots, k_{d-1}$ and n form a cycle of $|\pi|$.

If $d = 1$, that is $\pi(n) = n^{[j]}$ for some j with $0 \leq j \leq r-1$, then $\pi^m(n) = n^{[jm]} = n$ which implies that $r|jm$. Define $\pi' \in \mathcal{G}^m_{r,1,n-1}$ by ignoring the last digit of π. Then we have

$$\begin{aligned} \mathrm{fix}(\pi) &= \mathrm{fix}(\pi') + 1, \\ \mathrm{exc}_A(\pi) &= \mathrm{exc}_A(\pi'), \\ \mathrm{csum}(\pi) &= \mathrm{csum}(\pi') + j. \end{aligned}$$

If $d \geq 2$, we know that there are $A_{d-1,k}$ cyclic permutations in S_d with k excedances for $k \in [\![d-1]\!]$. For any cyclic permutation C of length d in S_d with

$$\mathrm{Exc}(C) = \{j \in [\![d-1]\!] | C(j) > j\}$$

such that $\mathrm{exc}(C)=k$, we can color the symbols in C with the color set $\{[0],[1],\dots,[r-1]\}$ and obtain the colored cyclic permutation C'. Suppose that $\mathrm{exc}_A(C')=k-i$, we know that $\mathrm{Exc}_A(C')\subseteq\mathrm{Exc}(C)$, which means that $\mathrm{exc}(C)-\mathrm{exc}_A(C)=i$, in other words, there are i number of symbols in $\mathrm{Exc}(C)$ with color numbers ranging from $[1]$ to $[r-1]$, so there are $\binom{k}{i}$ ways to do this.

Let $t=\mathrm{csum}(C')$ and t_ℓ be the color number of $\ell\in[\![d]\!]$, then we have the equation $t=t_1+t_2+\cdots+t_d$ with $0\le t_1,t_2,\cdots,t_d\le r-1$ such that

- $t_j=0$ for $j\in\mathrm{Exc}(C)$ and j has a color number $[0]$, and
- $1\le t_j\le r-1$ for all $j\in\mathrm{Exc}(C)-\mathrm{Exc}_A(C')$, so there are i number of such j's.

Therefore there are $U^{(i)}_{d-k,t}$ number of solutions of the above equation, totally, there are $\binom{k}{i}U^{(i)}_{d-k,t}$ ways to color the symbols in C such that $\mathrm{csum}(C')=t$ and $\mathrm{exc}_A(C')=k-i$, where $U^{(i)}_{d-k,t}$ is the coefficient of x^t in $(x+x^2+\cdots+x^{r-1})^i(1+x+\cdots+x^{r-1})^{d-k}$, which can be expressed as

$$\begin{aligned}
U^{(i)}_{d-k,t}&=[x^t](x+x^2+\cdots+x^{r-1})^i(1+x+\cdots+x^{r-1})^{d-k}\\
&=[x^t]\left(\frac{1-x^r}{1-x}-1\right)^i\left(\frac{1-x^r}{1-x}\right)^{d-k}\\
&=[x^t]\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\left(\frac{1-x^r}{1-x}\right)^{d+j-k}\\
&=\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\sum_{\ell\ge0}(-1)^\ell\binom{d+j-k}{\ell}\binom{d+j+t-k-\ell r-1}{t-\ell r}.
\end{aligned}$$

Let $C'=(i_1^{[t_1]},i_2^{[t_2]},\dots,i_d^{[t_d]})$, then $C'^d=(i_1^{[t]},i_2^{[t]},\dots,i_d^{[t]})$ with $t=t_1+t_2+\cdots+t_d$, hence $C'^m=(i_1^{[\frac{tm}{d}]},i_2^{[\frac{tm}{d}]},\dots,i_d^{[\frac{tm}{d}]})=1$ implies that $r|\frac{tm}{d}$. For any $\pi\in\mathcal{G}^m_{r,1,n}$ such that the symbol n lies in a cycle C' of length $d\ge2$ with $d|m$ (note that there are $\binom{n-1}{d-1}$ ways to choose the digits of such a cycle), define $\pi''\in\mathcal{G}^m_{r,1,n-d}$ in the following way: write π in its complete notation, i.e., as a matrix of two rows, see $(\star)$. The first row of π'' is $(1,2,\dots,n-d)$ while the second row is obtained from the second row of π by ignoring the digits in C' and the other digits are placed with the numbers $1,2,\dots,n-d$ in an order preserving way with respect to the second row of π. The parameters satisfy

$$\begin{aligned}
\mathrm{fix}(\pi)&=\mathrm{fix}(\pi''),\\
\mathrm{exc}_A(\pi)&=\mathrm{exc}_A(\pi'')+\mathrm{exc}_A(C'),\\
\mathrm{csum}(\pi)&=\mathrm{csum}(\pi'')+\mathrm{csum}(C').
\end{aligned}$$

The above consideration gives the following recurrence

$$\begin{aligned}
H^{(m)}_{r,1,n}(u,v,w)=H^{(m)}_{r,1,n-1}(u,v,w)&\sum_{\{t|0\le t<r,r|tm\}}uw^t+\\
&+\sum_{d|m,d\ge2}H^{(m)}_{r,1,n-d}(u,v,w)\binom{n-1}{d-1}A_{m,d}(v,w),
\end{aligned}$$

where

$$A_{m,d}(v,w)=\sum_{k=1}^{d-1}A_{d-1,k}\sum_{i=0}^{k}\binom{k}{i}v^{k-i}\sum_{r|\frac{tm}{d}}U^{(i)}_{d-k,t}w^{t}.$$

Rewriting the recurrence in terms of generating functions, we obtain that

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathcal{H}^{(m)}_{r,1}(x;u,v,w)=\sum_{n\geq 1}H^{(m)}_{r,1,n}(u,v,w)\frac{x^{n-1}}{(n-1)!}\\
&=\sum_{n\geq 1}\frac{x^{n-1}}{(n-1)!}H^{(m)}_{r,1,n-1}(u,v,w)\sum_{\{t|0\leq t<r,r|tm\}}uw^{t}+\\
&\quad+\sum_{d|m,d\geq 2}A_{m,d}(v,w)\frac{x^{d-1}}{(d-1)!}\sum_{n\geq d}\frac{x^{n-d}}{(n-d)!}H^{(m)}_{r,1,n-d}(u,v,w)\\
&=\mathcal{H}^{(m)}_{r,1}(x;u,v,w)\Big(\sum_{\{t|0\leq t<r,r|tm\}}uw^{t}+\sum_{d|m,d\geq 2}A_{m,d}(v,w)\frac{x^{d-1}}{(d-1)!}\Big).
\end{aligned}$$

Thus, the generating function $\mathcal{H}^{(m)}_{r,1}(x;u,v,w)$ satisfies

$$\frac{\frac{\partial}{\partial x}\mathcal{H}^{(m)}_{r,1}(x;u,v,w)}{\mathcal{H}^{(m)}_{r,1}(x;u,v,w)}=\sum_{\{t|0\leq t<r,r|tm\}}uw^{t}+\sum_{d|m,d\geq 2}A_{m,d}(v,w)\frac{x^{d-1}}{(d-1)!}.$$

Integrating with respect to x on both sides of the above differential equation, using the fact that $\mathcal{H}^{(m)}_{r,1}(0;u,v,w)=1$, we obtain the explicit expression for $\mathcal{H}^{(m)}_{r,1}(x;u,v,w)$ given in Theorem 3.1, and hence we complete the proof. □

Specially, if $m=p$ is a prime, then we have

Corollary 3.2. *Let $r\geq 1$ and p be a prime. The generating function $\mathcal{H}^{(p)}_{r,1}(x;u,v,w)$ is*

$$\exp\left\{ux\lambda_{r,p}(w)+\frac{x^{p}}{p!}\sum_{k=1}^{p-1}A_{p-1,k}\sum_{i=0}^{k}\binom{k}{i}v^{k-i}\sum_{j\geq 0}U^{(i)}_{p-k,jr}w^{jr}\right\},$$

where $\lambda_{r,p}(w)=\sum_{i=0}^{p-1}w^{\frac{ir}{p}}$ for $p\mid r$, and $\lambda_{r,p}(w)=1$ for $p\nmid r$.

For the sake of comparison, the cases $p=2$ and $p=3$ in Corollary 3.2 generate the explicit formulas for $\mathcal{H}^{(2)}_{r,1}(x;u,v,w)$ and $\mathcal{H}^{(3)}_{r,1}(x;u,v,w)$, that is

$$\begin{aligned}
\mathcal{H}^{(2)}_{r,1}(x;u,v,w)&=\exp(ux\lambda_{r,2}(w)+\frac{x^{2}}{2}(v+(r-1)w^{r})),\\
\mathcal{H}^{(3)}_{r,1}(x;u,v,w)&=\exp(ux\lambda_{r,3}(w)+\frac{x^{3}}{6}B_{3,3}(v,w)),
\end{aligned}$$

where $B_{3,3}(v,w)=v^{2}+v(1+3(r-1)w^{r})+(r^{2}-1)w^{r}+(r-1)(r-2)w^{2r}$.

Now let us compute the exponential generating function $\mathcal{H}^{(m)}_{r,s}(x;u,v,w)$ for the sequence $\{H^{(m)}_{r,s,n}(u,v,w)\}_{n\geq 0}$. For any $\sigma\in\mathcal{G}^{m}_{r,s,n}$, we have $\operatorname{csum}(\sigma)\equiv 0\pmod{s}$, so we should collect all the terms in which the exponent of w in $\mathcal{H}^{(m)}_{r,1}(u,v,w)$ is a multiplication of s. This observation can make us get the following:

Theorem 3.3. *For $r, m, s \geq 1$, let $\mathcal{H}^{(m)}_{r,1}(x;u,v,yw) = \sum_{n\geq 0} G_{m,r,n}(x;u,v,w)y^n$. Then*

$$\mathcal{H}^{(m)}_{r,s}(x;u,v,w) = \sum_{k\geq 0} G_{m,r,sk}(x;u,v,w).$$

Now let us focus on the case $m = 2$. Recall that

$$\mathcal{H}^{(2)}_{r,1}(x;u,v,w) = \begin{cases} e^{ux+\frac{1}{2}x^2(v+(r-1)w^r)}, & \text{if } r \text{ odd,} \\ e^{ux(1+w^{\frac{r}{2}})+\frac{1}{2}x^2(v+(r-1)w^r)}, & \text{if } r \text{ even .} \end{cases}$$

Then by Theorem 3.3, we can compute the explicit formula for $\mathcal{H}^{(2)}_{r,s}(x;u,v,w)$. Since $s \mid r$, we have two cases either r odd or r even.

- If r is an odd number, then it is clear that the exponent of y in each term of the expansions of $\mathcal{H}^{(2)}_{r,1}(x;u,v,yw)$ is always a multiplication of s. Hence,
$$\mathcal{H}^{(2)}_{r,s}(x;u,v,w) = \mathcal{H}^{(2)}_{r,1}(x;u,v,w).$$
- Similarly, if r is an even number and $s \mid \frac{r}{2}$, we have that
$$\mathcal{H}^{(2)}_{r,s}(x;u,v,w) = \mathcal{H}^{(2)}_{r,1}(x;u,v,w).$$
- Let r be any even number such that $s \nmid \frac{r}{2}$. Since $e^{ux(1+(yw)^{\frac{r}{2}})} = e^{ux}\sum_{k\geq 0}\frac{(ux(yw)^{\frac{r}{2}})^k}{k!}$ and $e^{\frac{1}{2}x^2(v+(r-1)(yw)^r)} = e^{\frac{1}{2}x^2 v}\sum_{k\geq 0}\frac{((r-1)x^2(yw)^r)^k}{2^k k!}$, then by collecting the coefficients of y in $\mathcal{H}^{(2)}_{r,1}(x;u,v,w)$ such that the exponent y is a multiplication of s, we get that

$$e^{\frac{1}{2}x^2(v+(r-1)(yw)^r)}\sum_{k\geq 0}\frac{(ux)^{2k}(yw)^{kr}}{(2k)!} = e^{ux+\frac{1}{2}x^2(v+(r-1)(yw)^r)}\frac{e^{uxw^{\frac{r}{2}}}+e^{-uxw^{\frac{r}{2}}}}{2}.$$

Therefore, the above cases give the following result.

Proposition 3.4. *We have*

$$\mathcal{H}^{(2)}_{r,s}(x;u,v,w) = \begin{cases} e^{ux+\frac{1}{2}x^2(v+(r-1)w^r)}, & \text{if } r \text{ odd,} \\ e^{ux(1+w^{\frac{r}{2}})+\frac{1}{2}x^2(v+(r-1)w^r)}, & \text{if } r \text{ even and } s \mid \frac{r}{2}, \\ e^{ux+\frac{1}{2}x^2(v+(r-1)w^r)}\frac{e^{uxw^{\frac{r}{2}}}+e^{-uxw^{\frac{r}{2}}}}{2}, & \text{if } r \text{ even and } s \nmid \frac{r}{2}. \end{cases}$$

Note that $\mathcal{H}^{(2)}_{r,s}(x;u,v,w)$ is the generating function for the number of involutions in $\mathcal{G}^{(2)}_{r,s,n}$. By expanding the generating functions, Bagno, Garber and Mansour [2] obtained the explicit formulas for the number of involutions in $\mathcal{G}^{(2)}_{r,s,n}$. But the expression in Proposition 5.7 [2] should be corrected by the third case of $\mathcal{H}^{(2)}_{r,s}(x;u,v,w)$ and hence Corollary 5.8 – 5.10 therein should be the following three corollaries, respectively.

Corollary 3.5. *The polynomial $H^{(2)}_{r,s,n}(u,v,w)$ is given by*

$$\sum_{k_1+2k_2+2k_3=n}\frac{n!}{k_1!(2k_2)!k_3!}\cdot\frac{u^{k_1+2k_2}w^{rk_2}(v+(r-1)w^r)^{k_3}}{2^{k_3}}.$$

Corollary 3.6. *Let $r \geq 1$. The number of colored involutions in $\mathcal{G}^{(2)}_{r,s,n}$ (r even, $s \nmid \frac{r}{2}$) with exactly k absolute fixed points and $\mathrm{exc}_A(\pi) = \ell$ is given by*

$$\sum_{k+2k_3=n, k_1+2k_2=k} \binom{k_3}{\ell} \cdot \frac{n!}{k_1!(2k_2)!k_3!} \cdot \frac{(r-1)^{k_3-\ell}}{2^{k_3}}.$$

Corollary 3.7. *The number of involutions $\pi \in \mathcal{G}^{(2)}_{r,s,n}$ (r even, $s \nmid \frac{r}{2}$) with $\mathrm{exc}^{\mathrm{Clr}}(\pi) = k$ is given by*

$$\sum_{k_1+2k_2+2k_3=n,\ r(k_2+k_3)=k} \frac{n!}{k_1!(2k_2)!k_3!} \cdot \left(\frac{r}{2}\right)^{\frac{k}{r}}.$$

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